

CONVERGENCE PROPERTIES OF HERMITIAN AND SKEW HERMITIAN SPLITTING METHODS

F. KYANFAR

DEPARTMENT OF APPLIED MATHEMATICS, SHAHID BAHONAR
UNIVERSITY OF KERMAN, KERMAN, IRAN.
E-MAIL: KYANFAR@UK.AC.IR

(Received: 11 April 2016, Accepted: 28 October 2016)

ABSTRACT. In this paper we consider the solutions of linear systems of saddle point problems. By using the spectrum of a quadratic matrix polynomial, we study the eigenvalues of the iterative matrix of the Hermitian and skew Hermitian splitting method.

AMS Classification: 65F10, 65F50, 15A06 .

Keywords: Splitting, Saddle point problem, Hermitian, Skew Hermitian.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. We consider the iterative solution of a large sparse non-Hermitian system of linear equations

$$(1) \quad Ax = b, \quad A \in M_n(\mathbb{C}), \quad A \neq A^*, \quad \text{and } x, b \in \mathbb{C}^n.$$

Based on the alternating splitting iteration [3], $A = H(A) + S(A)$, with $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2$ are Hermitian and skew-Hermitian parts of A ,

JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER

VOL. 3, NUMBER 1 (2014) 31-36.

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respectively. Let $\gamma > 0$ be a parameter. We consider the following splittings of A :

$$A = (\gamma I + H(A)) - (\gamma I - S(A)) \quad \text{and} \quad A = (\gamma I + S(A)) - (\gamma I - H(A)).$$

Here I denotes the identity matrix. The algorithm is obtained by alternating between these two splittings. Given an initial guess x_0 , the Hermitian and skew-Hermitian iteration computes a sequence $\{x_k\}$ as follows:

$$(2) \quad \begin{cases} (\gamma I + H(A))x_{k+\frac{1}{2}} = (\gamma I - S(A))x_k + b, \\ (\gamma I + S(A))x_{k+1} = (\gamma I - H(A))x_{k+\frac{1}{2}} + b. \end{cases}$$

Suppose

$$(3) \quad Q(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0$$

is a matrix polynomial, where $A_i \in M_n(\mathbb{C})$ ($i = 0, 1, \dots, m$), $A_m \neq 0$ and λ is a complex variable. The matrix polynomial $Q(\lambda)$ is called selfadjoint if all the coefficients A_i are Hermitian matrices. A complex number λ_0 is an eigenvalue of $Q(\lambda)$ if $\det Q(\lambda_0) = 0$. The spectrum of the matrix polynomial $Q(\lambda)$ is defined as

$$\sigma[Q(\lambda)] = \{\mu \in \mathbb{C} : \exists 0 \neq x \in \mathbb{C}^n, Q(\mu)x = 0\} = \{\mu \in \mathbb{C} : 0 \in \sigma(Q(\mu))\}.$$

The field of values of the matrix polynomial $Q(\lambda)$ is defined as

$$F[Q(\lambda)] = \{\mu \in \mathbb{C} : \exists 0 \neq x \in \mathbb{C}^n, x^* Q(\mu)x = 0\} = \{\mu \in \mathbb{C} : 0 \in F(Q(\mu))\},$$

where $\sigma(Q(\mu))$ and $F(Q(\mu))$ are the spectrum and the field of values of the matrix $Q(\mu)$, respectively. For more details see [?].

For $A, B \in M_n(\mathbb{C})$ with $0 \notin F(B)$, the ratio field of values is defined as follows:

$$(4) \quad R(A, B) = \left\{ \frac{x^* A x}{x^* B x} : x \in \mathbb{C}^n, \|x\| = 1 \right\}.$$

The ratio field of values turns out to be a special case of the numerical range of a matrix polynomial. Let $Q(\lambda) = A - \lambda B$. Then $R(A, B) = F[Q(\lambda)]$ if and only if $0 \notin F(B)$ (see [5]). In the next section by using the spectrum of a quadratic matrix polynomial, we study the eigenvalues of the iterative matrix of the Hermitian and skew Hermitian splitting method.

2. THE HSS ITERATION METHODS FOR SADDLE POINT PROBLEMS

We consider the iterative solutions of a saddle-point problem of the form

$$(5) \quad Ax = \begin{bmatrix} B & E \\ -E^* & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b,$$

where $B \in M_n(\mathbb{C})$ is a Hermitian positive definite, $E \in M_{n,m}(\mathbb{C})$ has full column rank, $n \geq m, f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$. The existence and uniqueness of the solutions of the system of linear equations (5) was guaranteed by Benzi and Golub in [2]. By alternating splitting method, we split the matrix A into its Hermitian and skew-Hermitian parts.

$$(6) \quad H(A) = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } S(A) = \begin{bmatrix} 0 & E \\ -E^* & 0 \end{bmatrix}.$$

We consider the *Hermitian and Skew-Hermitian Splitting* (HSS) iteration method to solve the saddle point problem (5). Asume $x_0 = (y_0, z_0)^t \in \mathbb{C}^{m+n}$ is an initial guess. The HSS method computes a sequence $\{x_{k+1} = (y_{k+1}, z_{k+1})^t\}, k = 0, 1, \dots$ by solving the linear subsystems:

$$(7) \quad \begin{cases} \left(\begin{bmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y^{k+\frac{1}{2}} \\ z^{k+\frac{1}{2}} \end{bmatrix} = \left(\begin{bmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{bmatrix} - \begin{bmatrix} 0 & E \\ -E^* & 0 \end{bmatrix} \right) \begin{bmatrix} y^k \\ z^k \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \\ \left(\begin{bmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{bmatrix} + \begin{bmatrix} 0 & E \\ -E^* & 0 \end{bmatrix} \right) \begin{bmatrix} y^{k+1} \\ z^{k+1} \end{bmatrix} = \left(\begin{bmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y^{k+\frac{1}{2}} \\ z^{k+\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}. \end{cases}$$

Therefore, the HSS iteration method can be obtained as follows:

$$(8) \quad \begin{bmatrix} y^{k+1} \\ z^{k+1} \end{bmatrix} = \mathcal{L}(\alpha, \beta) \begin{bmatrix} y^k \\ z^k \end{bmatrix} + \mathcal{G}(\alpha, \beta) \begin{bmatrix} f \\ g \end{bmatrix},$$

where the iteration matrix $\mathcal{L}(\alpha, \beta)$ of the HSS iteration method and the matrix $\mathcal{G}(\alpha, \beta)$ are as follows:

$$(9) \quad \mathcal{L}(\alpha, \beta) = \begin{bmatrix} \alpha(\alpha I_n + B) & (\alpha I_n + B)E \\ -\beta E^* & \beta^2 I_m \end{bmatrix}^{-1} \begin{bmatrix} \alpha(\alpha I_n - B) & -(\alpha I_n - B)E \\ \beta E^* & \beta^2 I_m \end{bmatrix},$$

$$\mathcal{G}(\alpha, \beta) = 2 \begin{bmatrix} \alpha I_n + B & 1/\alpha(\alpha I_n + B)E \\ -E^* & \beta I_m \end{bmatrix}^{-1}.$$

By [4, Theorem 2.1], we know that the HSS iteration method (8) is unconditionally convergent i.e. $\rho(\mathcal{L}(\alpha, \beta)) < 1, \forall \alpha, \beta > 0$.

In the following theorem, by the same notations as in [1, Lemma 3.1], we study the spectrum of the quadratic matrix polynomial

$$(10) \quad \mathcal{Q}_{\alpha,\beta}(\lambda) := \lambda^2(\alpha I + B) - 2\lambda\alpha(\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*) + (\alpha I - B).$$

Theorem 2.1. *Let $B \in M_n(\mathbb{C})$ be a Hermitian positive definite matrix, $E \in M_{n,m}(\mathbb{C})$ has full column rank, $n \geq m$ and let $\alpha, \beta > 0$ be given iteration parameters. Then, $\sigma[\mathcal{Q}_{\alpha,\beta}(\lambda)] \cap \{z \in \mathbb{C} : |z| = 1\} = \{1\}$, where the quadratic matrix polynomial $\mathcal{Q}_{\alpha,\beta}(\lambda)$ is as in (10).*

Proof. Let $\lambda_0 = e^{i\theta} \in \sigma[\mathcal{Q}_{\alpha,\beta}(\lambda)]$, $\theta \in [0, 2\pi)$. It is enough to show that $\mathcal{Q}_{\alpha,\beta}(e^{i\theta})$ is singular if and only if $\theta = 0$. We have $\mathcal{Q}_{\alpha,\beta}(e^{i\theta}) = e^{i\theta}[e^{i\theta}(\alpha I + B) - 2\alpha(\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*) + e^{-i\theta}(\alpha I - B)] = 2e^{i\theta}[\alpha \text{Cos}(\theta)I - \alpha(\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*) + i\text{Sin}(\theta)B]$. If $\theta \in (0, \pi) \cup (\pi, 2\pi)$, then $\Im(e^{-i\theta}\mathcal{Q}_{\alpha,\beta}(e^{i\theta})) = \text{Sin}(\theta)B$ is positive definite or negative definite. Therefore, if $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$, then $\mathcal{Q}_{\alpha,\beta}(e^{i\theta})$ is nonsingular. It is enough to consider $\theta \in \{0, \pi\}$ (i.e. $\lambda_0 = \pm 1$). $\mathcal{Q}_{\alpha,\beta}(\pm 1) = (\alpha I + B) \pm 2\alpha(\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*) + (\alpha I - B) = 2\alpha(I \pm (\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*))$. It is readily seen that $(\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*)$ is unitarily similar to the $n \times n$ diagonal matrix $D = \text{diag}\left(\frac{\alpha\beta - \mu_1}{\alpha\beta + \mu_1}, \dots, \frac{\alpha\beta - \mu_n}{\alpha\beta + \mu_n}\right)$, where μ_1, \dots, μ_n are the non-negative eigenvalues of EE^* . Hence $\mathcal{Q}_{\alpha,\beta}(1)$ and $\mathcal{Q}_{\alpha,\beta}(-1)$ are unitarily similar to the $n \times n$ diagonal matrices $\text{diag}\left(\frac{4\alpha\mu_1}{\alpha\beta + \mu_1}, \dots, \frac{4\alpha\mu_n}{\alpha\beta + \mu_n}\right)$, and $\text{diag}\left(\frac{4\alpha^2\beta}{\alpha\beta + \mu_1}, \dots, \frac{4\alpha^2\beta}{\alpha\beta + \mu_n}\right)$, respectively. Since $n > m$ we obtain that $\mu_n = 0$ and hence $\mathcal{Q}_{\alpha,\beta}(1)$ is singular and $\mathcal{Q}_{\alpha,\beta}(-1)$ is nonsingular. Therefore, $e^{i\theta} \in \sigma[\mathcal{Q}_{\alpha,\beta}(\lambda)]$ if and only if $\theta = 0$. This completes the proof. \square

The field of values of the matrix polynomial $\mathcal{Q}_{\alpha,\beta}(\lambda)$ is as follows:

$$F[\mathcal{Q}_{\alpha,\beta}(\lambda)] = \{\gamma \in \mathbb{C} : \mathbf{0} \in F(\mathcal{Q}_{\alpha,\beta}(\gamma))\} = \{\gamma \in \mathbb{C} : \exists \mathbf{0} \neq x \in \mathbb{C}^n, x^* \mathcal{Q}_{\alpha,\beta}(\gamma)x = 0\}.$$

Let $x \in \mathbb{C}^n$ be a unit vector. Define $\eta := x^* Bx$ and $\tilde{\mu} := \sqrt{x^* EE^* x}$. Then

$$(11) \quad F[\mathcal{Q}_{\alpha,\beta}(\lambda)] \subseteq \left\{ \frac{\alpha}{\alpha + \eta} \left(\frac{\alpha\beta - \tilde{\mu}^2}{\alpha\beta + \tilde{\mu}^2} \pm \sqrt{\left(\frac{\alpha\beta - \tilde{\mu}^2}{\alpha\beta + \tilde{\mu}^2} \right)^2 + \frac{\eta^2}{\alpha^2} - 1} \right) : \tilde{\mu}^2 \in F(EE^*), \eta \in F(B) \right\},$$

where $F(B) = [\eta_{\min}, \eta_{\max}]$ and $F(EE^*) = [\tilde{\mu}_{\min}^2, \tilde{\mu}_{\max}^2]$. If $n > m$, then $\tilde{\mu}_{\min} = 0$. By the same notations as in [1], we have

$$\varphi(\alpha, \beta; \eta, \tilde{\mu}) = \frac{\alpha}{\alpha + \eta} \left(\frac{\alpha\beta - \tilde{\mu}^2}{\alpha\beta + \tilde{\mu}^2} \pm \sqrt{\left(\frac{\alpha\beta - \tilde{\mu}^2}{\alpha\beta + \tilde{\mu}^2}\right)^2 + \frac{\eta^2}{\alpha^2} - 1} \right).$$

By Lemma [4, Theorem 2.1], we know that $\sigma(\mathcal{L}(\alpha, \beta)) \subseteq D, \forall \alpha, \beta > 0$, where $D = \{z \in \mathbb{C} : |z| < 1\}$. Also, by Theorem 2.1, we obtain that $\mathcal{Q}_{\alpha, \beta}(1)$ is a singular matrix, i.e. $1 \in \sigma[\mathcal{Q}_{\alpha, \beta}(\lambda)]$.

By using [1] and Theorem 2.1, we are revising [1, Lemma 3.1] as follows:

Theorem 2.2. *Consider the saddle-point problem (5). Let $B \in M_n(\mathbb{C})$ be Hermitian positive definite, $E \in M_{n,m}(\mathbb{C})$ has full column rank, $n \geq m$ and let $\alpha, \beta > 0$ be given the iteration parameters. Then*

$$\begin{aligned} \sigma[\mathcal{L}_{\alpha, \beta}(\lambda)] &= \sigma[\mathcal{Q}_{\alpha, \beta}(\lambda)] \setminus \{1\} \subseteq F[\mathcal{Q}_{\alpha, \beta}(\lambda)] \setminus \{1\}, \\ (ii) \ F[\mathcal{Q}_{\alpha, \beta}(\lambda)] &\subseteq \{\varphi(\alpha, \beta; \eta, \mu) : \eta \in F(B), \mu \in F(E^*E)\} \cup \{\varphi(\alpha, \beta; \eta, 0) : \eta \in F(B)\}. \end{aligned}$$

Moreover, if the matrices B and EE^* are commute with each other, then

$$(iii) \ \sigma[\mathcal{L}_{\alpha, \beta}(\lambda)] \subseteq \{\varphi(\alpha, \beta; \eta, \gamma) : \eta \in \sigma(B), \gamma \in \sigma(E^*E)\} \cup \{\varphi(\alpha, \beta; \eta, 0) : \eta \in \sigma(B)\}.$$

Proof. By the same method as in [1, Lemma 3.1] and Theorem 2.1, the part (i) holds. We know that $\sigma((\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*)) \subseteq R((\alpha\beta I - EE^*), (\alpha\beta I + EE^*))$, where $R(A, B)$ be as in (4). Since $(\alpha\beta I + EE^*)$ is positive definite, we obtain that the ratio field of values $R((\alpha\beta I - EE^*), (\alpha\beta I + EE^*))$ is convex, for more details see [5]. Hence

$$\begin{aligned} F((\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*)) &= \text{Conv}(\sigma((\alpha\beta I + EE^*)^{-1}(\alpha\beta I - EE^*))) \\ &\subseteq R((\alpha\beta I - EE^*), (\alpha\beta I + EE^*)). \end{aligned}$$

Easy computation shows that

$$F[\mathcal{Q}_{\alpha, \beta}(\lambda)] \subseteq \{\varphi(\alpha, \beta; \eta, \gamma) : \eta \in \sigma(B), \gamma \in \sigma(E^*E)\} \cup \{\varphi(\alpha, \beta; \eta, 0) : \eta \in \sigma(B)\}.$$

This completes the proof.

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