THE MEAN RESIDUAL LIFETIME OF PARALLEL SYSTEMS WITH TWO EXCHANGEABLE COMPONENTS UNDER THE GENERALIZED FARLIE-GUMBEL-MORGENSTERN MODEL

S. S. HASHEMI-BOSRA AND E. SALEHI
DEPARTMENT OF BASIC SCIENCES, BIRJAND UNIVERSITY OF TECHNOLOGY, BIRJAND 97198, IRAN
E-MAILS: SALEHI@BIRJANDUT.AC.IR

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Abstract. The parallel systems are special important types of coherent structures and have many applications in various areas. In this paper we consider a two-exchangeable-component parallel system for the Generalized Farlie-Gumbel-Morgenstern (Generalized FGM) distribution. We study the reliability properties of the residual lifetime of the system under the condition that both components of the system are operating at time t, and obtain an explicit expression for the mean residual lifetime (MRL) for such system. The asymptotic behavior of the proposed MRL function of the system is also investigated when the exchangeable lifetimes of components have a Generalized FGM bivariate exponential. Finally, we present some results for the Kendall's Tau correlation coefficient of Generalized FGM bivariate copula.

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1. Introduction

The mean residual lifetime function is one of the important measures in the reliability theory. During the last few years, many authors have studied properties of this function for coherent systems particularly parallel systems as well (see [1], [13] and [21]).

Much attention of authors has been paid to analyze the reliability of systems with independent components. Among others, we can imply to Asadi and Bayramoglu [2], Zhao et al. [27], Kochar and Xu [11] and Salehi and Asadi [23]. In reality, the components of the systems are dependent. Lately, many authors have been paid their attentions into this problem. In this regard, we can refer to Navaro et al. [18], Zhang [26], Sadegh [22], Navarro and Rubio [17], Rezapoor et al. [20], Jia et al. [9], Tavangar and Asadi [24], and Navarro and Gomis [15].

The copula function is a useful statistical tool to model the dependence structure among components of the system in the reliability theory (for example see [9] and [20]). The word *copula* was first used in a mathematical or statistical sense by Sklar in 1959. After that, copulas appeared in many literatures by several authors. In this field we refer to [10] and as a good reference [19].

Suppose $T_1, T_2, \ldots, T_n$ are dependent random variables representing the lifetimes of the components of a system. Assume that T_i has a continuous distribution F_i, density f_i and survival $\bar{F}_i$, $i = 1, 2, \ldots, n$. Denote by $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ the ordered lifetimes of the components. Two important special cases of coherent systems are parallel systems and series systems. A parallel system, consisting of n components, operates if and only if at least one component works. It is obvious that the lifetime of the system is $T_{n:n}$. As a dual of parallel system, the lifetime of series system is $T_{1:n}$. Under the exchangeability assumption of T_i's, the reliability function of the parallel system, denoted by $\bar{F}_{n:n}(t)$, is given by

$$\bar{F}_{n:n}(t) = P(T_{n:n} > t) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} P(T_{1:j} > t),$$

for more details see [5].

Let T be the lifetime of a component with reliability function $\bar{F}$. Assuming that the component has survived up to time t, the residual lifetime of T is defined as

$T_t = \{T - t|T > t\}$. The reliability function of T_t is equal to

$$\bar{F}(x|t) = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

The expected value of T_t called the mean residual lifetime or remaining lifetime of component, can be defined as follow for all t,

$$M(t) = E(T - t|T > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}.$$

The mean residual lifetime of a parallel system with n components, denoted by $M_n(t)$, is

$$\begin{aligned} M_n(t) &= E(T_{n:n} - t|T_{n:n} > t) \\ &= \frac{1}{\bar{F}_{n:n}(t)} \int_t^\infty \bar{F}_{n:n}(x)dx. \end{aligned}$$

Bairamov et al. [3] considered the MRL function of a parallel system consisting of i.i.d. components under the condition that all components are working at time t, and presented some significant results. Several authors have been studied the general case of this function for coherent systems. Among others, we refer to Asadi and Bayramoglu [2], Zhao et al. [27], Li and Zhao [13], Sadegh [21] and Kochar and Xu [11]. Here we study this function for a parallel system including two exchangeable components under the condition that both components are operating at time t, i.e. $\{T_{2:2} - t|T_{1:2} > t\}$. The expected value of this conditional random variable is also known as mean general residual lifetime (MGRL) in some literatures (e.g. see [13] and [27]). In following we explore some results on the limiting behavior of presented MRL function for Generalized FGM bivariate exponential distribution using an illustrative example.

This paper is organized as follows. Section 2 provides some concepts and necessary preliminaries. In Section 3, we consider the residual lifetime of parallel systems containing of two exchangeable components under the condition that both components of the system are working at time t. Under the Generalized FGM model, we give an explicit expression for the MGRL function of the system and explore its asymptotic behavior by using an example. Finally, we obtain a formula for the Kendall's Tau correlation coefficient of Generalized FGM bivariate copula and show that the extension range of Kendall's Tau for the Generalized FGM copula is wider than the one for the FGM copula in Section 4.

2. Preliminaries

In this section we give some definitions and concepts that are used in the next section. First, we introduce the copula function (see [19]).

Definition 2.1. A two-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ such that

1. for every $u, v \in [0,1]$,
$$C(u,1) = u \ , \quad C(1,v) = v \quad \text{and} \quad C(u,0) = 0 = C(0,v);$$
2. for every $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,
$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

In the following we present the definition of Generalized FGM copulas, for more details see [4].

Definition 2.2. C^{GFGM} is said to be the Generalized FGM copula on $[0,1]^2$, if for all $u, v \in [0,1]$,
$$C^{\text{GFGM}}(u,v) = uv\Big[1 + \theta\Big(1 - u^{\alpha}\Big)\Big(1 - v^{\alpha}\Big)\Big]^m,$$
where $\alpha > 0$, $m = 0, 1, 2, \ldots$, and
$$-\min\left\{1, \frac{1}{m\alpha^2}\right\} \leq \theta \leq \frac{1}{m\alpha},$$
are the dependent parameters.

Remark 2.1. When $\alpha = 1$ and $m = 1$, Generalized FGM bivariate copula convert to FGM bivariate copula that introduced by Morgenstern [14], Gumbel [7] and Farlie [6]. If $m = 0$ or $\theta = 0$, this model include the independent model.

From Eq. (2.6.1) in [19], the Generalized FGM survival copula denoted by $\hat{C}^{\text{GFGM}}$, is equal to

$$\hat{C}^{\text{GFGM}}(u,v) = u + v - 1 + C^{\text{GFGM}}(1-u, 1-v), \qquad 0 \leq u, v \leq 1. \tag{1}$$

Now, if $\bar{F}(x_1, x_2)$ is the Generalized FGM bivariate survival function of random vector (X_1, X_2), using (1) we have

$$\begin{aligned} \bar{F}(x_1,x_2) &= \bar{F}_1(x_1) + \bar{F}_2(x_2) - 1 \\ &+ (1-\bar{F}_1(x_1))(1-\bar{F}_2(x_2))\Big[1 + \theta\Big(1 - (1-\bar{F}_1(x_1))^{\alpha}\Big)\Big(1 - (1-\bar{F}_2(x_2))^{\alpha}\Big)\Big]^m, \end{aligned} \tag{2}$$

where $\bar{F}_i$ is the reliability function of X_i, $i = 1, 2$, and α, m and θ are defined in Definition 2.2.

3. The mean residual lifetime

In this section we consider a parallel system including two exchangeable components with lifetimes T_1 and T_2 where T_i has absolutely continuous distribution and survival function $F_i, \bar{F}_i$, $i = 1, 2$, respectively. The survival function of the general residual lifetime of the system denoted by $\psi(x|t)$, under the condition that both components of the system are working at time t, is equal to

$$\begin{aligned}\psi(x|t) &= P(T_{2:2} - t > x | T_{1:2} > t)\\ &= \frac{2\bar{F}(x+t, t) - \bar{F}(x+t, x+t)}{\bar{F}(t, t)}.\end{aligned}$$

For the Generalized FGM model, from (2) and under the assumption that the system components have the same distribution F, we can rewrite $\psi(x|t)$ as follows

$$\begin{aligned}\psi(x|t) = &\frac{1}{\bar{F}^2(t) + F^2(t)\sum_{i=1}^{m}\binom{m}{i}\theta^i\Big(1 - F^{\alpha}(t)\Big)^{2i}}\\ &\times\Bigg[2\bar{F}(t)\bar{F}(x+t) - F^2(x+t)\sum_{i=1}^{m}\binom{m}{i}\theta^i\Big(1 - F^{\alpha}(x+t)\Big)^{2i}\\ &-\bar{F}^2(x+t) + 2F(t)F(x+t)\sum_{i=1}^{m}\binom{m}{i}\theta^i(1 - F^{\alpha}(t))^i\Big(1 - F^{\alpha}(x+t)\Big)^{i}\Bigg].\end{aligned} \tag{3}$$

The mean general residual lifetime of the system can be defined as follows

$$\Psi(t) = E(T_{2:2} - t | T_{1:2} > t) = \int_0^{\infty}\psi(x|t)dx, \qquad \text{for } \ t > 0. \tag{4}$$

Theorem 3.1. Suppose that T_1 and T_2 have a Generalized FGM bivariate distribution with parameters α, θ and m. Let $\Psi(t)$ be the MGRL function of a parallel system consisting of two exchangeable components that they have common distribution function F. Then, for $t > 0$,

$$\begin{aligned}\Psi(t) = &\frac{1}{1 + \phi^2(t)\sum_{i=1}^{m}\binom{m}{i}\theta^i\Big(1 - F^{\alpha}(t)\Big)^{2i}}\Bigg[2M_1(t) - M_2(t)\\ &+\frac{2\phi(t)\sum_{i=1}^{m}\binom{m}{i}\theta^i\Big(1 - F^{\alpha}(t)\Big)^{i}\mu_1^i(t)}{\bar{F}(t)} - \frac{\sum_{i=1}^{m}\binom{m}{i}\theta^i\mu_2^i(t)}{\bar{F}^2(t)}\Bigg],\end{aligned}$$

where $\phi(t) = \frac{F(t)}{\bar{F}(t)}$, $M_l(t)$ is the mean residual lifetime of the series system with l independent and identical components for $l = 1, 2$, and for $i = 1, 2, \ldots, m$, $j = 1, 2$,

$$\mu_j^i(t) = \int_0^\infty \left[F(x+t)\Big(1 - F^\alpha(x+t)\Big)^i \right]^j dx.$$

Proof: From (3), (4) and using simple algebra, the proof can be obtained easily.

Example 3.1. Let T_1 and T_2 denote the lifetimes of two components which are connected in a parallel system. Assume that the exchangeable lifetimes T_1 and T_2 have Generalized FGM bivariate exponential distribution with parameters α, m, θ and $\lambda > 0$. Using Theorem 3.1, we can obtain an explicit expression for $\Psi(t)$. The graphs of MGRL functions for different parameters are presented in Figures 1 and 2.

It is shown in Figure 1 that the shape of MGRL function depends on the parameter θ. In this case there exists a change point obtained by solving $\Psi'(t) = 0$, that is denoted by t^*. If $\theta > 0$, for $t < t^*$, the MGRL function is increasing, and for $t > t^*$, the MGRL function is decreasing. For the another case, if $\theta < 0$, for $t < t^*$, the MGRL function is decreasing and when $t > t^*$, the MGRL function is increasing. For large value of t, $\Psi(t)$ is constant that is

$$\lim_{t\to\infty} \Psi(t) = \frac{3}{2\lambda}.$$

This is the value of MGRL function in the independent case, (see [16]). These results are shown in Figure 1 for two cases with different values of θ. In case a) $\lambda = 1$, $m = 2$ and $\alpha = 1.5$; and case b) $\lambda = 0.5$, $m = 2$ and $\alpha = 2$.

Remark 3.1. In the Generalized FGM bivariate model, if $m = 1$ and $\alpha = 1$, $\Psi(t)$ in Example 3.1 deduce to the mean general residual lifetime of a parallel system consisting of two exchangeable components for FGM family, which is already derived by Ucer and Gurler [25].

Corollary 3.1. Under the Generalized FGM model for the parallel system consisting of two arbitrary components, it is easy to obtain $\Psi(t)$ in the same way of Theorem 3.1. Also, if the lifetimes of components have exponential distribution with parameters $\lambda_i > 0$, $i = 1, 2$, then

$$\lim_{t\to\infty} \Psi(t) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

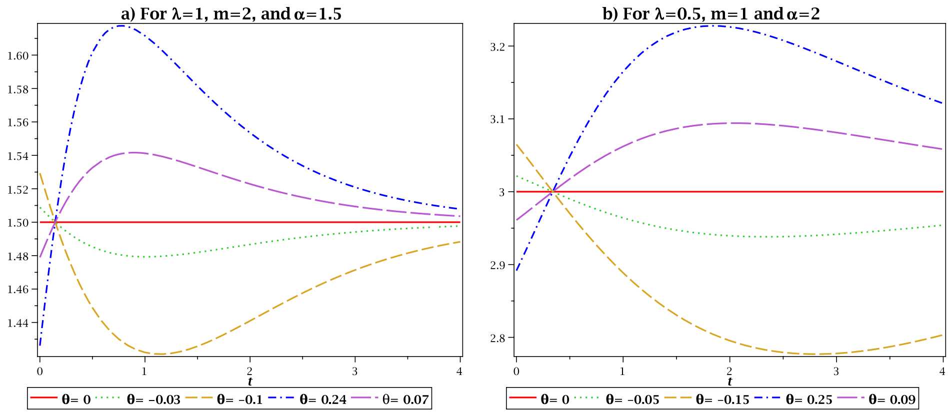

FIGURE 1. The curves of $\Psi(t)$ of parallel system consisting of two components for the Generalized FGM distribution with exponential marginals for different values of θ.

As shown in Figure 2, the parameters m and α are the scale parameters. We plot the MGRL functions for two values of $\theta = 0.1, -0.03$, in Figure 2.

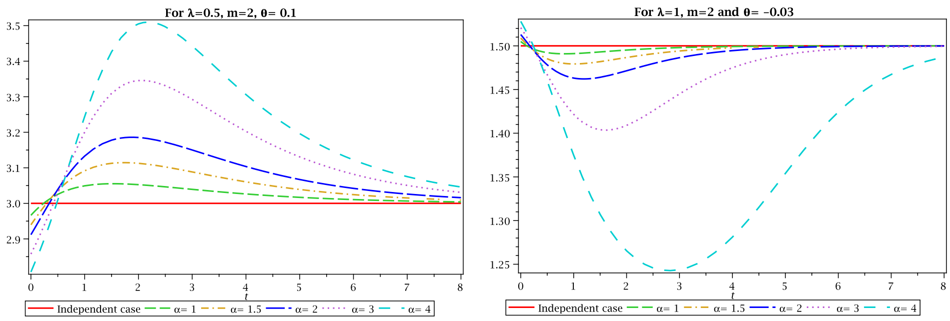

FIGURE 2. The curves of $\Psi(t)$ of parallel system consisting of two components for the Generalized FGM distribution with exponential marginals for different values of α.

4. KENDALL'S TAU FOR THE GENERALIZED FGM COPULA

One of the most commonly used nonparametric measures of association for two random variables is *Kendall's Tau* (τ) introduced in terms of concordance. Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors.

The population version of *Kendall's Tau* is defined as the difference between the probability of concordance and the probability of discordance, i.e.

$$\tau = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

To introduce the sample version, let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ denote a random sample of n observations from a vector (X, Y) of continuous random variables. So that the *Kendall's Tau* is represented as follow

$$\tau = \frac{C - D}{C + D} = \frac{C - D}{\binom{n}{2}},$$

where C and D are the number of concordant pairs and discordant pairs, respectively. For more details, we refer to [8] and [12], and as a good reference [19].

In this section, we present an explicit expression of the dependence measures *Kendall's Tau* for the Generalized FGM bivariate copula. Also, we show that the admissible range of *Kendall's Tau* for the Generalized FGM copula is wider than the one for the FGM copula. First, we need to give an useful theorem to prove the result (see [19] for a proof).

Theorem 4.1. Let X and Y be continuous random variables whose copula is C. Then the population version of *Kendall's Tau* for X and Y is given by

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1,$$

or equivalently

$$\tau = 1 - 4 \int_0^1 \int_0^1 \frac{\partial}{\partial u} C(u, v) \frac{\partial}{\partial v} C(u, v) du dv.$$

In the following theorem, we reformulate the *Kendall's Tau* correlation coefficient for the Generalized FGM bivariate copula.

Theorem 4.2. Let (X, Y) be a random vector having a Generalized FGM bivariate distribution. Then, the *Kendall's Tau* equals to

$$\begin{aligned} \tau \quad = \quad 1 - 4 \Bigg(\frac{1}{\alpha^2} \sum_{i=0}^{2m} \binom{2m}{i} \theta^i \Big[B(i+1, \frac{2}{\alpha}) \Big]^2 + m^2 \sum_{i=0}^{2m-2} \binom{2m-2}{i} \theta^{i+2} \Big[B(i+2, 1 + \frac{2}{\alpha}) \Big]^2 \\ - \frac{2m}{\alpha} \sum_{i=0}^{2m-1} \binom{2m-1}{i} \theta^{i+1} B(i+2, \frac{2}{\alpha}) B(i+1, 1 + \frac{2}{\alpha}) \Bigg), \end{aligned}$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Proof. Using Theorem 4.1 and taking the partial derivatives of Generalized FGM copula, we have

$$\begin{aligned}\tau &= 1-4\int_0^1\int_0^1 \frac{\partial}{\partial u}C^{\mathrm{GFGM}}(u,v)\frac{\partial}{\partial v}C^{\mathrm{GFGM}}(u,v)dudv\\ &= 1-4\int_0^1\int_0^1 \Big(1+\theta(1-u^\alpha)(1-v^\alpha)\Big)^{2m}\\ &\qquad\times\Bigg[uv-\frac{m\alpha\theta uv\Big(v^\alpha(1-u^\alpha)+u^\alpha(1-v^\alpha)\Big)}{1+\theta(1-u^\alpha)(1-v^\alpha)}\\ &\qquad\qquad+\frac{m^2\alpha^2\theta^2 v^{\alpha+1}u^{\alpha+1}(1-u^\alpha)(1-v^\alpha)}{\Big[1+\theta(1-u^\alpha)(1-v^\alpha)\Big]^2}\Bigg]dudv.\end{aligned}$$

Now, using the Binomial expansion, i.e.

$$\Big(1+\theta(1-u^\alpha)(1-v^\alpha)\Big)^m=\sum_{i=0}^m\binom{m}{i}\Big[\theta(1-u^\alpha)(1-v^\alpha)\Big]^i,$$

and the fact that

$$\int_0^1\int_0^1 u(1-u^\alpha)^i v(1-v^\alpha)^i dudv=\left[\frac{\Gamma(i+1)\Gamma(\frac{2}{\alpha})}{\alpha\Gamma(i+1+\frac{2}{\alpha})}\right]^2,$$

the proof is complete.

Remark 4.1. The *Kendall's Tau* correlation coefficient of Generalized FGM bivariate copula is computed for some admissible range of associated parameter θ in Table 1. It is known that the range of *Kendall's Tau* correlation coefficient for the FGM copula is between $[-0.22, 0.22]$, see [19]. However, by Table 1, we indicated that the extension range of *Kendall's Tau* for the Generalized FGM bivariate copula is wider than the one for the FGM copula. According to Table 1, for the Generalized FGM copula, the strongest positive of *Kendall's Tau* correlation coefficient attains $\tau\cong 0.29$ and also the weakest negative of *Kendall's Tau* correlation coefficient is $\tau\cong -0.35$.

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TABLE 1. The admissible values of dependent parameter θ and *Kendall's Tau* for the Generalized FGM copula with different values of m and α.

m	α	θ		τ	
		Lower bound	Upper bound	Lower bound	Upper bound
1	0.001	-1.0000	1000.0	-4.99×10^{-7}	4.995×10^{-4}
2	0.01	-1.0000	50.000	-9.9×10^{-5}	0.0049
3	0.1	-1.0000	3.3333	-0.0134	0.0466
10	0.1	-1.0000	1.0000	-0.0437	0.0471
25	0.1	-1.0000	0.4000	-0.1037	0.0472
50	0.1	-1.0000	0.2000	-0.1923	0.0473
100	0.1	-1.0000	0.1000	-0.3403	0.0474
300	0.1	-0.3333	0.0333	-0.33989	0.04732
400	0.1	-0.2500	0.0250	-0.33987	0.04737
1	1	-1.0000	1.0000	-0.2222	0.2222
1	2	-0.2500	0.5000	-0.1249	0.2499
2	1	-0.5000	0.5000	-0.2088	0.2366
2	2	-0.1250	0.2500	-0.1215	0.2643
3	1	-0.3333	0.3333	-0.2051	0.2423
3	2	-0.0833	0.1666	-0.1204	0.2696
3	3	-0.0370	0.1111	-0.0782	0.2557
4	2	-0.0625	0.1250	-0.1199	0.2725
4	3	-0.0277	0.0833	-0.0779	0.2578
5	2	-0.0500	0.1000	-0.1196	0.2743
5	4	-0.0125	0.0500	-0.0546	0.2372
10	2	-0.0250	0.0500	-0.1190	0.2779
10	4	-0.00625	0.0250	-0.0545	0.2394
10	10	-0.0010	0.0100	-0.0138	0.1442
25	20	-0.0001	0.0020	-0.0041	0.0844
50	2	-0.0050	0.0100	-0.1185	0.2809
50	50	-8×10^{-6}	0.0004	-0.0007	0.0373
2	100	-5×10^{-5}	0.0050	-0.00019	0.01927
3	100	-3×10^{-5}	0.0033	-0.00017	0.0191
100	100	-10^{-6}	0.0001	-0.00019	0.0193
300	1	-0.0033	0.0033	-0.1965	0.2522
300	2	-0.00083	0.00167	-0.1180	0.2822
400	1	-0.0025	0.0025	-0.1983	0.2552
400	2	-6.25×10^{-4}	0.00125	-0.1185	0.2816
400	3	-2.778×10^{-4}	0.00083	-0.0775	0.2637
400	20	-6.25×10^{-6}	1.25×10^{-4}	-0.00412	0.0845