GAUSS-SIDEL AND SUCCESSIVE OVER RELAXATION ITERATIVE METHODS FOR SOLVING SYSTEM OF FUZZY SYLVESTER EQUATIONS

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ABSTRACT. In this paper, we present Gauss-Sidel and successive over relaxation (SOR) iterative methods for finding the approximate solution system of fuzzy Sylvester equations (SFSE), $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. Finally, the proposed iterative methods are illustrated by solving an example.

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1. INTRODUCTION

Solution of a system of linear equations plays a crucial role in almost every field of sciences and engineering. In many applications, some of the system parameters are represented by fuzzy numbers rather then crisp numbers. Therefore, it is important to develop mathematical models and numerical procedures to general system of

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fuzzy linear equations (SFLE). In [2] it is considered a general model for solving a SFLE whose coefcient matrix is crisp and its right hand side is an arbitrary fuzzy vector. They stated some conditions for the existence of a unique fuzzy solution to SFLE by embedding method and converting the original system to a crisp linear system of equations. Later on many authors have studied SFLE. The numrical methods for SFLE were proposed by Allahviranloo [14, 15]. Dehghan and Hashemi [3] have applied several iterative methods for solving SFLE. Wang and Zheng [5] have studied some block iterative methods to solve SFLE. It is well-known that the Sylvester matrix equation is of the form

$$AX + XB = C, \tag{1}$$

where A is an $n \times n$ crisp matrix, B is an $m \times m$ crisp matrix, C is an $n \times m$ fuzzy matrix and $X$is an $n \times m$ unknown matrix. This matrix equation plays an important role in control theory, signal processing, ltering, model reduction, image restoration, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, and block-diagonalization of matrices. For example, Benner [12, 13]; Datta and Datta [4]; Hyland and Bernstein [7]; Laub et al. [1]. Standard solution methods for Sylvester equations of the form (1) are the BartelsStewart method in [11] and the HessenbergSchur method in [6]. The methods are based on the transforming the coefcient matrices to Schur or Hessenberg form and then solving the corresponding linear system of equations directly by a backward substitution process. Therefore, these methods are classied as direct methods.

In this paper, we discuss on certain cases system of fuzzy Sylvester equations (1) where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. This paper is organized as follows:

In section 2, we introduce some basic definitions and results on fuzzy numbers and system of fuzzy Sylvester equations. In section 3, we propose the Gauss-Sidel and SOR iterative methods for finding the approximate solution of the system of fuzzy Sylvester equations. In section 4, we illustrate the proposed iterative methods by solving an example. Conclusions are drawn in section 5.

2. Preliminaries

A fuzzy number U is an ordered pair of functions $(\underline{U}(\alpha), \overline{U}(\alpha)), 0 \leq \alpha \leq 1$, which satisfies the following conditions:
1. $\underline{U}(\alpha)$ is a continuous, monotonically increasing function on $[0, 1]$.
2. $\overline{U}(\alpha)$ is a continuous, monotonically decreasing function on $[0, 1]$.
3. $\underline{U}(\alpha) \leq \overline{U}(\alpha)$ on $[0, 1]$.

For arbitrary fuzzy numbers $U = (\underline{U}(\alpha), \overline{U}(\alpha))$, $V = (\underline{V}(\alpha), \overline{V}(\alpha))$, and a scalar k, we define addition, subtraction and scalar multiplication by k as:
1. $(U + V)(\alpha) = (\underline{U}(\alpha) + \underline{V}(\alpha), \overline{U}(\alpha)) + \overline{V}(\alpha))$,
2. $(U - V)(\alpha) = (\underline{U}(\alpha) - \overline{V}(\alpha), \overline{U}(\alpha)) - \underline{V}(\alpha))$,
3. $(kU)(\alpha)= \begin{cases} (k\underline{U}(\alpha), k\overline{U}(\alpha)) & k \geq 0 \\ (k\overline{U}(\alpha), k\underline{U}(\alpha)) & k < 0. \end{cases}$
4. $U = V iff \underline{U}(\alpha) = \underline{V}(\alpha), \overline{U}(\alpha) = \overline{V}(\alpha).$ (2)

The system of linear equations

$$AX + XB = C, \tag{3}$$

is called system of fuzzy Sylvester equations if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times m$ crisp matrices and $C = (c_{ij})$ is an $m \times m$ fuzzy matrix. The system of linear equations $AX + XB = C$ is called system of fully fuzzy Sylvester equations if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are $m \times m$ fuzzy matrices. A fuzzy matrix $X(\alpha) = (x_{ij}(\alpha)) = (\underline{x}_{ij}(\alpha), \overline{x}_{ij}(\alpha))$, $1 \leq i, j \leq m$, $0 \leq \alpha \leq 1$, is called the solution of fuzzy Sylvester equations, if

$$\underline{\left(\sum_{k=1}^{m} a_{ik}x_{kj} + \sum_{k=1}^{m} x_{ik}b_{kj}\right)}(\alpha) = \underline{\sum_{k=1}^{m} a_{ik}x_{kj}}(\alpha) + \underline{\sum_{k=1}^{m} x_{ik}b_{kj}}(\alpha) =$$

$$\sum_{k=1}^{m} \underline{a_{ik}x_{kj}}(\alpha) + \sum_{k=1}^{m} \underline{x_{ik}b_{kj}}(\alpha) = \underline{c}_{ij}(\alpha),$$

$$\overline{\left(\sum_{k=1}^{m} a_{ik}x_{kj} + \sum_{k=1}^{m} x_{ik}b_{kj}\right)}(\alpha) = \overline{\sum_{k=1}^{m} a_{ik}x_{kj}}(\alpha) + \overline{\sum_{k=1}^{m} x_{ik}b_{kj}}(\alpha) =$$

$$\sum_{k=1}^{m} \overline{a_{ik}x_{kj}}(\alpha) + \sum_{k=1}^{m} \overline{x_{ik}b_{kj}}(\alpha) = \overline{c}_{ij}(\alpha). \tag{4}$$

In particular, if $b_{kj} \geq 0, 1 \leq k \leq m, 1 \leq j \leq m$ and $a_{ik} \geq 0$, $1 \leq i \leq m, 1 \leq k \leq m$, we easily get

$$\underline{\left(\sum_{k=1}^{m} a_{ik}x_{kj} + \sum_{k=1}^{m} x_{ik}b_{kj}\right)}(\alpha) = \underline{\sum_{k=1}^{m} a_{ik}x_{kj}}(\alpha) + \underline{\sum_{k=1}^{m} x_{ik}b_{kj}}(\alpha) =$$

$$\sum_{k=1}^{m} a_{ik}\underline{x}_{kj}(\alpha) + \sum_{k=1}^{m} \underline{x}_{ik}(\alpha)b_{kj} = \underline{c}_{ij}(\alpha),$$

$$\overline{\left(\sum_{k=1}^{m} a_{ik}x_{kj} + \sum_{k=1}^{m} x_{ik}b_{kj}\right)}(\alpha) = \overline{\sum_{k=1}^{m} a_{ik}x_{kj}}(\alpha) + \overline{\sum_{k=1}^{m} x_{ik}b_{kj}}(\alpha) =$$

$$\sum_{k=1}^{m} a_{ik}\overline{x}_{kj}(\alpha) + \sum_{k=1}^{m} \overline{x}_{ik}(\alpha)b_{kj} = \overline{c}_{ij}(\alpha). \tag{5}$$

Consider the ij-th equation of the system (3):

$$\sum_{k=1}^{m} a_{ik}(\underline{x}_{kj}, \overline{x}_{kj}) + \sum_{k=1}^{m} (\underline{x}_{ik}, \overline{x}_{ik})b_{kj} = (\underline{c}_{ij}, \overline{c}_{ij}), 1 \leq i, j \leq m. \tag{6}$$

From (6) we have two $m^2 \times m^2$ crisp linear systems for all $1 \leq i, j \leq m$ that there can be extended to an $2m^2 \times 2m^2$ crisp linear system as follows:

$$SZ + ZT = Y, \tag{7}$$

Where

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} + \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix} = \begin{pmatrix} \underline{C} & \overline{C} \\ \overline{C} & \underline{C} \end{pmatrix}. \tag{8}$$

Thus system of fuzzy linear equations (3) is extended to a system of crisp linear equations (8) where $A = S_1 + S_2, S_1 \geq 0, S_2 \leq 0$ and $B = T_1 + T_2, T_1, \geq 0, T_2 \leq 0$. System of equations (8) can be written as follows:

$$\begin{cases} S_1\underline{X} + S_2\overline{X} + \underline{X}T_1 + \overline{X}T_2 = \underline{C}, \\ S_2\underline{X} + S_1\overline{X} + \underline{X}T_2 + \overline{X}T_1 = \overline{C}. \end{cases} \tag{9}$$

If $A = S_1 - S_2, S_1 \geq 0, S_2 \geq 0$ and $B = T_1 - T_2, T_1 \geq 0, T_2 \geq 0$, system of equations (8) become:

$$\begin{cases} S_1\underline{X} - S_2\overline{X} + \underline{X}T_1 - \overline{X}T_2 = \underline{C}, \\ -S_2\underline{X} + S_1\overline{X} + \overline{X}T_1 - \underline{X}T_2 = \overline{C}. \end{cases} \tag{10}$$

The matrices $A = (a_{ij})$ and $B = (b_{ij}), 1 \leq i, j \leq m$, in system of equations (3) are both positive definite iff the matrices S and T in system of equations (7) are both positive definite.

Proof. see [9]. □

The matrices $A = (a_{ij})$ and $B = (b_{ij}), 1 \leq i, j \leq m$, with $a_{ii} > 0$ and $b_{ii} > 0$ as in system of equations (3) are both strictly diagonally dominant iff the matrices S and T in system of equations (7) are both strictly diagonally dominant.

Proof. see [14]. □

3. The Gauss-Sidel and SOR iterative methods

In this section, we use the Gauss-Sidel and SOR iterative methods to solve system of equations (3). First we present the Gauss-Sidel iterative method.

Without loss of generality, suppose that in system of equations (8), $s_{ii} > 0$ and $t_{ii} > 0$ for all $1 \leq i \leq 2m$. Let $S = L + D + U$ where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 \\ S_2 & L_1 \end{pmatrix}, U = \begin{pmatrix} U_1 & S_2 \\ 0 & U_1 \end{pmatrix},$$

and also $T = L' + D' + U'$ where

$$D' = \begin{pmatrix} D'_1 & 0 \\ 0 & D'_1 \end{pmatrix}, L' = \begin{pmatrix} L'_1 & 0 \\ T_2 & L'_1 \end{pmatrix}, U' = \begin{pmatrix} U'_1 & T_2 \\ 0 & U'_1 \end{pmatrix},$$

$(D_1)_{ii} = s_{ii} > 0, 1 \leq i \leq m$ and $(D'_1)_{ii} = t_{ii} > 0, 1 \leq i \leq m$ and suppose $S_1 = L_1 + D_1 + U_1$ and $T_1 = L'_1 + D'_1 + U'_1$. In the Gauss-Sidel method, from the structure of $SZ + ZT = Y$ we have

$$\begin{pmatrix} D_1 + L_1 & 0 \\ 0 & D_1 + L_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} + \begin{pmatrix} U_1 & S_2 \\ S_2 & U_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} +$$

$$\begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} \begin{pmatrix} D_1' + L_1' & 0 \\ 0 & D_1' + L_1' \end{pmatrix} + \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} \begin{pmatrix} U_1' & T_2 \\ T_2 & U_1' \end{pmatrix} = \begin{pmatrix} \underline{C} & \overline{C} \\ \overline{C} & \underline{C} \end{pmatrix}. \tag{11}$$

then

$$\begin{cases} (D_1 + L_1)\underline{X} + \underline{X}(D_1' + L_1') = \underline{C} - U_1\underline{X} - S_2\overline{X} - \underline{X}U_1' - \overline{X}T_2, \\ (D_1 + L_1)\overline{X} + \overline{X}(D_1' + L_1') = \overline{C} - U_1\overline{X} - S_2\underline{X} - \underline{X}T_2 - \overline{X}U_1'. \end{cases} \tag{12}$$

The elements of $X^{(k+1)}(\alpha) = (\underline{X}^{(k+1)}(\alpha), \overline{X}^{(k+1)}(\alpha))$, $0 \leq \alpha \leq 1$ are

$$\begin{aligned} \underline{x}_{ij}^{(k+1)}(\alpha) = \frac{1}{s_{ii} + t_{jj}} \Big[& \underline{c}_{ij}(\alpha) - \sum_{l=1}^{i-1} s_{il}\underline{x}_{lj}^{(k+1)}(\alpha) - \sum_{l=i+1}^{m} s_{il}\underline{x}_{lj}^{(k)}(\alpha) - \sum_{l=1}^{m} s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha) \\ & - \sum_{l=1}^{j-1} \underline{x}_{il}^{(k)}(\alpha)t_{lj} - \sum_{l=j+1}^{m} \underline{x}_{il}^{(k+1)}(\alpha)t_{lj} - \sum_{l=1}^{m} \overline{x}_{il}^{(k)}(\alpha)t_{l,m+j} \Big], \\ \overline{x}_{ij}^{(k+1)}(\alpha) = \frac{1}{s_{ii} + t_{jj}} \Big[& \overline{c}_{ij}(\alpha) - \sum_{l=1}^{i-1} s_{il}\overline{x}_{lj}^{(k+1)}(\alpha) - \sum_{l=i+1}^{m} s_{il}\overline{x}_{lj}^{(k)}(\alpha) - \sum_{l=1}^{m} s_{i,m+l}\underline{x}_{lj}^{(k)}(\alpha) \\ & - \sum_{l=1}^{j-1} \overline{x}_{il}^{(k)}(\alpha)t_{lj} - \sum_{l=j+1}^{m} \overline{x}_{il}^{(k+1)}(\alpha)t_{lj} - \sum_{l=1}^{m} \underline{x}_{il}^{(k)}(\alpha)t_{l,m+j} \Big], \end{aligned}$$

$$k = 0, 1, \ldots, \qquad 1 \leq i, j \leq m. \tag{13}$$

The stopping criterion with tolerance $\varepsilon > 0$ is

$$\frac{||\overline{X}^{(k+1)} - \overline{X}^{(k)}||}{||\overline{X}^{(k+1)}||} < \varepsilon, \qquad \frac{||\underline{X}^{(k+1)} - \underline{X}^{(k)}||}{||\underline{X}^{(k+1)}||} < \varepsilon, \qquad k = 0, 1, \ldots.$$

However , the rate of convergence of the Gauss-Sidel iteration can, in certain cases, be improved by introducing a parameter ω, known as the relaxation parameter. The following modified Gauss-Sidel iteration is known as the successive over relaxation iteration or, in short, SOR iteration, if $\omega > 1$.

Now we present the SOR iterative method.

Consider system of equations (13), we have

$$\begin{aligned} \underline{x}_{ij}^{(k+1)}(\alpha) = \underline{x}_{ij}^{(k)}(\alpha) - \underline{x}_{ij}^{(k)}(\alpha) + \frac{1}{s_{ii} + t_{jj}} \Big[& \underline{c}_{ij}(\alpha) - \sum_{l=1}^{i-1} s_{il}\underline{x}_{lj}^{(k+1)}(\alpha) - \sum_{l=i+1}^{m} s_{il}\underline{x}_{lj}^{(k)}(\alpha) - \\ \sum_{l=1}^{m} s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha) - \sum_{l=1}^{j-1} \underline{x}_{il}^{(k)}(\alpha)t_{lj} - & \sum_{l=j+1}^{m} \underline{x}_{il}^{(k+1)}(\alpha)t_{lj} - \sum_{l=1}^{m} \overline{x}_{il}^{(k)}(\alpha)t_{l,m+j} \Big]. \end{aligned}$$

then

$$\underline{x}_{ij}^{(k+1)}(\alpha)=\underline{x}_{ij}^{(k)}(\alpha)+\frac{1}{s_{ii}+t_{jj}}\Big[\underline{c}_{ij}(\alpha)-\sum_{l=1}^{i-1}s_{il}\underline{x}_{lj}^{(k+1)}(\alpha)-\sum_{l=i}^{m}s_{il}\underline{x}_{lj}^{(k)}(\alpha)-$$

$$\sum_{l=1}^{m}s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha)-\sum_{l=1}^{j}\underline{x}_{il}^{(k)}(\alpha)t_{lj}-\sum_{l=j+1}^{m}\underline{x}_{il}^{(k+1)}(\alpha)t_{lj}-\sum_{l=1}^{m}\overline{x}_{il}^{(k)}(\alpha)t_{l,m+j}\Big]. \tag{14}$$

With efficiency ω, system of equations (14) can be written as:

$$\underline{x}_{ij}^{(k+1)}(\alpha)=\underline{x}_{ij}^{(k)}(\alpha)+\frac{\omega}{s_{ii}+t_{jj}}\Big[\underline{c}_{ij}(\alpha)-\sum_{l=1}^{i-1}s_{il}\underline{x}_{lj}^{(k+1)}(\alpha)-\sum_{l=i}^{m}s_{il}\underline{x}_{lj}^{(k)}(\alpha)-$$

$$\sum_{l=1}^{m}s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha)-\sum_{l=1}^{j}\underline{x}_{il}^{(k)}(\alpha)t_{lj}-\sum_{l=j+1}^{m}\underline{x}_{il}^{(k+1)}(\alpha)t_{lj}-\sum_{l=1}^{m}\overline{x}_{il}^{(k)}(\alpha)t_{l,m+j}\Big].$$

$$1\le i,j\le m, 0\le\alpha\le 1, k=0,1,\dots \quad . \tag{15}$$

Thus we write as:

$$\underline{x}_{ij}^{(k+1)}(\alpha)=\frac{\omega}{s_{ii}+t_{jj}}\Big[\underline{c}_{ij}(\alpha)-\sum_{l=1}^{i-1}s_{il}\underline{x}_{lj}^{(k+1)}(\alpha)-\sum_{l=i+1}^{m}s_{il}\underline{x}_{lj}^{(k)}(\alpha)-\sum_{l=1}^{m}s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha)$$

$$-\sum_{l=1}^{j-1}\underline{x}_{il}^{(k)}(\alpha)t_{lj}-\sum_{l=j+1}^{m}\underline{x}_{il}^{(k+1)}(\alpha)t_{lj}-\sum_{l=1}^{m}\overline{x}_{il}^{(k)}(\alpha)t_{l,m+j}\Big]+(1-\omega)\underline{x}_{ij}^{(k)}. \tag{16}$$

The elements of $X^{(k+1)}(\alpha)=(\underline{X}^{(k+1)}(\alpha),\overline{X}^{(k+1)}(\alpha))$ are

$$\underline{x}_{ij}^{(k+1)}(\alpha)=\frac{\omega}{s_{ii}+t_{jj}}\Big[\underline{c}_{ij}(\alpha)-\sum_{l=1}^{i-1}s_{il}\underline{x}_{lj}^{(k+1)}(\alpha)-\sum_{l=i+1}^{m}s_{il}\underline{x}_{lj}^{(k)}(\alpha)-\sum_{l=1}^{m}s_{i,m+l}\overline{x}_{lj}^{(k)}(\alpha)$$

$$-\sum_{l=1}^{j-1}\underline{x}_{il}^{(k)}(\alpha)t_{lj}-\sum_{l=j+1}^{m}\underline{x}_{il}^{(k+1)}(\alpha)t_{lj}-\sum_{l=1}^{m}\overline{x}_{il}^{(k)}(\alpha)t_{l,m+j}\Big]+(1-\omega)\underline{x}_{ij}^{(k)},$$

$$\overline{x}_{ij}^{(k+1)}(\alpha)=\frac{\omega}{s_{ii}+t_{jj}}\Big[\overline{c}_{ij}(\alpha)-\sum_{l=1}^{i-1}s_{il}\overline{x}_{lj}^{(k+1)}(\alpha)-\sum_{l=i+1}^{m}s_{il}\overline{x}_{lj}^{(k)}(\alpha)-\sum_{l=1}^{m}s_{i,m+l}\underline{x}_{lj}^{(k)}(\alpha)$$

$$-\sum_{l=1}^{j-1}\overline{x}_{il}^{(k)}(\alpha)t_{lj}-\sum_{l=j+1}^{m}\overline{x}_{il}^{(k+1)}(\alpha)t_{lj}-\sum_{l=1}^{m}\overline{x}_{il}^{(k)}(\alpha)t_{l,m+j}\Big]+(1-\omega)\overline{x}_{ij}^{(k)},$$

$$1\le i,j\le m, 1\le\alpha\le m, k=0,1,\dots. \tag{17}$$

The stopping criterion with tolerance $\varepsilon > 0$ is

$$\frac{||\overline{X}^{(k+1)} - \overline{X}^{(k)}||}{||\overline{X}^{(k+1)}||} < \varepsilon, \qquad \frac{||\underline{X}^{(k+1)} - \underline{X}^{(k)}||}{||\underline{X}^{(k+1)}||} < \varepsilon, \qquad k = 0, 1, \ldots.$$

4. Numerical examples

In this section we illustrate the methods in Section 3 by solving one example. Consider the system of fuzzy Sylvester equations $AX + XB = C$ where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, C = \begin{pmatrix} (0,1,2) & (1,2,3) \\ (1,2,3) & (0,1,2) \end{pmatrix}.$$

We extend the A matrix to the S matrix and the B matrix to the T matrix. As follow:

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

The exact solution is given by

$x_{11} = (0.6935 + 0.2857\alpha, 1.2649 - 0.2857\alpha)$,
$x_{12} = (0.4018 + 0.2857\alpha, 0.9732 - 0.2857\alpha)$,
$x_{21} = (0.1280 + 0.1429\alpha, 0.4137 - 0.1429\alpha)$,
$x_{22} = (-0.0804 + 0.1429\alpha, 0.2054 - 0.1429\alpha)$.

By the Gauss-Sidel method the approximate solution with 4 iterates and $\varepsilon = 10^{-2}$ is

$x_{11} = (0.6946 + 0.2849\alpha, 1.2645 - 0.2849\alpha)$,
$x_{12} = (0.4018 + 0.2857\alpha, 0.9732 - 0.2857\alpha)$,
$x_{21} = (0.1277 + 0.1431\alpha, 0.4138 - 0.1431\alpha)$,
$x_{22} = (-0.0804 + 0.1429\alpha, 0.2054 - 0.1429\alpha)$.

And the SOR method the approximate solution with 8 iterates and $\varepsilon = 10^{-2}$ is

$x_{11} = (0.6863 + 0.2895\alpha, 1.2653 - 0.2895\alpha)$,
$x_{12} = (0.4011 + 0.2855\alpha, 0.9720 - 0.2855\alpha)$,
$x_{21} = (0.1368 + 0.1412\alpha, 0.4193 - 0.1412\alpha)$,
$x_{22} = (-0.0788 + 0.1427\alpha, 0.2067 - 0.1427\alpha)$.

The exact and approximate solutions are brought $\alpha = 0, \alpha = 0.5, \alpha = 1$ with $\varepsilon = 10^{-2}$ in the following tables.

$\alpha = 0.0$	Exact solution	Gauss-Sidel method	SOR method
$(\underline{x}_{11}, \overline{x}_{11})$	$(0.6935, 1.2649)$	$(0.6946, 1.2645)$	$(0.6863, 1.2653)$
$(\underline{x}_{12}, \overline{x}_{12})$	$(0.4018, 0.9732)$	$(0.4018, 0.9732)$	$(0.4011, 0.9720)$
$(\underline{x}_{21}, \overline{x}_{21})$	$(0.1280, 0.4137)$	$(0.1277, 0.4138)$	$(0.1368, 0.4193)$
$(\underline{x}_{22}, \overline{x}_{22})$	$(-0.0804, 0.2054)$	$(-0.0804, 0.2054)$	$(-0.0788, 0.2067)$

$\alpha = 0.5$	Exact solution	Gauss-Sidel method	SOR method
$(\underline{x}_{11}, \overline{x}_{11})$	$(0.8364, 1.1221)$	$(0.8370, 1.1221)$	$(0.8311, 1.1206)$
$(\underline{x}_{12}, \overline{x}_{12})$	$(0.5446, 0.8303)$	$(0.5446, 0.8303)$	$(0.5438, 0.8293)$
$(\underline{x}_{21}, \overline{x}_{21})$	$(0.1995, 0.3423)$	$(0.1993, 0.3423)$	$(0.2074, 0.3487)$
$(\underline{x}_{22}, \overline{x}_{22})$	$(-0.0089, 0.1340)$	$(-0.0089, 0.1340)$	$(-0.0074, 0.1354)$

$\alpha = 1.0$	Exact solution	Gauss-Sidel method	SOR method
$(\underline{x}_{11}, \overline{x}_{11})$	$(0.9792, 0.9792)$	$(0.9795, 0.9796)$	$(0.9758, 0.9758)$
$(\underline{x}_{12}, \overline{x}_{12})$	$(0.6875, 0.6875)$	$(0.6875, 0.6875)$	$(0.6866, 6866)$
$(\underline{x}_{21}, \overline{x}_{21})$	$(0.2709, 0.2709)$	$(0.2708, 2708)$	$(0.2780, 0.2781)$
$(\underline{x}_{22}, \overline{x}_{22})$	$(0.0625, 0.0625)$	$(0.0625, 0.0625)$	$(0.0639, 0.0640)$

5. Conclusion

In this paper, we apply Gauss-Sidel and SOR iterative methods for finding the approximate solution of a system of fuzzy Sylvester equations of the form $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix.

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