A WEIGHTED LINEAR REGRESSION MODEL FOR IMPERCISE RESPONSE

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Abstract. A weighted linear regression model with impercise response and p-real explanatory variables is analyzed. The LR fuzzy random variable is introduced and a metric is suggested for coping with this kind of variables. A least square solution for estimating the parameters of the model is derived. The result are illustrated by the means of some case studies.

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1. Introduction

Classical regression analysis is helpful in ascertaining the probable form of the relationship between variables, and usually the ultimate objective is to predict, or estimate, the value of one variable corresponding to given values of other variables. The method usually employed for obtaining the "regression surface" is known as the method of least squares and the parameters are estimated by minimizing the sum of squares of the difference between observed and predicted values.

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A fuzzy linear regression model(FLR) was first introduced by Tanaka et al. [18]. Their method, in which the observed data are crisp, has been developed in different directions by several authors(see for example[4, 11, 17, 19]). Tanaka et al.'s approach is essentially based on transforming the problem of fitting a fuzzy model on a data set to a linear programming problem.

Another approach to fuzzy regression is introduced by Celmins [3] and Diamond [6], using a generalized least squares method. In the fuzzy least squares approach, the optimal model is usually derived based on a metric on the space of fuzzy numbers. For more on this approach and some applications see, for example, [1, 5, 15, 21]. Coppi et al. [5] have proposed a linear regression model with crisp inputs and LR fuzzy response. The basic idea consists in modeling the centers of the response variable by means of a classical regression model, and simultaneously modeling the left and the right spreads of the response through simple linear regression on its estimated center. The study in Coppi et al. [5] is mainly descriptive, and the authors impose a non-negativity condition in the numerical minimization problem to avoid negative estimated spreads. Ferraro et al.[9] proposed an alternative model to overcome the non-negativity condition by means of modeling a transformation of left and right spreads.

Different kinds of weighted fuzzy regression models were introduced in several studies, see for instance, [2] and [19]. We modify Ferraro et al.[9] model for weighted regression. Numerical examples shows that the modification result is lower standard deviation errors.

This paper is organized as follows: in Section 2 modeling imprecise response using LR fuzzy random variables is formalized and Ferraro et al. model [9] is briefly discussed. In Section 3, a weighted linear regression model for imprecise response in both simple and multiple case is proposed and the estimators of the parameters are obtained. In Section 4, numerical examples are provided and compared with Ferraro et al.[9] model. Finally, Section 5 provides a conclusion.

2. **Modeling the imprecise data**

2.1. **Fuzzy sets**

. Let X be a universal set. A fuzzy set A of X is defined by its membership function $A : X \to [0, 1]$ In practice, there are experiments whose results can be described by means of fuzzy sets of a particular class, determined by three values: the center, the left spread and the right spread. this type of fuzzy datum is called LR fuzzy number and is defined as follows:

$$A(x) = \begin{cases} L(\frac{A^m - x}{A^l}) & x \leq A^m \\ R(\frac{x - A^m}{A^r}) & x \geq A^m, \end{cases}$$

where $A^m \in \mathbb{R}$ is the center, $A^l \in \mathbb{R}^+$ and $A^r \in \mathbb{R}^+$ are, respectively, the left and the right spread and, L and R are functions such that $L(0) = R(0) = 1$ and $L(x) = R(x) = 0, \forall x \in \mathbb{R}\backslash[0,1]$. If $A^r = A^l$ the fuzzy number A is referred to as symmetrical [9].

Remark 2.1. An interval I is a particular kind of LR fuzzy set that can be characterized by means of the extremes $[infI, supI]$ or, by means of $midI = [supI + infI]/2$ and $sprI = [supI - infI]/2$ [9].

Definition 2.1. Yang and Ko [20] have defined a distance D^2_{LR} between two LR fuzzy numbers $A, B \in \mathcal{F}_{LR}$ as follows:

(1)
$$\begin{aligned} D^2_{LR}(A,B) &= (A^m - B^m)^2 + ((A^m - \lambda A^l) - (B^m - \lambda B^l))^2 + ((A^m + \rho A^r) - (B^m + \rho B^r))^2 \\ &= 3(A^m - B^m)^2 + \lambda^2(A^l - B^l)^2 + \rho^2(A^r - B^r)^2 - 2\lambda(A^m - B^m)(A^l - B^l) \\ &+ 2\rho(A^m - B^m)(A^r - B^r). \end{aligned}$$

where $\mathcal{F}_{LR}$ is the class of fuzzy numbers and $\lambda = \int_0^1 L^{-1}(\omega)d(\omega)$ and $\rho = \int_0^1 R^{-1}(\omega)d(\omega)$ represent the influence of the shape of the membership function on the distance. The $(\mathcal{F}_{LR}, D^2_{LR})$ is a metric space [9].

2.2. **Fuzzy random variables**

. Kwakernak [14], Puri and Ralescu [16] and Kelement et al. [13] have introduced the concept of fuzzy random variable (FRV) as an extension of random variables as well as random sets.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. The mapping $X : \Omega \to \mathcal{F}_{LR}$ is an FRV. In the case of LR FRVs, this is equivalent to $(X^m, X^l, X^r) : \Omega \to (\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ beings a random vector [9].

2.3. **A linear regression model for imprecise response**

. Ferraro et al. [9] introduced a linear regression model for imprecise response. Where they proposed using a transformation of left and right spreads to overcome the non-negativity condition.

Consider a random experiment in which LR fuzzy observations on the variables $Y, X_1, X_2, \ldots, X_p$ on n statistical units are $\{Y_i, \underline{X}_i\}_{i=1,\ldots,n}$, where $\underline{X}_i = (X_{1i}, X_{2i}, \ldots, X_{pi})$, or in compact form $(\underline{Y}, X)$, where $\underline{Y}$ is the $n \times 1$ vector of observations Y and X is the $n \times p$ matrix of the observations on $\underline{X}$. Then for two invertible functions $g : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$:

$$
\left\{
\begin{array}{l}
Y^m = \underline{X}'\underline{a}_m + b_m + \epsilon_m \\
g(Y^l) = \underline{X}'\underline{a}_l + b_l + \epsilon_l \\
h(Y^r) = \underline{X}'\underline{a}_r + b_r + \epsilon_r.
\end{array}
\right.
\tag{2}
$$

where ϵ_m, ϵ_l and ϵ_r are real-valued random variables with $E(\epsilon_m|\underline{X}) = E(\epsilon_l|\underline{X}) = E(\epsilon_r|\underline{X}) = 0$ and $\underline{a}_m = (a_{m1}, \ldots, a_{mp})'$, $\underline{a}_l = (a_{l1}, \ldots, a_{lp})'$ and $\underline{a}_r = (a_{r1}, \ldots, a_{rp})'$ are $(p \times 1)$-vectors of the parameters related to the vector $\underline{X}$. The covariance matrix of the vector of the explanatory variables $\underline{X}$ will be denoted by $\Sigma_{\underline{X}}$ and Σ will stand for the covariance matrix of $(\epsilon_m, \epsilon_l, \epsilon_r)$, whose variances are strictly positive and finite. Since the expected value of ϵ_m, ϵ_l and ϵ_r given $\underline{X}$ are equal to 0, hence ϵ_m, ϵ_l and ϵ_r are uncorrelated with the explanatory variables [9].

Theorem 2.1. Under the assumptions of model (2), the LS-estimators of the model are:

$$
\begin{aligned}
\hat{\underline{a}}_m &= (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{\underline{Y}^m}, \\
\hat{\underline{a}}_l &= (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{g(\underline{Y}^l)}, \\
\hat{\underline{a}}_r &= (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{h(\underline{Y}^r)}, \\
\hat{b}_m &= \overline{Y^m} - \overline{\underline{X}'}\hat{\underline{a}}_m, \\
\hat{b}_l &= \overline{g(Y^l)} - \overline{\underline{X}'}\hat{\underline{a}}_l, \\
\hat{b}_r &= \overline{h(Y^r)} - \overline{\underline{X}'}\hat{\underline{a}}_r,
\end{aligned}
$$

where, as usual, $\overline{Y}^m$, $\overline{g(Y^l)}$, $\overline{h(Y^r)}$ and $\overline{\underline{X}}$ are, respectively, the sample means of Y^m, $g(Y^l)$, $h(Y^r)$ and $\underline{X}$

$$\begin{aligned}
\widetilde{\underline{Y^m}} &= \underline{Y}^m - \underline{1}\overline{Y^m},\\
\widetilde{g(\underline{Y^l})} &= g(\underline{Y}^l) - \underline{1}\overline{g(Y^l)},\\
\widetilde{h(\underline{Y^r})} &= h(\underline{Y}^r) - \underline{1}\overline{h(Y^r)}
\end{aligned}$$

are the centered values of the response and

$$\widetilde{X} = X - \underline{1}\overline{\underline{X}'},$$

the centered matrix of the explanatory variables [9].

3. A weighted linear regression model for imprecise response

In this section we introduce a weighted linear regression model in both simple and multiple case. This model is based on Ferraro et al. model [9].

Consider a random experiment in which an LR fuzzy response variable Y and a real explanatory variable X observed on n statistical units are $\{Y_i, X_i\}_{i=1,\dots,n}$ and Y is determined by (Y^m, Y^l, Y^r). Suppose $g : (0,\infty) \to \mathbb{R}$ and $h : (0,\infty) \to \mathbb{R}$ are invertible, a weighted simple linear regression model can be represented in the following way:

$$\left\{\begin{array}{c}
w_m^{\frac{1}{2}} Y^m = w_m^{\frac{1}{2}} a_m X + w_m^{\frac{1}{2}} b_m + w_m^{\frac{1}{2}} \varepsilon_m\\
w_l^{\frac{1}{2}} g(Y^l) = w_l^{\frac{1}{2}} a_l X + w_l^{\frac{1}{2}} b_l + w_l^{\frac{1}{2}} \varepsilon_l\\
w_r^{\frac{1}{2}} h(Y^r) = w_r^{\frac{1}{2}} a_r X + w_r^{\frac{1}{2}} b_r + w_r^{\frac{1}{2}} \varepsilon_r
\end{array}\right. . \tag{3}$$

where w_m,w_l and w_r are respectively, the weights of Y^m, Y^l and Y^r. In order to get the estimators of the regression parameters the least squares (LS) criterion will be used.

Theorem 3.1. The least square estimator of parameters for model (3) are

$$\hat{a}_m = \frac{\sum_{i=1}^n w_{mi} X_i Y_i^m - \frac{\sum_{i=1}^n w_{mi} X_i \sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{mi}}}{\sum_{i=1}^n w_{mi}(X_i - \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{mi}})^2}, \quad \hat{b}_m = \frac{\sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{mi}} - \hat{a}_m \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{mi}}$$

$$\hat{a}_l = \frac{\sum_{i=1}^n w_{li} X_i g(Y_i^l) - \frac{\sum_{i=1}^n w_{li} X_i \sum_{i=1}^n w_{li} g(Y_i^l)}{\sum_{i=1}^n w_{li}}}{\sum_{i=1}^n w_{li}(X_i - \frac{\sum_{i=1}^n w_{li} X_i}{\sum_{i=1}^n w_{li}})^2}, \quad \hat{b}_l = \frac{\sum_{i=1}^n w_{li} g(Y_i^l)}{\sum_{i=1}^n w_{li}} - \hat{a}_l \frac{\sum_{i=1}^n w_{li} X_i}{\sum_{i=1}^n w_{li}}$$

$$\hat{a}_r = \frac{\sum_{i=1}^n w_{ri} X_i h(Y_i^r) - \frac{\sum_{i=1}^n w_{ri} X_i \sum_{i=1}^n w_{ri} h(Y_i^r)}{\sum_{i=1}^n w_{ri}}}{\sum_{i=1}^n w_{ri}(X_i - \frac{\sum_{i=1}^n w_{ri} X_i}{\sum_{i=1}^n w_{ri}})^2}, \quad \hat{b}_r = \frac{\sum_{i=1}^n w_{ri} h(Y_i^r)}{\sum_{i=1}^n w_{ri}} - \hat{a}_r \frac{\sum_{i=1}^n w_{ri} X_i}{\sum_{i=1}^n w_{ri}}.$$

Proof. For estimating $\hat{a}_m$, $\hat{a}_l$, $\hat{a}_r$, $\hat{b}_m$, $\hat{b}_l$ and $\hat{b}_r$, we first minimize Yang-Ko metric [20] as follows:

(4)
$$min\Delta^2_{\lambda\rho} = min \sum_{i=1}^n D^2_{\lambda\rho}((w_{mi}^{\frac{1}{2}} Y_i^m, w_{li}^{\frac{1}{2}} g(Y_i^l), w_{ri}^{\frac{1}{2}} h(Y_i^r)), (w_{ri}^{\frac{1}{2}} (Y_i^m)^*, w_{li}^{\frac{1}{2}} g^*(Y_i^l), w_{ri}^{\frac{1}{2}} h^*(Y_i^r))$$

where $w_{mi}^{\frac{1}{2}}(Y_i^m)^* = w_{mi}^{\frac{1}{2}} a_m X_i + w_{mi}^{\frac{1}{2}} b_m$, $w_{li}^{\frac{1}{2}} g^*(Y_i^l) = w_{li}^{\frac{1}{2}} a_l X_i + w_{li}^{\frac{1}{2}} b_l$ and $w_{ri}^{\frac{1}{2}} h^*(Y_i^r) = w_{ri}^{\frac{1}{2}} a_r X_i + w_{ri}^{\frac{1}{2}} b_r$ are predicted values. The function to minimize becomes

$$\begin{aligned} \Delta^2_{\lambda\rho} = &\sum_{i=1}^n [3 w_{mi}^{\frac{1}{2}} (Y_i^m - a_m X_i - b_m)^2] \\ &+ \sum_{i=1}^n [\lambda^2 w_{li}^{\frac{1}{2}} (g(Y_i^l) - a_l X_i - b_l)^2 + \rho^2 w_{ri}^{\frac{1}{2}} (h(Y_i^r) - a_r X_i - b_r)^2] \\ &+ \sum_{i=1}^n [-2\lambda w_{mi}^{\frac{1}{2}} (Y_i^m - a_m X_i - b_m) w_{li}^{\frac{1}{2}} (g(Y_i^l) - a_l X_i - b_l)] \\ &+ \sum_{i=1}^n [+2\rho w_{mi}^{\frac{1}{2}} (Y_i^m - a_m X_i - b_m) w_{ri}^{\frac{1}{2}} (h(Y_i^r) - a_r X_i - b_r)]. \end{aligned} \tag{5}$$

To estimate b_l and b_r, we equate the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to b_l and b_r, to zero. Hence:

$$\begin{aligned} \frac{\partial \Delta^2_{\lambda\rho}}{\partial b_l} = 0 \iff b_l = &\frac{\sum_{i=1}^n w_{li} g(Y_i^l)}{\sum_{i=1}^n w_{li}} - a_l \frac{\sum_{i=1}^n w_{li} X_i}{\sum_{i=1}^n w_{li}} - \frac{1}{\lambda} \frac{\sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{li}} \\ &+ \frac{a_m}{\lambda} \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{li}} + \frac{b_m \sum_{i=1}^n w_{mi}}{\lambda \sum_{i=1}^n w_{li}}. \end{aligned} \tag{6}$$

$$\frac{\partial \Delta^2_{\lambda\rho}}{\partial b_r} = 0 \iff b_r = \frac{\sum_{i=1}^n w_{ri} h(Y_i^r)}{\sum_{i=1}^n w_{ri}} - a_r \frac{\sum_{i=1}^n w_{ri} X_i}{\sum_{i=1}^n w_{ri}} + \frac{1}{\rho} \frac{\sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{ri}}$$

$$- \frac{a_m}{\rho} \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{ri}} - \frac{b_m \sum_{i=1}^n w_{mi}}{\rho \sum_{i=1}^n w_{ri}}. \tag{7}$$

To estimate b_m we have to take in to account that b_l and b_r obtained above are expressed as function of b_m. Thus, by substituting (6) and (7) in (5) and equating to zero the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to b_m we get

$$\frac{\partial \Delta^2_{\lambda\rho}}{\partial b_m} = 0 \iff b_m = \frac{\sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{mi}} - a_m \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{mi}}.$$

As result we obtain the following solutions that depend on the parameters, a_m, a_l and a_r.

$$\hat{b}_m = \frac{\sum_{i=1}^n w_{mi} Y_i^m}{\sum_{i=1}^n w_{mi}} - \hat{a}_m \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{mi}},$$
$$\hat{b}_l = \frac{\sum_{i=1}^n w_{li} g(Y_i^l)}{\sum_{i=1}^n w_{li}} - \hat{a}_l \frac{\sum_{i=1}^n w_{li} X_i}{\sum_{i=1}^n w_{li}},$$
$$\hat{b}_r = \frac{\sum_{i=1}^n w_{ri} h(Y_i^r)}{\sum_{i=1}^n w_{ri}} - \hat{a}_r \frac{\sum_{i=1}^n w_{ri} X_i}{\sum_{i=1}^n w_{ri}},$$

the centered values of X_i based on weights of center and spreads are

$$\widetilde{X_{mi}} = X_i - \frac{\sum_{i=1}^n w_{mi} X_i}{\sum_{i=1}^n w_{mi}},$$
$$\widetilde{X_{li}} = X_i - \frac{\sum_{i=1}^n w_{li} X_i}{\sum_{i=1}^n w_{li}},$$
$$\widetilde{X_{ri}} = X_i - \frac{\sum_{i=1}^n w_{ri} X_i}{\sum_{i=1}^n w_{ri}},$$

so the objective function can be written as follows

$$
\begin{aligned}
\Delta^2_{\lambda\rho} = & \sum_{i=1}^{n}[3(\widetilde{Y_i^m} - a_m\widetilde{X_{mi}})^2] \\
& + \sum_{i=1}^{n}[\lambda^2(\widetilde{g(Y_i^l)} - a_l\widetilde{X_{li}})^2 + \rho^2(\widetilde{h(Y_i^r)} - a_r\widetilde{X_{ri}})^2] \\
& + \sum_{i=1}^{n}[-2\lambda(\widetilde{Y_i^m} - a_m\widetilde{X_{mi}})(\widetilde{g(Y_i^l)} - a_l\widetilde{X_{ri}})] \\
& + \sum_{i=1}^{n}[+2\rho(\widetilde{Y_i^m} - a_m\widetilde{X_{mi}})(\widetilde{h(Y_i^r)} - a_r\widetilde{X_{ri}})].
\end{aligned} \tag{8}
$$

By equating to zero the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to a_l and a_r we obtain

(9)
$$
\frac{\partial\Delta^2_{\lambda\rho}}{\partial a_l} = 0 \iff a_l = \frac{\sum_{i=1}^n w_{li}\widetilde{X_{li}}\widetilde{g(Y_i^l)}}{\sum_{i=1}^n w_{li}\widetilde{X_i^2}} - \frac{1}{\lambda}\frac{\sum_{i=1}^n w_{mi}\widetilde{Y_i^m}\widetilde{X_{mi}}}{\sum_{i=1}^n w_{li}\widetilde{X_i^2}} + \frac{\sum_{i=1}^n w_{mi}\widetilde{X_{mi}^2}}{\sum_{i=1}^n w_{li}\widetilde{X_{li}^2}}\frac{a_m}{\lambda}
$$

(10)
$$
\frac{\partial\Delta^2_{\lambda\rho}}{\partial a_r} = 0 \iff a_r = \frac{\sum_{i=1}^n w_{ri}\widetilde{X_{ri}}\widetilde{h(Y_i^r)}}{\sum_{i=1}^n w_{ri}\widetilde{X_i^2}} + \frac{1}{\rho}\frac{\sum_{i=1}^n w_{mi}\widetilde{Y_i^m}\widetilde{X_{mi}}}{\sum_{i=1}^n w_{ri}\widetilde{X_i^2}} - \frac{\sum_{i=1}^n w_{mi}\widetilde{X_{mi}^2}}{\sum_{i=1}^n w_{ri}\widetilde{X_{ri}^2}}\frac{a_m}{\rho},
$$

where

$$
\begin{aligned}
\widetilde{Y_i^m} &= Y_i^m - \frac{\sum_{i=1}^n w_{mi}Y_i^m}{\sum_{i=1}^n w_{mi}} \\
\widetilde{g(Y_i^l)} &= g(Y_i^l) - \frac{\sum_{i=1}^n w_{li}g(Y_i^l)}{\sum_{i=1}^n w_{li}} \\
\widetilde{h(Y_i^r)} &= h(Y_i^r) - \frac{\sum_{i=1}^n w_{ri}h(Y_i^r)}{\sum_{i=1}^n w_{ri}}.
\end{aligned}
$$

Substituting (9) and (10) into (8) and by equating the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to a_m to zero, we obtain the estimation of a_m as follows:

$$
\frac{\partial\Delta^2_{\lambda\rho}}{\partial a_m} = 0 \iff \hat{a}_m = \frac{\sum_{i=1}^n w_{mi}X_iY_i^m - \frac{\sum_{i=1}^n w_{mi}X_i\sum_{i=1}^n w_{mi}Y_i^m}{\sum_{i=1}^n w_{mi}}}{\sum_{i=1}^n w_{mi}(X_i - \frac{\sum_{i=1}^n w_{mi}X_i}{\sum_{i=1}^n w_{mi}})^2} \tag{11}
$$

Finally, by substituting (11) into (9) and (10) the solutions of LS problem are obtained. $\square$

Consider a random experiment in which LR fuzzy random variables $Y, X_1, X_2, \ldots, X_p$ observed on n statistical units are $\{Y_i, \underline{X}_i\}_{i=1,\ldots,n}$, where $\underline{X}_i = (X_{1i}, X_{2i}, \ldots, X_{pi})$. Or, in compact form $(\underline{Y}, X)$, where $\underline{Y}$ is the $n \times 1$ vector of observations of Y and X is the $n \times p$ matrix of the observation of $\underline{X}$. Suppose $g : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ are invertible. The weighted multiple linear regression model is introduced as follows:

$$\left\{ \begin{array}{l} W_m^{\frac{1}{2}} \underline{Y}^m = W_m^{\frac{1}{2}} X \underline{a}_m + W_m^{\frac{1}{2}} \underline{1} b_m + W_m^{\frac{1}{2}} \underline{\epsilon}_m \\ W_l^{\frac{1}{2}} g(\underline{Y}^l) = W_l^{\frac{1}{2}} X \underline{a}_l + W_l^{\frac{1}{2}} \underline{1} b_l + W_l^{\frac{1}{2}} \underline{\epsilon}_l \\ W_r^{\frac{1}{2}} h(\underline{Y}^r) = W_r^{\frac{1}{2}} X \underline{a}_r + W_r^{\frac{1}{2}} \underline{1} b_r + W_r^{\frac{1}{2}} \underline{\epsilon}_r, \end{array} \right. \tag{12}$$

where $\underline{\epsilon}_m$, $\underline{\epsilon}_L$ and $\underline{\epsilon}_r$ are the $(n \times 1)$ vectors of real-valued random variables and $\underline{a}_m$, $\underline{a}_l$ and $\underline{a}_r$ are the $(p \times 1)$ vectors of the parameters related to X and W_m, W_l and W_r are respectively, $(n \times n)$ diagonal matrices of related to center and spreads.

Theorem 3.2. The LS estimators of the parameters of model (12) are

$$\begin{aligned} \hat{\underline{a}}_m &= (\widetilde{X}_m' W_m \widetilde{X}_m)^{-1} \widetilde{X}_m' W_m \widetilde{Y}^m \\ \hat{\underline{a}}_l &= (\widetilde{X}_l' W_l \widetilde{X}_l)^{-1} \widetilde{X}_l' W_l \widetilde{g(Y^l)} \\ \hat{\underline{a}}_r &= (\widetilde{X}_r' W_r \widetilde{X}_r)^{-1} \widetilde{X}_r' W_r \widetilde{h(Y)^r} \\ \hat{b}_m &= (\underline{1}' W_m \underline{1})^{-1} \left((\underline{1}' W_m \underline{Y}^m) - (\underline{1}' W_m X) \underline{a}_m \right) \\ \hat{b}_l &= (\underline{1}' W_l \underline{1})^{-1} \Big((\underline{1}' W_l g(\underline{Y}^l)) - (\underline{1}' W_l X) \underline{a}_l \Big) \\ \hat{b}_r &= (\underline{1}' W_r \underline{1})^{-1} \left((\underline{1}' W_r h(\underline{Y}^r)) - (\underline{1}' W_r X) \underline{a}_r \right), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\underline{Y}^m} &= (\underline{Y}^m - \underline{1}(\underline{1}' W_m \underline{1})^{-1} \underline{1}' W_m \underline{Y}^m) \\ \widetilde{g(\underline{Y}^l)} &= (g(\underline{Y}^l) - \underline{1}(\underline{1}' W_l \underline{1})^{-1} \underline{1}' W_l g(\underline{Y}^l)) \\ \widetilde{h(\underline{Y}^r)} &= (h(\underline{Y}^r) - \underline{1}(\underline{1}' W_r \underline{1})^{-1} \underline{1}' W_r h(\underline{Y}^r)), \end{aligned}$$

are the centered values of response and the centered values of X on the base of weight matrices of center and spreads are

$$\begin{aligned} \widetilde{X}_m &= (X - \underline{1}(\underline{1}' W_m \underline{1})^{-1} \underline{1}' W_m X) \\ \widetilde{X}_l &= (X - \underline{1}(\underline{1}' W_l \underline{1})^{-1} \underline{1}' W_l X) \\ \widetilde{X}_r &= (X - \underline{1}(\underline{1}' W_r \underline{1})^{-1} \underline{1}' W_r X). \end{aligned}$$

Proof. In this case, using the Yang-Ko metric $\Delta^2_{\lambda\rho}$ written in vector terms, the LS problem consists in looking for $\hat{a}_m$, $\hat{a}_l$, $\hat{a}_r$, $\hat{b}_m$, $\hat{b}_l$ and $\hat{b}_r$ in the order to

$$min\Delta^2_{\lambda\rho} = minD^2_{\lambda\rho}((W_m^{\frac{1}{2}}\underline{Y}^m,\, W_l^{\frac{1}{2}}g(\underline{Y}^l),\, W_r^{\frac{1}{2}}h(\underline{Y}^r)), (W_m^{\frac{1}{2}}(\underline{Y}^m)^*,\, W_l^{\frac{1}{2}}g^*(\underline{Y}^l),\, W_r^{\frac{1}{2}}h^*(\underline{Y}^r))$$

where $W_m^{\frac{1}{2}}(\underline{Y}^m)^* = W_m^{\frac{1}{2}}X\underline{a}_m + W_m^{\frac{1}{2}}\underline{1}b_m$, $W_l^{\frac{1}{2}}g^*(\underline{Y}^l) = W_l^{\frac{1}{2}}X\underline{a}_l + W_l^{\frac{1}{2}}\underline{1}b_l$ and $W_r^{\frac{1}{2}}h^*(\underline{Y}^r) = W_r^{\frac{1}{2}}X\underline{a}_r + W_r^{\frac{1}{2}}\underline{1}b_r$ are $n\times 1$ vectors of the predicted values. The function to minimize is

$$\begin{aligned}\Delta^2_{\lambda\rho} =& \| W_m^{\frac{1}{2}}\underline{Y}^m - W_m^{\frac{1}{2}}(\underline{Y}^m)^* \|^2 + \| (W_m^{\frac{1}{2}}\underline{Y}^m - \lambda W_l^{\frac{1}{2}}g(\underline{Y}^l)) - (W_m^{\frac{1}{2}}(\underline{Y}^m)^* - \lambda W_l^{\frac{1}{2}}g^*(\underline{Y}^l)) \|^2 \\ &+ \| (W_m^{\frac{1}{2}}\underline{Y}^m + \rho W_r^{\frac{1}{2}}h(\underline{Y}^r)) - (W_m^{\frac{1}{2}}(\underline{Y}^m)^* + \rho W_r^{\frac{1}{2}}h^*(\underline{Y}^r)) \|^2\end{aligned}$$

which becomes

$$\begin{aligned}\Delta^2_{\lambda\rho} =& 3(W_m^{\frac{1}{2}}\underline{Y}^m - W_m^{\frac{1}{2}}X\underline{a}_m - W_m^{\frac{1}{2}}\underline{1}b_m)'(W_m^{\frac{1}{2}}\underline{Y}^m - W_m^{\frac{1}{2}}X\underline{a}_m - W_m^{\frac{1}{2}}\underline{1}b_m) \\ &+ \lambda^2(W_l^{\frac{1}{2}}g(\underline{Y}^l) - W_l^{\frac{1}{2}}X\underline{a}_l - W_l^{\frac{1}{2}}\underline{1}b_l)'(W_l^{\frac{1}{2}}g(\underline{Y}^l) - W_l^{\frac{1}{2}}X\underline{a}_l - W_l^{\frac{1}{2}}\underline{1}b_l) \\ &+ \rho^2(W_r^{\frac{1}{2}}h(\underline{Y}^r) - W_r^{\frac{1}{2}}X\underline{a}_r - W_r^{\frac{1}{2}}\underline{1}b_r)'(W_r^{\frac{1}{2}}h(\underline{Y}^r) - W_r^{\frac{1}{2}}X\underline{a}_r - W_r^{\frac{1}{2}}\underline{1}b_r) \\ &- 2\lambda(W_m^{\frac{1}{2}}\underline{Y}^m - W_m^{\frac{1}{2}}X\underline{a}_m - W_m^{\frac{1}{2}}\underline{1}b_m)'(W_l^{\frac{1}{2}}g(\underline{Y}^l) - W_l^{\frac{1}{2}}X\underline{a}_l - W_l^{\frac{1}{2}}\underline{1}b_l) \\ &+ 2\rho(W_m^{\frac{1}{2}}\underline{Y}^m - W_m^{\frac{1}{2}}X\underline{a}_m - W_m^{\frac{1}{2}}\underline{1}b_m)'(W_r^{\frac{1}{2}}h(\underline{Y}^r) - W_r^{\frac{1}{2}}X\underline{a}_r - W_r^{\frac{1}{2}}\underline{1}b_r).\end{aligned} \tag{13}$$

To estimate b_l and b_r, we equate the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to b_l and b_r to zero, that is

$$\begin{aligned}b_l =& (\underline{1}'W_l\underline{1})^{-1}(\underline{1}'W_lg(\underline{Y}^l) - \underline{1}'W_lX\underline{a}_l - \lambda^{-1}\underline{1}'W_l\underline{Y}^m \\ &+ \lambda^{-1}\underline{1}'W_lX\underline{a}_m + \lambda^{-1}\underline{1}'W_l\underline{1}b_m),\end{aligned} \tag{14}$$

$$\begin{aligned}b_r =& (\underline{1}'W_r\underline{1})^{-1}(\underline{1}'W_rh(\underline{Y}^r) - \underline{1}'W_rX\underline{a}_r + \rho^{-1}\underline{1}'W_r\underline{Y}^m \\ &- \rho^{-1}\underline{1}'W_rX\underline{a}_m - \rho^{-1}\underline{1}'W_r\underline{1}b_m).\end{aligned} \tag{15}$$

Substituting (14) and (15) in (13) and equating the partial derivative of $\Delta^2_{\lambda\rho}$ with respect to b_m to zero, we obtained

$$b_m = (\underline{1}'W_m\underline{1})^{-1}\big((\underline{1}'W_m\underline{Y}^m) - (\underline{1}'W_mX)\underline{a}_m\big) \tag{16}$$

Hence:

$$
\begin{aligned}
b_m &= (\underline{1}'\,W_m\underline{1})^{-1}\big((\underline{1}'\,W_m\underline{Y}^m) - (\underline{1}'\,W_m X)\underline{a}_m\big)\\
b_l &= (\underline{1}'\,W_l\underline{1})^{-1}\Big((\underline{1}'\,W_l g(\underline{Y}^l)) - (\underline{1}'\,W_l X)\underline{a}_l\Big)\\
b_r &= (\underline{1}'\,W_r\underline{1})^{-1}\left((\underline{1}'\,W_r h(\underline{Y}^r)) - (\underline{1}'\,W_r X)\underline{a}_r\right),
\end{aligned}
$$

the centered values of X on the base of weight matrices of center and spreads are

$$
\begin{aligned}
\widetilde{X}_m &= (X - \underline{1}(\underline{1}'\,W_m\underline{1})^{-1}\underline{1}'\,W_m X)\\
\widetilde{X}_l &= (X - \underline{1}(\underline{1}'\,W_l\underline{1})^{-1}\underline{1}'\,W_l X)\\
\widetilde{X}_r &= (X - \underline{1}(\underline{1}'\,W_r\underline{1})^{-1}\underline{1}'\,W_r X).
\end{aligned}
$$

So the objective function can be written as follows:

$$
\begin{aligned}
\Delta^2_{\lambda\rho} =&3(W_m^{\frac{1}{2}}\widetilde{Y}^m - W_m^{\frac{1}{2}}\widetilde{X}_m\underline{a}_m)'(W_m^{\frac{1}{2}}\widetilde{Y}^m - W_m^{\frac{1}{2}}\widetilde{X}_m\underline{a}_m)\\
&+ \lambda^2(W_l^{\frac{1}{2}}\widetilde{g(Y)^l} - W_l^{\frac{1}{2}}\widetilde{X}_l\underline{a}_l)'(W_l^{\frac{1}{2}}\widetilde{g(Y)^l} - W_l^{\frac{1}{2}}\widetilde{X}_l\underline{a}_l)\\
&+ \rho^2(W_r^{\frac{1}{2}}\widetilde{h(Y)^r} - W_r^{\frac{1}{2}}\widetilde{X}_l\underline{a}_r)'(W_r^{\frac{1}{2}}\widetilde{h(Y)^r} - W_r^{\frac{1}{2}}\widetilde{X}_r\underline{a}_r)\\
&- 2\lambda(W_m^{\frac{1}{2}}\widetilde{Y}^m - W_m^{\frac{1}{2}}\widetilde{X}_m\underline{a}_m)'(W_l^{\frac{1}{2}}\widetilde{g(Y)^l} - W_l^{\frac{1}{2}}\widetilde{X}_l\underline{a}_l)\\
&+ 2\rho(W_m^{\frac{1}{2}}\widetilde{Y}^m - W_m^{\frac{1}{2}}\widetilde{X}_m\underline{a}_m)'(W_r^{\frac{1}{2}}\widetilde{h(Y)^r} - W_r^{\frac{1}{2}}\widetilde{X}_r\underline{a}_r).
\end{aligned}
\tag{17}
$$

Finally, by equating the partial derivative of (17) with respect to a_l and a_r to zero, by simple calculations substituting a_l and a_r in (17) and equating to zero the partial derivative of (17) with respect to a_m, we get

$$
\begin{aligned}
\underline{a}_m &= (\widetilde{X}_m'\,W_m\widetilde{X}_m)^{-1}\widetilde{X}_m'\,W_m\widetilde{Y}^m\\
\underline{a}_l &= (\widetilde{X}_l'\,W_l\widetilde{X}_l)^{-1}\widetilde{X}_l'\,W_l\widetilde{g(Y^l)}\\
\underline{a}_r &= (\widetilde{X}_r'\,W_r\widetilde{X}_r)^{-1}\widetilde{X}_r'\,W_r\widetilde{h(Y)^r}.
\end{aligned}
$$

Hence, the LS estimators are as follows:

$$\begin{aligned}
\underline{\hat{a}}_m &= (\widetilde{X}_m' W_m \widetilde{X}_m)^{-1} \widetilde{X}_m' W_m \widetilde{Y}^m \\
\underline{\hat{a}}_l &= (\widetilde{X}_l' W_l \widetilde{X}_l)^{-1} \widetilde{X}_l' W_l \widetilde{g(Y^l)} \\
\underline{\hat{a}}_r &= (\widetilde{X}_r' W_r \widetilde{X}_r)^{-1} \widetilde{X}_r' W_r \widetilde{h(Y)^r} \\
\hat{b}_m &= (\underline{1}' W_m \underline{1})^{-1} \left((\underline{1}' W_m \underline{Y}^m) - (\underline{1}' W_m X)\underline{a}_m\right) \\
\hat{b}_l &= (\underline{1}' W_l \underline{1})^{-1} \Big((\underline{1}' W_l g(\underline{Y}^l)) - (\underline{1}' W_l X)\underline{a}_l\Big) \\
\hat{b}_r &= (\underline{1}' W_r \underline{1})^{-1} \left((\underline{1}' W_r h(\underline{Y}^r)) - (\underline{1}' W_r X)\underline{a}_r\right),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\underline{Y}^m} &= (\underline{Y}^m - \underline{1}(\underline{1}' W_m \underline{1})^{-1} \underline{1}' W_m \underline{Y}^m) \\
\widetilde{g(\underline{Y}^l)} &= (g(\underline{Y}^l) - \underline{1}(\underline{1}' W_l \underline{1})^{-1} \underline{1}' W_l g(\underline{Y}^l)) \\
\widetilde{h(\underline{Y}^r)} &= (h(\underline{Y}^r) - \underline{1}(\underline{1}' W_r \underline{1})^{-1} \underline{1}' W_r h(\underline{Y}^r)).
\end{aligned}$$

□

Remark 3.1. [10] The prediction errors are weighted by dividing each prediction error by a factor proportional to the corresponding subpopulations standard deviation. This ensures that the method of estimation will give more weight to observations from subpopulations with smaller standard deviations because these observations are more reliable, and less weight will be given to observations from subpopulation with larger standard deviations because these observations are less reliable. So $w_i = (\frac{1}{g(x_i)})^2$, where $g(x_i)$ is proportional to the corresponding subpopulation standard deviation.

4. **Numerical examples**

To illustrate the application of the weighted regression model introduced in this work we consider the following examples and compare them with Ferraro et al. method. In this example we are interested in analyzing the dependence relationship of the Retail Trade Sales (in millions of dollars) of the U.S. in 2002 by kind of business on establishments (see http://www.census.gov/econ/www/). The Retail Trade Sales are intervals in the period January 2002 through December 2002 (see Table 1). For each interval we consider the center and the spreads and we apply the proposed weighted regression model in order to evaluate the dependence relationship. We transform the spreads by means of the logarithmic transformation. We consider weights of center and spreads, behaving as $unif(0,1)$ random variables. The parameters are estimated by means of our method and Ferraro et al. method and the accuracy of the estimators are evaluated by means of a bootstrap procedure with 800 replications. As seen in table 2 in many cases the standard errors of our method are less than Ferraro et al. standard error. We consider the exam-

TABLE 1. The retail trade sales and the number of employees of 22 kinds of business in the U.S. in 2002.

Kind of business	Retail trade sales	Establishments
Automotive parts, acc., and tire stores	(4638,5759)	57698
Furniture stores	(4054,4685)	28244
Home furnishings stores	(2983,5032)	36960
Household appliance stores	(1035,1387)	10330
Computer and software stores	(1301,1860)	10134
Building mat. and supplies dealers	(14508,20727)	67190
Hardware stores	(1097,1691)	15103
Beer, wine, and liquor stores	(2121,3507)	28957
Pharmacies and drug stores	(11964,14741)	40234
Gasoline stations	(16763,23122)	121446
Means clothing stores	(532,1120)	9437
Family clothing stores	(3596,9391)	24539
Shoe stores	(1464,2485)	28499
Jewelry stores	(1304,5810)	28625
Sporting goods stores	(1748,3404)	22239
Book stores	(968,1973)	10860
Discount dept. stores	(9226,17001)	5650
Department stores	(5310,14057)	3705
Warehouse clubs and superstores	(13162,22089)	2912
All other gen. merchandize stores	(2376,4435)	28456
Miscellaneous store retailers	(7862,10975)	129464
Fuel dealers	(1306,3145)	11079

TABLE 2. Estimation of the parameters of models and estimation of their standard errors.

	Estimated value		Estimate of standard error	
Estimator	Ferraro et al. method	Our method	Ferraro et al. method	Our method
$\hat{a}_m$	0.08216	0.1211	0.0536	0.0459
$\hat{a}_l$	$7.4339e^{-6}$	$2.0800e^{-6}$	$7.5074e^{-6}$	$6.06960e^{-6}$
$\hat{a}_r$	$7.4339e^{-6}$	$2.0800e^{-6}$	$7.5074e^{-6}$	$6.06960e^{-6}$
$\hat{b}_m$	3843	2177	1977	1853
$\hat{b}_l$	6.6717	6.9601	0.3611	0.3745
$\hat{b}_r$	6.6717	6.9601	0.3611	0.3745

ple of a dataset studied by Coppi et al. [5] having multivariate inputs and their corresponding non-symetric triangular fuzzy outputs. This dataset consists of 21 observations of atmospheric concentration of carbon monoxide (CO) with six independent meteorological variables: x_1=temperature, x_2=relative, x_3=atmospheric pressure, x_4=rain, x_5=radiation and x_6=wind speed, observed in the city of Rome (see table 3). For obtaining diagonal matrices of weight, we consider some groups from close data of center and spreads then obtain the variance of each group and means of X corresponding to the groups and fit lines between variances and means by means of least square approach as follows

$$\begin{aligned}
\hat{S}^2_{\underline{Y}^m} &= 5.011 + 0.005\bar{X}_1 + 0.003\bar{X}_2 - 0.005\bar{X}_3 - 0.27\bar{X}_4 + 1.60\bar{X}_5 + 0.007\bar{X}_6 \\
\hat{S}^2_{g(\underline{Y}^l)} &= 8.72 - 0.04\bar{X}_1 + 0.01\bar{X}_2 - 0.008\bar{X}_3 - 0.17\bar{X}_4 + 1.06\bar{X}_5 + 0.06\bar{X}_6 \\
\hat{S}^2_{h(\underline{Y}^r)} &= -6.40 + 0.033\bar{X}_1 + 0.012\bar{X}_2 + 0.006\bar{X}_3 + 0.15\bar{X}_4 - 3.13\bar{X}_5 + 0.067\bar{X}_6.
\end{aligned}$$

By substituting explanatory variables in the equations, the estimation of variances are obtained. The diagonal matrix of inverse of these values are considered as weight matrices. The parameters are estimated by means of our method and Ferraro et al. method and the accuracy of the estimators are evaluated by means of a bootstrap procedure with 800 replications. As seen in table 4 in almost all cases, the standard errors of our method are less than Ferraro et al.xz[9].

TABLE 3. Numerical data of example 4.2.

x_1	x_2	x_3	x_4	x_5	x_6	Y^m	Y^l	Y^r
19.04	82.27	1009.90	0.90	0.15	2.87	1.15	0.84	0.60
17.66	88.70	1017.50	0.90	016	1.12	2.98	2.83	2.02
18.15	82.51	1025.60	0.02	0.14	0.85	3.92	1.97	1.97
18.43	79.46	1032.10	0.00	0.16	0.45	4.65	1.89	2.24
20.67	68.85	1027.10	0.00	0.22	0.91	3.98	2.13	2.13
21.64	79.39	1020	0.02	0.29	1.07	3.35	2.63	1.58
18.85	88.87	1018.20	1.30	0.24	0.69	3.13	2.39	1.71
16.16	88.92	1020.90	0.09	0.16	0.40	4.15	2.41	2.41
17.02	83.52	1028	0.00	0.13	0.83	3.96	2.48	2.10
14.72	87.51	1025.30	0.05	0.04	2.09	4.07	2.01	2.38
15.89	86.04	1018.50	0.04	0.04	0.90	3.30	2.16	1.83
17.83	91.77	1015.60	0.93	0.06	1.22	4.02	2.35	2.78
20.96	83.62	1012.30	0.02	0.13	3	2.06	3.19	1.59
17.07	73.10	1020.40	0.02	0.13	1.75	1.37	0.66	0.66
14.71	80.38	1028.60	0.09	0.01	0.73	3.35	1.73	1.73
20.41	87.98	1026.40	0.24	0.11	1.87	1.45	1.15	0.97
20.13	90.13	1023.10	0.02	0.17	2.39	2.74	1.41	1.41
15.64	64.95	1022.40	0.00	0.08	1.25	2.44	1.95	1.39
13.22	80.16	1021.60	0.01	0.00	1.02	2.79	2.23	1.59
12.98	86.14	1023.40	0.44	0.00	0.70	3.31	1.73	1.24
13.10	89.12	1028.60	0.01	0.00	1.17	4.02	2.96	2.11

5. Conclusion

In this paper we have introduced a weighted linear regression model for imprecise response based on Ferraro et al. method. This method is especially attractive since the standard errors of estimators are obtained are lower than Ferraro et al. standard errors. So better regression lines are fitted by using of proposed method.

TABLE 4. Estimation the parameters of models and estimation of their standard errors.

Estimator	Estimated value		Estimate of standard error	
Estimator	Ferraro et al. method	Our method	Ferraro et al. method	Our method
$\hat{a}_{m1}$	-0.0257	0.0247	0.2648	0.2626
$\hat{a}_{m2}$	0.0515	0.0620	0.03244	0.03361
$\hat{a}_{m3}$	0.0334	0.0277	0.06702	0.06443
$\hat{a}_{m4}$	-0.5573	-0.4754	0.8316	0.6667
$\hat{a}_{m5}$	1.919	0.3823	7.6451	7.5998
$\hat{a}_{m6}$	-0.7615	-0.7598	0.5221	0.4857
$\hat{a}_{l1}$	0.008072	0.0023	0.1826	0.1801
$\hat{a}_{l2}$	0.0353	0.03288	0.03556	0.03113
$\hat{a}_{l3}$	-0.0522	-0.0537	0.05627	0.05315
$\hat{a}_{l4}$	-0.6688	-0.6552	0.8328	0.6946
$\hat{a}_{l5}$	0.6760	0.9807	5.2489	4.89022
$\hat{a}_{l6}$	-0.4564	-0.4749	0.4153	0.4095
$\hat{a}_{r1}$	0.0525	0.07588	0.1736	0.1643
$\hat{a}_{r2}$	0.0331	0.02794	0.02393	0.02229
$\hat{a}_{r3}$	-0.0021	0.0050	0.04322	0.04005
$\hat{a}_{r4}$	-0.2814	-0.2170	0.5323	0.5202
$\hat{a}_{r5}$	-1.0500	-1.7336	5.07622	4.6291
$\hat{a}_{r6}$	-0.4540	-0.4826	0.3690	0.3479
$\hat{b}_m$	-33.98	-29.7787	69.2177	66.3558
$\hat{b}_l$	53.037	54.8956	56.8805	55.7944
$\hat{b}_r$	1.3221	-6.1468	44.1435	40.8383

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