HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATIONS ON FUZZY NORMED LINER SPACES

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Abstract. In this paper, we use the definition of fuzzy normed spaces given by Bag and Samanta and the behaviors of solutions of the additive functional equation are described. The Hyers-Ulam stability problem of this equation is discussed and theorems concerning the Hyers-Ulam-Rassias stability of the equation are proved on fuzzy normed linear space.

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1. Introduction

In 1992, Felbin [6] has offered an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [10]. He has shown that every finite dimensional normed linear space has a completion. Then Xiao and Zhu [16] have modified the definition of this fuzzy norm and studied the topological properties of fuzzy normed linear spaces. Another fuzzy norm is defined by Bag and Samanta [3]. Bag and Samanta [4] have defined concepts of weakly fuzzy boundedness, strongly fuzzy boundedness, fuzzy continuity, strongly fuzzy

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continuity, weakly fuzzy continuity, sequentially fuzzy continuity and fuzzy norm of linear operators with an associated fuzzy norm defined in [3].

Defining, in some way, the class of approximate solutions of the given functional equation one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation. Such a problem was formulated by Ulam in 1940 (cf. [15]) and solved in the next year for the Cauchy functional equation by Hyers [8]. In 1950, Aoki [1] and in 1978, Rassias [13] proved a generalization of Hyers theorem for additive and linear mappings, respectively:

Theorem 1.1. *Let f be an approximately additive mapping from a normed vector space X into a Banach space Y, i.e. f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1$. Then the mapping $A : X \longrightarrow Y$ defined by $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|A(x) - f(x)\| \leq (2\varepsilon/2 - 2^p)\|x\|^p$$

for all $x \in X$.

The result of Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassias theorem was obtained by Gavruta [7] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

Moreover, some authors introduce some reasonable versions of fuzzy approximately additive functions on fuzzy normed spaces.(see [2, 5, 11, 12]).

In this paper, we use the definition of fuzzy normed spaces given in [14] to exhibit some reasonable notions of fuzzy approximately additive functions in fuzzy normed spaces and prove that under some suitable conditions, an approximately additive function f from a fuzzy normed space X into a fuzzy Banach space Y can be approximated in a fuzzy sense by an additive mapping A from X to Y. This will able us to establish some versions of (generalized) Hyers-Ulam-Rassias stability in the fuzzy normed linear space setting.

2. Preliminaries

We give below some basic preliminaries required for this paper.

Definition 2.1. (Xiao and Zhu [9]) A mapping $\tilde{\eta}:\mathbb{R}\longrightarrow[0,1]$ is called a fuzzy real number with α-level set $[\tilde{\eta}]_\alpha=\{t:\tilde{\eta}(t)\geq\alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0\in\mathbb{R}$ such that $\tilde{\eta}(t_0)=1$.
(N2) for each $\alpha\in(0,1]$, there exist real numbers $\eta_\alpha^-\leq\eta_\alpha^+$ such that the $\alpha-$level set $[\tilde{\eta}]_\alpha$ is equal to the closed interval $[\eta_\alpha^-,\eta_\alpha^+]$.
The set of all fuzzy real numbers is denoted by $F(R)$. Since each $r\in\mathbb{R}$ can be considered as the fuzzy real number $\tilde{r}\in F(R)$ defined by

$$\tilde{r}(t)=\begin{cases}1 & , \quad t=r\\ 0 & , \quad t\neq r,\end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(R)$.

Definition 2.2. (Kaleva and Seikkala [10]) The arithmetic operations $+$, $-$, $\times$ and $/$ on $F(R)\times F(R)$ are defined by

$$\begin{aligned}(\tilde{\eta}+\tilde{\gamma})(t) &= \sup_{t=x+y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\ (\tilde{\eta}-\tilde{\gamma})(t) &= \sup_{t=x-y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\ (\tilde{\eta}\times\tilde{\gamma})(t) &= \sup_{t=xy}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\ (\tilde{\eta}/\tilde{\gamma})(t) &= \sup_{t=x/y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 2.3. (Kaleva and Seikkala [10]) Let $\tilde{\eta}\in F(R)$. If $\tilde{\eta}(t)=0$, for all $t<0$, then $\tilde{\eta}$ is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $F^+(R)$.

Lemma 2.4. *(Kaleva and Seikkala [10]) Let $\tilde{\eta},\tilde{\gamma}\in F(R)$ and $[\tilde{\eta}]_\alpha=[\eta_\alpha^-,\eta_\alpha^+]$, $[\tilde{\gamma}]_\alpha=[\gamma_\alpha^-,\gamma_\alpha^+]$. Then*
i) $[\tilde{\eta}+\tilde{\gamma}]_\alpha=[\eta_\alpha^-+\gamma_\alpha^-,\eta_\alpha^++\gamma_\alpha^+]$
ii) $[\tilde{\eta}-\tilde{\gamma}]_\alpha=[\eta_\alpha^--\gamma_\alpha^+,\eta_\alpha^+-\gamma_\alpha^-]$
iii) $[\tilde{\eta}\times\tilde{\gamma}]_\alpha=[\eta_\alpha^-\gamma_\alpha^-,\eta_\alpha^+\gamma_\alpha^+]$ *for* $\tilde{\eta},\tilde{\gamma}\in F^+(R)$
iv) $[1/\tilde{\eta}]_\alpha=[\frac{1}{\eta_\alpha^+},\frac{1}{\eta_\alpha^-}]$ *if* $\eta_\alpha^->0$

Definition 2.5. (Kaleva and Seikkala [10]) Let $\tilde{\eta}, \tilde{\gamma} \in F(R)$ and $[\tilde{\eta}]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\tilde{\gamma}]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0, 1]$. Define a partial ordering by $\tilde{\eta} \leq \tilde{\gamma}$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0, 1]$. Strict inequality in $F(R)$ is defined by $\tilde{\eta} < \tilde{\gamma}$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0, 1]$.

Lemma 2.6. *Let $\tilde{\eta} \in F(R)$. Then $\tilde{\eta} \in F^+(R)$ if and only if $\tilde{0} \leq \tilde{\eta}$.*

Definition 2.7. (Bag and Samanta) Let X be a linear space over $\mathbb{R}$ (real number). Let N be a fuzzy subset of $X \times \mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$
$(N1)$ $N(x, t) = 0$, for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x, t) = 1$, for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx, t) = N(x, t/|c|)$, for all $t \in \mathbb{R}$,
$(N4)$ $N(x + u, s + t) \geq min\{N(x, s), N(u, t)\}$, for all $s, t \in \mathbb{R}$,
$(N5)$ $N(x, .)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.
Then N is called a fuzzy norm on X.

We assume that
$(N6)$ $N(x, t) > 0$, for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x, .)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$,

Definition 2.8. (Bag and Samanta [4]) Let (X, N) be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

3. Functional Equations

In this section, we study four types of fuzzy versions of the Hyers-Ulam-Rassias theorems(3.1.,3.4.,3.8.,3.11.) and some corollaries.

Theorem 3.1. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces and $f : X \longrightarrow Y$ be a function such that*

$$N_2(f(x + y) - f(x) - f(y), \delta_\alpha^-) \geq \alpha, \text{ for all } x, y \in X,$$

for some $\tilde{0} < \tilde{\delta} \in F(R)$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

exists for each $x \in X$ and $A : X \longrightarrow Y$ is the unique additive function such that

$$N_2(A(x) - f(x), \delta_\alpha^-) \geq \alpha, \text{ for all } x \in X. \tag{1}$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$ then A is linear.

Proof. We have

$$N_2(f(x+y) - f(x) - f(y), \delta_\alpha^-) \geq \alpha, \text{ for all } x, y \in X.$$

Hence $N_2(f(2x) - 2f(x), \delta_\alpha^-) \geq \alpha$, for all $x \in X$. Replacing x by $x/2$ in this inequality, we obtain that

$$\begin{aligned} N_2(1/2f(x) - f(x/2), \delta_\alpha^-/2) &= N_2(2(1/2f(x) - f(x/2)), \delta_\alpha^-) \\ &= N_2(f(x) - 2f(x/2), \delta_\alpha^-) \\ &\geq \alpha, \text{ for all } x \in X. \end{aligned}$$

By induction on n, we have

$$N_2(2^{-n}f(x) - f(2^{-n}x), (1 - 2^{-n})\delta_\alpha^-) \geq \alpha, \text{ for all } x \in X.$$

So

$$N_2(2^{-n}f(2^n x) - f(x), (1 - 2^{-n})\delta_\alpha^-) \geq \alpha, \text{ for all } x \in X.$$

We obtain that $N_2(2^{-n}f(2^n x) - 2^{-m}f(2^m x), (2^{-m} - 2^{-n})\delta_\alpha^-) \geq \alpha$, for all $x \in X$, $n > m$ and $\alpha \in (0, 1)$.
Let $t > 0$ and $\alpha \in (0, 1)$ be given. Hence there is $K > 0$ such that

$$N_2(2^{-n}f(2^n x) - 2^{-m}f(2^m x), t) \geq N_2(2^{-n}f(2^n x) - 2^{-m}f(2^m x), (2^{-m} - 2^{-n})\delta_\alpha^-)$$

$$\geq \alpha, \text{ for all } n, m \geq K.$$

So $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$$

exists for each $x \in X$.
Let $x, y \in X$. Suppose that $t > 0$ and $\alpha \in (0, 1)$. We have

$$N_2(2^{-n}f(2^n x + 2^n x) - 2^{-n}f(2^n x) - 2^{-n}f(2^n y), 2^{-n}\delta_\alpha^-) \geq \alpha, \text{ for all } n \in \mathbb{N}.$$

Then there is $K > 0$ such that

$$\begin{aligned} N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t) &\geq \\ N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),2^{-n}\delta_\alpha^-) &\geq \alpha, \text{ for all } n \geq K. \end{aligned}$$

So

$$\lim_{n\to\infty} N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t)=1, \text{ for all } t>0.$$

Since

$$\lim_{n\to\infty} N_2(2^{-n}f(2^nx)+2^{-n}f(2^ny)-A(x)-A(y),t)=1, \text{ for all } t>0,$$

and

$N_2(2^{-n}f(2^nx+2^nx)-A(x)-A(y),t) \geq \min\{N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t/2), N_2(A(x)+A(y)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t/2)\}$, for all $t>0$,
it follows that

$$\lim_{n\to\infty} N_2(2^{-n}f(2^nx+2^nx)-A(x)-A(y),t)=1 \text{ for all } t>0.$$

Now we have

$$N_2(A(x+y)-A(x)-A(y),t) \geq$$

$$\min\{N_2(2^{-n}f(2^nx+2^nx)-A(x+y),t/2), N_2(A(x)+A(y)-2^{-n}f(2^nx+2^nx),t/2)\},$$

for all $t>0$. As $n \longrightarrow \infty$, we get

$$N_2(A(x+y)-A(x)-A(y),t)=1, \text{ for all } t>0.$$

Hence $A(x+y)=A(x)+A(y)$, so A is an additive function.
Let $x \in X$. We have

$$N_2(2^{-n}f(2^nx)-A(x),2^{-n}\delta_\alpha^-) \geq$$

$$\min\{N_2(2^{-m}f(2^mx)-A(x),2^{-n}\delta_\alpha^- - \epsilon), N_2(2^{-n}f(2^nx)-2^{-m}f(2^mx),\epsilon)\},$$

for all $m,n \in \mathbb{N}$. Since $\{2^{-n}f(2^nx)\}$ is Cuachy and $A(x)=\lim_{n\to\infty} 2^{-n}f(2^nx)$, there is $K>0$ such that $N_2(2^{-n}f(2^nx)-A(x),2^{-n}\delta_\alpha^-) \geq \alpha$, for all $n>K$. Hence

$$N_2(f(x)-A(x),\delta_\alpha^-) \geq$$

$$\min\{N_2(2^{-n}f(2^nx)-A(x),2^{-n}\delta_\alpha^-), N_2(2^{-n}f(2^nx)-f(x),(1-2^{-n})\delta_\alpha^-)\} \geq \alpha,$$

for all $n>K$. Thus $N_2(f(x)-A(x),\delta_\alpha^-) \geq \alpha$.
Let $A' : X \longrightarrow Y$ be another additive function satisfying in (1) and $A(y) \neq A'(y)$

for some $y \in X$. Then there exists $t_0 > 0$ such that $N_2(A(y) - A'(y), t_0) < 1$. Suppose that $N_2(A(y) - A'(y), t_0) < \alpha < 1$. For any integer $n > 2\delta_\alpha^-/t_0$, we see that

$$\begin{aligned}
\alpha &> N_2(A(y) - A'(y), t_0) \\
&\geq N_2(A(y) - A'(y), 2\delta_\alpha^-/n) \\
&= N_2(A(ny) - A'(ny), 2\delta_\alpha^-) \\
&\geq \min\{N_2(A(ny) - f(ny), \delta_\alpha^-), N_2(f(ny) - A'(ny), \delta_\alpha^-)\} \\
&\geq \alpha,
\end{aligned}$$

this is contradiction. Hence, A is the unique additive function satisfying the inequality (1).
Assume that f is continuous at y. If A is not continuous at a point y, then there exist $t_0 > 0$, $0 < \epsilon < 1$ and a sequence $\{x_n\} \subseteq X$ converging to zero such that $N_2(A(x_n), t_0) < 1 - \epsilon$, for all $n \in \mathbb{N}$. Let m be an integer greater than $3\delta_{1-\epsilon}^-/t_0$. Then

$$N_2(A(mx_n), 3\delta_{1-\epsilon}^-) = N_2(A(x_n), 3\delta_{1-\epsilon}^-/m) \leq N_2(A(x_n), t_0) < 1 - \epsilon, \text{ for all } n \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned}
1 - \epsilon &> N_2(A(mx_n), 3\delta_{1-\epsilon}^-) \\
&= N_2(A(mx_n + y) - A(y), 3\delta_{1-\epsilon}^-) \\
&\geq \min\{N_2(A(mx_n + y) - f(mx_n + y), \delta_{1-\epsilon}^-), N_2(f(mx_n + y) - f(y), \delta_{1-\epsilon}^-), \\
&\quad N_2(A(y) - f(y), \delta_{1-\epsilon}^-)\} \\
&\geq \min\{1 - \epsilon, N_2(f(mx_n + y) - f(y), \delta_{1-\epsilon}^-)\}, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Since $f(mx_n + y) \longrightarrow f(y)$ as $n \longrightarrow \infty$. This contradiction means that the continuity of f at a point in X implies the continuity of A on X. For a fixed $x \in X$, if $f(tx)$ is continuous in t, then it follows from the above consideration that $A(tx)$ is continuous in t, hence A is linear. □

Example 3.2. Let $(X, \|.\|)$ be a Banach spaces and $f : X \longrightarrow X$ be a function such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta, \text{ for all } x, y \in X,$$

for some $0 < \delta \in \mathbb{R}$. Define a fuzzy norms N as follows:

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\tilde{\delta} \in F(R)$ and $[\tilde{\delta}]_\alpha = [\alpha\delta, \delta]$. Suppose that $x, y \in X$ and $\alpha \in (0,1)$. If $\|f(x+y) - f(x) - f(y)\| < \alpha\delta$ then $N(f(x+y) - f(x) - f(y), \alpha\delta) = 1 \geq \alpha$. If $\|f(x+y) - f(x) - f(y)\| \geq \alpha\delta$ then

$$N(f(x+y) - f(x) - f(y), \alpha\delta) = \alpha\delta/\|f(x+y) - f(x) - f(y)\| \geq \alpha.$$

Hence $N(f(x+y) - f(x) - f(y), \alpha\delta) \geq \alpha$, for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

exists for each $x \in X$ and $A : X \longrightarrow Y$ is the unique additive function such that

$$N(A(x) - f(x), \alpha\delta) \geq \alpha, \text{ for all } x \in X.$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$ then A is linear.

Corollary 3.3. *Under the hypotheses of Theorem 3.1, if f is continuous at a single point of X, then A is continuous everywhere in X.*

Theorem 3.4. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces and $f : X \longrightarrow Y$ be a function such that*

$$\begin{aligned} N_1(x, t^{1/p}) \ \geq \ \alpha \text{ and } N_1(y, s^{1/p}) \ &\geq \ \alpha \text{ implies that} \\ N_2(f(x+y) - f(x) - f(y), \theta_\alpha^-(t+s)) \ &\geq \ \alpha, \text{ for all } x, y \in X, \end{aligned}$$

for some $\tilde{0} < \tilde{\theta} \in F(R)$, $p \in [0,1)$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N_1(x, t^{1/p}) \geq \alpha \text{ implies that } N_2(A(x) - f(x), (2\theta_\alpha^-/2 - 2^p)t)) \geq \alpha, \text{ for all } x \in X.$$

Moreover, if f is continuous at $0 \in X$ then A is linear.

Proof. Let $x \in X$. If $N_1(x, t^{1/p}) \geq \alpha$ then $N_2(f(2x) - 2f(x), \theta_\alpha^-(2t)) \geq \alpha$, thus $N_2(2^{-1}f(2x) - f(x), \theta_\alpha^- t) \geq \alpha$, by induction on n, we have

$$N_2(2^{-n}f(2^n x) - f(x), \theta_\alpha^- t \sum_{m=0}^{n-1} 2^{m(p-1)}) \geq \alpha, \text{ for all } n \in \mathbb{N},$$

hence

$$N_2(2^{-n}f(2^nx) - f(x), (2\theta_\alpha^-/(2-2^p))t) \geq \alpha, \text{ for all } n \in \mathbb{N}.$$

Thus

$$N_2(2^{-m}f(2^mx) - 2^{-n}f(2^nx), 2^{n(p-1)}(2\theta_\alpha^-/(2-2^p))t) \geq \alpha, \text{ for all } m > n > 0.$$

Let $s > 0$ and $0 < \varepsilon < 1$. Then there is $K > 0$ such that $s \geq 2^{n(p-1)}(2\theta_\alpha^-/(2-2^p))t$, for all $n \geq K$. Hence

$$N_2(2^{-m}f(2^mx) - 2^{-n}f(2^nx), s) \geq \alpha, \text{ for all } m > n > K.$$

So $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^nx)$$

exists for each $x \in X$.
Moreover, we have

$$N_2(A(x) - f(x), (2\theta_\alpha^-/(2-2^p))t) \geq$$

$$\min\{N_2(2^{-n}f(2^nx) - f(x), \theta_\alpha^- t\sum_{k=0}^{n-1} 2^{k(p-1)}), N_2(2^{-n}f(2^nx) - 2^{-m}f(2^mx), \epsilon),$$

$$N_2(2^{-m}f(2^mx) - A(x), (2\theta_\alpha^-/(2-2^p))t - \theta_\alpha^- t\sum_{k=0}^{n-1} 2^{k(p-1)} - \epsilon)\}, \text{ for all } m, n \in \mathbb{N}.$$

Since $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence and $A(x) = \lim_{n\to\infty} 2^{-n}f(2^nx)$,

$$N_2(A(x) - f(x), (2\theta_\alpha^-/(2-2^p))t) \geq \alpha.$$

Hence

$N_1(x, t^{1/p}) \geq \alpha$ implies that $N_2(A(x) - f(x), (2\theta_\alpha^-/2 - 2^p)t)) \geq \alpha,$ for all $x \in X$.

Let $t > 0$ and $0 < \epsilon < 1$ be given. Hence there is $K > 0$ such that

$N_1(x, 2^{(n(1-p)-1)/p}t^{1/p}) \geq 1 - \epsilon$ and $N_1(y, 2^{(n(1-p)-1)/p}t^{1/p}) \geq 1 - \epsilon,$ for all $n \geq K$.

So

$N_1(2^nx, 2^{(n-1)/p}t^{1/p}) \geq 1 - \epsilon$ and $N_1(2^ny, 2^{(n-1)/p}t^{1/p}) \geq 1 - \epsilon,$ for all $n \geq K$.

Thus

$$\begin{aligned} N_2(f(2^nx+2^nx)-f(2^nx)-f(2^ny),2^nt) &= \\ N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t) &\geq 1-\epsilon, \end{aligned}$$

for all $n \geq K$. Hence

$$\lim_{n\to\infty} N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t)=1, \text{ for all } t>0.$$

We have

$$\begin{gathered} N_2(A(x+y)-A(x)-A(y),t) \geq \\ \min\{N_2(2^{-n}f(2^nx+2^nx)-A(x+y),t/3), \\ N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t/3), \\ N_2(2^{-n}f(2^nx)+2^{-n}f(2^ny)-A(x)-A(y),t/3)\}, \end{gathered}$$

for all $t>0$, as $n \longrightarrow \infty$, we obtain that

$$N_2(A(x+y)-A(x)-A(y),t)=1, \text{ for all } t>0.$$

Hence $A(x+y)=A(x)+A(y)$, so A is an additive function.
We now want to prove that A is such a unique additive function. Assume that there exists another one, denoted by $A' : X \longrightarrow Y$ such that

$N_1(x,t^{1/p}) \geq \alpha$ implies that $N_2(A'(x)-f(x),(2\theta_\alpha^-/(2-2^p))t)) \geq \alpha$, for all $x \in X$.

Let $x \in X$, $t>0$ and $\alpha \in (0,1)$. Since $\lim_{t\to\infty} N_1(x,t)=1$, there exists $K>0$ such that $N_1(x,(((2-2^p)/4\theta_\alpha^-)n^{1-p}t)^{1/p}) \geq \alpha$, for all $n>K$. Hence

$$N_1(nx,(((2-2^p)/4\theta_\alpha^-)nt)^{1/p}) \geq \alpha, \text{ for all } n>K.$$

$N_2(A(nx)-f(nx),nt/2)) \geq \alpha$ and $N_2(A'(nx)-f(nx),nt/2)) \geq \alpha$, for all $n>K$.

So

$$N_2(1/n(A(nx)-f(nx)),t/2)) \geq \alpha \text{ and } N_2(1/n(A'(nx)-f(nx)),t/2)) \geq \alpha,$$

for all $n>K$. Now we get

$$\begin{gathered} N_2(A(x)-A'(x)),t)) = N_2(1/n(A(nx)-A'(nx)),t)) \geq \\ \min\{N_2(1/n(A(nx)-f(nx)),t/2)),N_2(1/n(f(nx)-A'(nx)),t/2))\} \geq \alpha, \end{gathered}$$

for all $n > K$. Thus $N_2(A(x) - A'(x), t) \geq \alpha$. As $\alpha \longrightarrow 1^-$, we get $N_2(A(x) - A'(x), t) = 1$. So $A(x) = A'(x)$. then $A = A'$.
Let f be continuous at $0 \in X$. Suppose that $\{x_n\} \subseteq X$ and $x_n \longrightarrow 0$. Let $t > 0$ and $\epsilon \in (0, 1)$. Then there exists $K > 0$ such that

$$N_2(f(x_n), t/2) \geq 1 - \epsilon \text{ and } N_1(x_n, (((2 - 2^p)/4\theta^-_{1-\epsilon})t)^{1/p}) \geq 1 - \epsilon, \text{ for all } n > K.$$

Thus

$$N_2(f(x_n), t/2) \geq 1 - \epsilon \text{ and } N_2(A(x_n) - f(x_n), t/2) \geq 1 - \epsilon, \text{ for all } n > K.$$

Now we have

$$\begin{aligned} N_2(A(x_n), t) &\geq \min\{N_2(A(x_n) - f(x_n), t/2), N_2(f(x_n), t/2)\} \\ &\geq 1 - \epsilon, \text{ for all } n > K. \end{aligned}$$

So $\lim_{n\to\infty} N_2(A(x_n), t) = 1$. Then A is continuous at 0, which implies that A is linear. □

Example 3.5. Let $(X, \|.\|)$ be a Banach spaces and $f : X \longrightarrow X$ be a function such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \text{ for all } x, y \in X,$$

for some $0 < \theta \in \mathbb{R}$. Define a fuzzy norms N as follows:

$$N(x, t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\tilde{\theta} \in F(R)$ and $[\tilde{\theta}]_\alpha = [\alpha^{1-p}\theta, \theta]$. Suppose that $x, y \in X$, $s, t \in \mathbb{R}$, $\alpha \in (0, 1)$, $N(x, t^{1/p}) \geq \alpha$ and $N(y, s^{1/p}) \geq \alpha$. Now we show that

$$N(f(x + y) - f(x) - f(y), \alpha^{1-p}\theta(t + s)) \geq \alpha.$$

If $\|f(x + y) - f(x) - f(y)\| < \alpha^{1-p}\theta(t + s)$ then

$$N(f(x + y) - f(x) - f(y), \alpha^{1-p}\theta(t + s)) = 1 \geq \alpha.$$

If $\alpha^{1-p}\theta(t + s) \leq \|f(x + y) - f(x) - f(y)\|$.
Case 1: Let $t^{1/p} \leq \|x\|$ and $s^{1/p} \leq \|y\|$. We have $t^{1/p}/\|x\| = N(x, t^{1/p}) \geq \alpha$ and $s^{1/p}/\|y\| = N(y, s^{1/p}) \geq \alpha$. Hence $\alpha^p\|x\|^p \leq t$ and $\alpha^p\|y\|^p \leq s$. So

$$\alpha^p\|f(x + y) - f(x) - f(y)\| \leq \alpha^p\theta(\|x\|^p + \|y\|^p) \leq \theta(t + s).$$

Thus

$N(f(x+y)-f(x)-f(y),\alpha^{1-p}\theta(t+s)) = \alpha^{1-p}(t+s)/\|f(x+y)-f(x)-f(y)\| \geq \alpha.$

Case 2: Let $\|x\| < t^{1/p}$ and $s^{1/p} \leq \|y\|$. We have $\alpha^p\|x\|^p \leq \|x\|^p \leq t$ and $s^{1/p}/\|y\| = N(y, s^{1/p}) \geq \alpha$. Hence $\alpha^p\|x\|^p \leq t$ and $\alpha^p\|y\|^p \leq s$. Similar to case 1, we obtain that

$$N(f(x+y)-f(x)-f(y),\alpha^{1-p}\theta(t+s)) \geq \alpha.$$

Case 3: Let $\|x\| < t^{1/p}$ and $\|y\| < s^{1/p}$. So $\alpha^p\|x\|^p \leq \|x\|^p < t$ and $\alpha^p\|y\|^p \leq \|y\|^p < s$. Similar to case 1, We get

$$N(f(x+y)-f(x)-f(y),\alpha^{1-p}\theta(t+s)) \geq \alpha.$$

Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$N(x,t^{1/p}) \geq \alpha$ implies that $N(A(x)-f(x),(2\alpha^{1-p}\theta/2-2^p)t)) \geq \alpha,$ for all $x \in X$.

Moreover, if f is continuous at $0 \in X$ then A is linear.

Corollary 3.6. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces and $f : X \longrightarrow Y$ be a function such that*

$$N_2(f(x+y)-f(x)-f(y),\theta^-_\alpha(t+s)) \geq 1/2(N_1(x,t^{1/p})+N_1(y,s^{1/p})),$$

for all $x, y \in X$, for some $\tilde{0} < \tilde{\theta} \in F(R)$ and $p \in [0,1)$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N_2(A(x)-f(x),(2\theta^-_\alpha/2-2^p)t)) \geq N_1(x,t^{1/p}),\ \textit{for all } x \in X.$$

Moreover, if f is continuous at $0 \in X$ then A is linear.

Corollary 3.7. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces and $f : X \longrightarrow Y$ be a function such that*

$$N_2(f(x+y)-f(x)-f(y),\theta^-_\alpha(t+s)) \geq \min\{N_1(x,t^{1/p}),N_1(y,s^{1/p})\},$$

for all $x, y \in X$, for some $\tilde{0} < \tilde{\theta} \in F(R)$ and $p \in [0,1)$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N_2(A(x)-f(x),(2\theta^-_\alpha/2-2^p)t)) \geq N_1(x,t^{1/p}),\ \textit{for all } x \in X.$$

Moreover, if f is continuous at $0 \in X$ then A is linear.

Theorem 3.8. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces satisfying $(N7)$ and $\varphi : X \times X \longrightarrow F^+(R)$ be a function such that*

$$\phi(x, y) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi(2^k x, 2^k y),$$

is convergent for all $x, y \in X$. Also let $f : X \longrightarrow Y$ be a function satisfying

$$N_2(f(x+y) - f(x) - f(y), (\varphi(x, y))^-_\alpha) \geq \alpha, \text{ for all } x, y \in X.$$

Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N_2(A(x) - f(x), (\phi(x, x))^-_\alpha) \geq \alpha, \text{ for all } x \in X.$$

Proof. Let $\alpha \in (0, 1)$ and $x, y \in X$. We have

$$N_2(f(x+y) - f(x) - f(y), (\varphi(x, y))^-_\alpha) \geq \alpha,$$

then

$$N_2((1/2)f(2x) - f(x), (1/2)(\varphi(x, x))^-_\alpha) \geq \alpha.$$

By induction on n, we obtain that

$$N_2((2^{-n})f(2^n x) - f(x), \sum_{k=0}^{n-1} 2^{-k-1}(\varphi(2^k x, 2^k x))^-_\alpha) \geq \alpha, \text{ for all } n > 0.$$

Hence

$$N_2(2^{-m}f(2^m x) - 2^{-n}f(2^n x), \sum_{k=m}^{n-1} 2^{-k-1}(\varphi(2^k x, 2^k x))^-_\alpha) \geq \alpha, \text{ for all } n > m > 0.$$

Let $t > 0$. Since $\phi(x, x) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi(2^k x, 2^k x)$ is convergent for all $x \in X$, for every $\epsilon > 0$, there exists $K_\epsilon > 0$ such that

$$N_2(2^{-m}f(2^m x) - 2^{-n}f(2^n x), t) \geq$$
$$N_2(2^{-m}f(2^m x) - 2^{-n}f(2^n x), \sum_{k=m}^{n-1} 2^{-k-1}(\varphi(2^k x, 2^k x))^-_\alpha) \geq \alpha, \text{ for all } n > m > K_\epsilon.$$

Thus $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$$

exists for each $x \in X$.

Let $x \in X$. Then

$$N_2(A(x) - f(x), (\phi(x, x))^-_\alpha) \geq$$

$$\min\{N_2(f(x)-2^{-n}f(2^nx),\sum_{k=0}^{n-1}2^{-k-1}(\varphi(2^kx,2^kx))_\alpha^-),$$

$$N_2(A(x)-2^{-m}f(2^mx),(\phi(x,x))_\alpha^- - \sum_{k=0}^{n-1}2^{-k-1}(\varphi(2^kx,2^kx))_\alpha^- - \epsilon),$$

$$N_2(2^{-m}f(2^mx)-2^{-n}f(2^nx)\epsilon))\}.$$

Since $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence and $A(x)=\lim_{n\to\infty}2^{-n}f(2^nx)$,

$$N_2(A(x)-f(x),(\phi(x,x))_\alpha^-)\geq\alpha.$$

Let $x,y\in X$. We have

$$N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),2^{-n}(\varphi(2^nx,2^ny))_\alpha^-)\geq\alpha,$$

for all $\alpha\in(0,1)$ and all $n\in\mathbb{N}$. Since $\phi(x,y)=\sum_{k=0}^{\infty}2^{-k-1}\varphi(2^kx,2^ky)$ is convergent, $\lim_{n\to\infty}2^{-n}(\varphi(2^nx,2^ny))_\alpha^-=0$. Let $t>0$ and $0<\alpha<1$ be given. Hence there is $K>0$ such that

$$\begin{aligned} N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t) &\geq \\ N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),2^{-n}(\varphi(2^nx,2^ny))_\alpha^-) &\geq \alpha, \end{aligned}$$

for all $n\geq K$. Hence

$$\lim_{n\to\infty}N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t)=1,\ \text{for all } t>0.$$

We have

$$N_2(A(x+y)-A(x)-A(y),t)\geq$$

$$\min\{N_2(2^{-n}f(2^nx+2^nx)-A(x+y),t/3),$$

$$N_2(2^{-n}f(2^nx+2^nx)-2^{-n}f(2^nx)-2^{-n}f(2^ny),t/3),$$

$$N_2(2^{-n}f(2^nx)+2^{-n}f(2^ny)-A(x)-A(y),t/3)\},$$

for all $t>0$, as $n\longrightarrow\infty$, we obtain that

$$N_2(A(x+y)-A(x)-A(y),t)=1,\ \text{for all } t>0.$$

Hence $A(x+y)=A(x)+A(y)$, so A is an additive function.
We now want to prove that A is such a unique additive function. Assume that there exists another one, denoted by $A':X\longrightarrow Y$ such that

$$N_2(A'(x)-f(x),(\phi(x,x))_\alpha^-)\geq\alpha,\ \text{for all } x\in X.$$

Let $x \in X$ and $t > 0$. Then there exists $K > 0$ such that $t \geq \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x)_{\alpha}^{-}$, for all $n \geq K$. So

$$\begin{aligned} N_2(A'(x) - A(x), t) &\geq N_2(A'(x) - A(x), \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x)_{\alpha}^{-}) \\ &= N_2(A'(x) - A(x), 1/2^n(2(\phi(2^n x, 2^n x))_{\alpha}^{-})) \\ &= N_2(A'(2^n x) - A(2^n x), 2(\phi(2^n x, 2^n x)_{\alpha}^{-})) \end{aligned}$$

$$\geq \min\{N_2(A'(2^n x) - f(2^n x), \phi(2^n x, 2^n x)_{\alpha}^{-}), N_2(f(2^n x) - A(2^n x), \phi(2^n x, 2^n x)_{\alpha}^{-})\}$$

$$\geq \alpha, \text{ for all } n \geq K.$$

As $\alpha \longrightarrow 1^{-}$ we get

$$N_2(A(x) - A'(x), t) = 1.$$

So $A(x) = A'(x)$, for all $x \in X$. then $A = A'$.

□

Example 3.9. Let $(X, \|.\|)$ be a Banach spaces and $\varphi_0 : X \times X \longrightarrow [0, +\infty)$ be a function satisfying

$$\phi_0(x, y) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi_0(2^k x, 2^k y) < +\infty,$$

for all $x, y \in X$. Suppose that a function $f : X \longrightarrow X$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi_0(x, y), \text{ for all } x, y \in X.$$

Define a fuzzy norms N as follows:

$$N(x, t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\varphi : X \times X \longrightarrow F^{+}(R)$ and $[\varphi(x, y)]_{\alpha} = [\alpha\varphi_0(x, y), \varphi_0(x, y)]$. Suppose that $x, y \in X$ and $\alpha \in (0, 1)$. If $\|f(x + y) - f(x) - f(y)\| < \alpha\varphi_0(x, y)$ then

$$N(f(x + y) - f(x) - f(y), \alpha\varphi_0(x, y)) = 1 \geq \alpha.$$

If $\|f(x + y) - f(x) - f(y)\| \geq \alpha\varphi_0(x, y)$ then

$$N(f(x + y) - f(x) - f(y), \alpha\varphi_0(x, y)) = \alpha\varphi_0(x, y)/\|f(x + y) - f(x) - f(y)\| \geq \alpha.$$

Hence $N(f(x+y)-f(x)-f(y),\alpha\varphi_0(x,y))\geq\alpha$, for all $x,y\in X$. Then there exists a unique additive function $A:X\longrightarrow Y$ with

$$N(A(x)-f(x),\alpha\phi_0(x,x))\geq\alpha,\ \text{for all } x\in X.$$

Theorem 3.10. *Let (X,N_1) and (Y,N_2) be fuzzy Banach spaces satisfying (N7) and $\varphi:X\times X\longrightarrow F^+(R)$ be a function such that*

$$\phi(x,y)=\sum_{k=0}^{\infty}2^k\varphi(2^{-k}x,2^{-k}y),$$

is convergent for all $x,y\in X$. Also let $f:X\longrightarrow Y$ be a function satisfying

$$N_2(f(x+y)-f(x)-f(y),(\varphi(x,y))_\alpha^-)\quad\geq\quad\alpha,\ \textit{for all } x,y\in X.$$

Then there exists a unique additive function $A:X\longrightarrow Y$ such that

$$N_2(A(x)-f(x),(\phi(x,x))_\alpha^-)\quad\geq\quad\alpha,\ \textit{for all } x\in X.$$

Proof. Proof is similar to proof of Theorem 3.8. □

Theorem 3.11. *Let (X,N_1) and (Y,N_2) be fuzzy Banach spaces satisfying (N7) and $\varphi:X\times X\longrightarrow F^+(R)$ be a function such that*

$$\phi(x)=\sum_{i=1}^{\infty}2^{-i}(\varphi(2^{i-1}x,0)+\varphi(0,2^{i-1}x)+\varphi(2^{i-1}x,2^{i-1}x)),$$

is convergent for all $x\in X$, and

$$\lim_{n\to\infty}2^{-n}\varphi(2^nx,2^ny)=0,\ \textit{for all } x,y\in X.$$

Also let $f,g,h:X\longrightarrow Y$ be functions satisfying

$$N_2(f(x+y)-g(x)-h(y),(\varphi(x,y))_\alpha^-)\quad\geq\quad\alpha,\ \textit{for all } x,y\in X.$$

Then there exists a unique additive function $A:X\longrightarrow Y$ such that

$$N_2(g(0),t)\geq\alpha,\ \ N_2(h(0),s)\geq\alpha,$$

implies that

$$N_2(A(x)-f(x),t+s+(\phi(x))_\alpha^-)\geq\alpha,$$
$$N_2(A(x)-g(x),t+2s+(\phi(x))_\alpha^-+(\varphi(x,0))_\alpha^-)\geq\alpha,$$
$$N_2(A(x)-h(x),2t+s+(\phi(x))_\alpha^-+(\varphi(0,x))_\alpha^-)\geq\alpha,$$

for all $x\in X$.

Proof. Let $x \in X$ and $N_2(g(0), t) \geq \alpha,\ N_2(h(0), s) \geq \alpha$. We have

$$N_2(f(2x) - g(x) - h(x), (\varphi(x,x))^-_\alpha) \geq \alpha,$$

$$N_2(f(x) - g(x) - h(0), (\varphi(x,0))^-_\alpha) \geq \alpha,$$

$$N_2(A(x) - g(0) - h(x), (\varphi(0,x))^-_\alpha) \geq \alpha.$$

Hence

$$N_2(f(2x) - g(x) - h(x), (\varphi(x,y))^-_\alpha) \geq \alpha,$$

$$N_2(f(x) - g(x), s + (\varphi(x,0))^-_\alpha) \geq$$

$$\min\{N_2(f(x) - g(x) - h(0), (\varphi(x,0))^-_\alpha), N_2(h(0), s)\} \geq \alpha,$$

and

$$N_2(f(x) - h(x), t + (\varphi(0,x))^-_\alpha) \geq$$

$$\min\{N_2(f(x) - g(0) - h(x), (\varphi(0,x))^-_\alpha), N_2(g(0), t)\} \geq \alpha.$$

We define

$$u(x) = t + s + (\varphi(x,0))^-_\alpha + (\varphi(0,x))^-_\alpha + (\varphi(x,x))^-_\alpha, \text{ for all } x, \in X.$$

Now we obtain that

$$N_2(f(2x) - 2f(x), u(x)) \geq \min\{N_2(f(2x) - g(x) - h(x), (\varphi(x,y))^-_\alpha),$$

$$N_2(f(x) - g(x), s + (\varphi(x,0))^-_\alpha), N_2(f(x) - h(x), t + (\varphi(0,x))^-_\alpha)\} \geq \alpha,$$

for all $x \in X$. Thus

$$N_2(2^{-n} f(2^n x) - 2^{-m} f(2^m x),$$

$$(t+s)\sum_{i=m}^{n-1} 2^{-i-1} + \sum_{i=m}^{n-1} 2^{-i-1}((\varphi(2^i x, 0))^-_\alpha + (\varphi(0, 2^i x))^-_\alpha + (\varphi(2^i x, 2^i x))^-_\alpha)) \geq$$

$$N_2(2^{-n} f(2^n x) - 2^{-m} f(2^m x),$$

$$\sum_{i=m}^{n-1} 2^{-i-1}(t + s + (\varphi(2^i x, 0))^-_\alpha + (\varphi(0, 2^i x))^-_\alpha + (\varphi(2^i x, 2^i x))^-_\alpha)) =$$

$$N_2(2^{-n} f(2^n x) - 2^{-m} f(2^m x), \sum_{i=m}^{n-1} 2^{-i-1} u(2^i x)) \geq$$

$$\min\{N_2(2^{-m} f(2^m x) - 2^{-m-1} f(2^{m+1} x), 2^{-m-1} u(2^m x)), ...,$$

$$N_2(2^{-(n-1)} f(2^{n-1} x) - 2^{-n} f(2^n x), 2^{-n} u(2^{n-1} x))\} \geq \alpha,$$

for all $n > m > 0$. Since

$$\phi(x) = \sum_{i=1}^{\infty} 2^{-i}(\varphi(2^{i-1}x, 0) + \varphi(0, 2^{i-1}x) + \varphi(2^{i-1}x, 2^{i-1}x)),$$

is convergent for all $x \in X$ it follows that $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$$

exists for each $x \in X$.

Let $x \in X$, $r > 0$, $N_2(g(0), t) \geq \alpha$ and $N_2(h(0), s) \geq \alpha$. Since

$$N_2(A(x) - 2^{-n}g(2^n x), r) \geq \min\{N_2(2^{-n}f(2^n x) - A(x), r - 2^{-n}(s + (\varphi(x, 0))^-_\alpha)),$$
$$N_2(2^{-n}f(2^n x) - 2^{-n}g(2^n x), 2^{-n}(s + (\varphi(x, 0))^-_\alpha))\}, \text{ for all } n \in \mathbb{N}.$$

Then $A(x) = \lim_{n\to\infty} 2^{-n}g(2^n x)$. Similarly, $A(x) = \lim_{n\to\infty} 2^{-n}h(2^n x)$, for all $x \in X$. Let $x, y \in X$ and $r > 0$, we have

$$N_2(A(x+y) - A(x) - A(y), r) \geq \min\{N_2(A(x) + A(y) - 2^{-n}g(2^n x) - 2^{-n}h(2^n y), \epsilon),$$
$$N_2(A(x+y) - 2^{-n}f(2^n x + 2^n y), r - 2^{-n}(\varphi(2^n x, 2^n y))^-_\alpha - \epsilon),$$
$$N_2(2^{-n}f(2^n x + 2^n y) - 2^{-n}g(2^n x) - 2^{-n}h(2^n y), 2^{-n}(\varphi(2^n x, 2^n y))^-_\alpha)\},$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$. As $n \longrightarrow \infty$, then

$$N_2(A(x+y) - A(x) - A(y), r) \geq \alpha, \text{ for all } \alpha \in (0, 1).$$

Hence $A(x+y) = A(x) + A(y)$, for all $x, y \in X$, so A is an additive function.

Let $x \in X$ and $N_2(g(0), t) \geq \alpha$, $N_2(h(0), s) \geq \alpha$. Applying an induction argument on n, we get that

$$N_2(f(2^n x) - 2^n f(x), \sum_{i=1}^{n} 2^{i-1}u(2^{n-i}x)) \geq \alpha, \text{ for all } x \in X.$$

Thus

$$N_2(A(x) - f(x), t + s + \phi(x)^-_\alpha) \geq$$
$$\min\{N_2(A(x) - 2^{-n}f(2^n x), t + s + \phi(x)^-_\alpha - \sum_{i=1}^{n} 2^{i-1-n}u(2^{n-i}x)),$$
$$N_2(f(x) - 2^{-n}f(2^n x), \sum_{i=1}^{n} 2^{i-1-n}u(2^{n-i}x))\} \geq$$
$$\min\{\alpha, N_2(A(x) - 2^{-n}f(2^n x), (1 - \sum_{i=1}^{n} 2^{i-1-n})(t+s) +$$

$$\phi(x)^-_\alpha - \sum_{i=1}^{n} 2^{-i}((\varphi(2^{i-1}x,0))^-_\alpha + (\varphi(0,2^{i-1}x))^-_\alpha + (\varphi(2^{i-1}x,2^{i-1}x))^-_\alpha))\},$$

for all $n \in \mathbb{N}$. As $n \longrightarrow \infty$, then

$$N_2(A(x) - f(x), t + s + \phi(x)^-_\alpha) \geq \alpha,$$

similarly

$$N_2(A(x) - g(x), t + 2s + (\phi(x))^-_\alpha + (\varphi(x,0))^-_\alpha) \geq \alpha,$$

$$N_2(A(x) - h(x), 2t + s + (\phi(x))^-_\alpha + (\varphi(0,x))^-_\alpha) \geq \alpha,$$

for all $x \in X$.

We now want to prove that A is such a unique additive function. Assume that there exists another one, denoted by $A' : X \longrightarrow Y$ such that

$$N_2(g(0), t) \geq \alpha,\ N_2(h(0), s) \geq \alpha,$$

implies that

$$N_2(A'(x) - f(x), t + s + (\phi(x))^-_\alpha) \geq \alpha,$$

$$N_2(A'(x) - g(x), t + 2s + (\phi(x))^-_\alpha + (\varphi(x,0))^-_\alpha) \geq \alpha,$$

$$N_2(A'(x) - h(x), 2t + s + (\phi(x))^-_\alpha + (\varphi(0,x))^-_\alpha) \geq \alpha,$$

for all $x \in X$. Let $x \in X$, $r > 0$ and $N_2(g(0), t) \geq \alpha,\ N_2(h(0), s) \geq \alpha$. Then there is $K > 0$ such that

$$N_2(A'(x) - A(x), r) \geq N_2(A'(x) - A(x),$$

$$1/2^{n-1}(t+s)+2\sum_{i=n+1}^{\infty} 2^{-i}((\varphi(2^{i-1}x,0))^-_\alpha+(\varphi(0,2^{i-1}x))^-_\alpha+(\varphi(2^{i-1}x,2^{i-1}x))^-_\alpha)) \geq$$

$$\min\{N_2(2^{-n}A'(2^n x) - 2^{-n}f(2^n x),$$

$$1/2^n(t+s) + \sum_{i=n+1}^{\infty} 2^{-i}((\varphi(2^{i-1}x,0))^-_\alpha + (\varphi(0,2^{i-1}x))^-_\alpha + (\varphi(2^{i-1}x,2^{i-1}x))^-_\alpha)),$$

$$N_2(2^{-n}f(2^n x) - 2^{-n}A(2^n x),$$

$$1/2^n(t+s) + \sum_{i=n+1}^{\infty} 2^{-i}((\varphi(2^{i-1}x,0))^-_\alpha + (\varphi(0,2^{i-1}x))^-_\alpha + (\varphi(2^{i-1}x,2^{i-1}x))^-_\alpha))\} =$$

$$\min\{N_2(2^{-n}A'(2^n x) - 2^{-n}f(2^n x), 1/2^n(t+s+(\phi(2^n x))^-_\alpha),$$

$$N_2(2^{-n}f(2^n x) - 2^{-n}A(2^n x), 1/2^n(t+s+(\phi(2^n x))^-_\alpha))\} \geq \alpha,\ \text{for all } n > K.$$

Thus

$$N_2(A(x) - A'(x), r) \geq \alpha,\ \text{for all } \alpha \in (0,1).$$

As $\alpha \longrightarrow 1^-$ we get

$$N_2(A(x) - A'(x), t) = 1.$$

So $A(x) = A'(x)$, for all $x \in X$. then $A = A'$. $\square$

Example 3.12. Let $(X, \|.\|)$ be a Banach spaces and $\varphi_0 : X \times X \longrightarrow [0, +\infty)$ be a function satisfying

$$\phi_0(x) = \sum_{i=1}^{\infty} 2^{-i}(\varphi_0(2^{i-1}x, 0) + \varphi_0(0, 2^{i-1}x) + \varphi_0(2^{i-1}x, 2^{i-1}x)) < +\infty,$$

for all $x \in X$, and

$$\lim_{n\to\infty} 2^{-n}\varphi_0(2^n x, 2^n y) = 0, \text{ for all } x, y \in X.$$

Suppose that functions $f, g, h : X \longrightarrow X$ satisfy the inequality

$$\|f(x+y) - g(x) - h(y)\| \leq \varphi_0(x, y), \text{ for all } x, y \in X.$$

Define a fuzzy norms N as follows:

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\varphi : X \times X \longrightarrow F^+(R)$ and $[\varphi(x,y)]_\alpha = [\alpha\varphi_0(x,y), \varphi_0(x,y)]$. Suppose that $x, y \in X$ and $\alpha \in (0,1)$. If $\|f(x+y) - g(x) - h(y)\| < \alpha\varphi_0(x,y)$ then

$$N(f(x+y) - g(x) - h(y), \alpha\varphi_0(x,y)) = 1 \geq \alpha.$$

If $\|f(x+y) - g(x) - h(y)\| \geq \alpha\varphi_0(x,y)$ then

$$N(f(x+y) - f(x) - f(y), \alpha\varphi_0(x,y)) = \alpha\varphi_0(x,y)/\|f(x+y) - g(x) - h(y)\| \geq \alpha.$$

Hence $N(f(x+y) - g(x) - h(y), \alpha\varphi_0(x,y)) \geq \alpha$, for all $x, y \in X$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N(g(0), t) \geq \alpha, \ N(h(0), s) \geq \alpha,$$

implies that

$$N(A(x) - f(x), t + s + \alpha\phi_0(x)) \geq \alpha,$$

$$N(A(x) - g(x), t + 2s + \alpha(\phi_0(x) + \varphi_0(x, 0)) \geq \alpha,$$

$$N(A(x) - h(x), 2t + s + \alpha(\phi_0(x) + \varphi_0(0, x)) \geq \alpha,$$

for all $x \in X$.

Corollary 3.13. *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces and $\tilde{\theta} \in F^+(R)$ and $p \in [0, 1)$ be constants. Also let $f, g, h : X \longrightarrow Y$ be functions such that*

$$\begin{aligned} N_1(x, t^{1/p}) \ \geq \ \alpha \text{ and } N_1(y, s^{1/p}) &\geq \alpha \text{ implies that} \\ N_2(f(x+y) - g(x) - h(y), \theta^-_\alpha(t+s)) &\geq \alpha, \text{ for all } x, y \in X, \end{aligned}$$

Then there exists a unique linear function $A : X \longrightarrow Y$ such that

$$N_2(g(0), t) \geq \alpha, \ N_2(h(0), s) \geq \alpha \text{ and } N_1(x, r^{1/p}) \geq \alpha,$$

implies that

$$\begin{aligned} N_2(A(x) - f(x), t + s + (4/2 - 2^p)\theta^-_\alpha r)) &\geq \alpha, \\ N_2(A(x) - g(x)), t + 2s + (6 - 2^p/2 - 2^p)\theta^-_\alpha r)) &\geq \alpha, \\ N_2(A(x) - h(x)), t + s + (6 - 2^p/2 - 2^p)\theta^-_\alpha r)) &\geq \alpha, \end{aligned}$$

for all $x \in X$.

4. Conclusion

We use the definition of fuzzy normed spaces given in [3] to study some appropriate notions of fuzzy approximately additive functions in fuzzy normed spaces and prove that under some suitable conditions, an approximately additive function f from a fuzzy Banach space (X, N_1) into a fuzzy Banach space (Y, N_2) can be approximated in a fuzzy sense by an additive mapping A. We discuss four types of fuzzy versions of the Hyers-Ulam-Rassias theorems and some corollaries. Also by example we show that these theorems are extension of classical analysis to fuzzy analysis. Moreover, there are very functional equation which are not studied in fuzzy normed linear spaces.

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