ENTROPY OF GEODESIC FLOWS ON SUBSPACES OF HECKE SURFACE WITH ARITHMETIC CODE

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Abstract. There are different ways to code the geodesic flows on surfaces with negative curvature. Such code spaces give a useful tool to verify the dynamical properties of geodesic flows. Here we consider special subspaces of geodesic flows on Hecke surface whose arithmetic codings varies on a set with infinite alphabet. Then we will compare the topological complexity of them by computing their topological entropies.

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1. Introduction

Let $\mathcal{H} = \{z : z = x + iy,\ y > 0\}$ be the upper half plane with hyperbolic metric. By this metric, the geodesics on $\mathcal{H}$ are semicircles and lines perpendicular to the x-axis. Let $T\mathcal{H}$ be the unit tangent bundle of $\mathcal{H}$ and $u \in T\mathcal{H}$ be the unit vector tangent to a geodesic γ. The geodesic flow φ^t is a homeomorphism which moves u along the geodesic at a hyperbolic distance t.

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We denote by $G_\alpha = \langle z+\alpha, \frac{-1}{z}\rangle$, $\alpha = 2\cos\frac{\pi}{q}$ and $q \geq 3$, the *Hecke triangle group*. This group is a discrete subgroup of $PSL(2,\mathbb{R})$. The quotient of $\mathcal{H}$ by G_α is called the *Hecke triangle surface* and we will denote it by $\mathcal{H}_\alpha$. So, $\mathcal{H}$ is the universal covering of $\mathcal{H}_\alpha$. To any geodesic on $\mathcal{H}_\alpha$, infinitely many geodesics on $\mathcal{H}$ are associated.

To investigate the dynamical properties of geodesic flows on $\mathcal{H}_\alpha$, we lift them to $\mathcal{H}$. To each oriented geodesic $\gamma = (w, u) \in \mathcal{H}$, one can correspond a point (w, u) in wu-plane. This means any geodesic in $\mathcal{H}_\alpha$ is related to infinitely many points in the wu-plane. To obtain a one to one relation, we consider the action of G_α on the wu-plane. This gives different types of fundamental domains called $\mathbb{T}_\alpha \subset \{(w, u) : |w| \geq 1, \ |u| \leq 1\}$.

Let $T_\alpha(z) = z+\alpha$ and $S(z) = \frac{-1}{z}$. For any $\mathbb{T}_\alpha$, let $\mathbb{S}_\alpha = S(\mathbb{T}_\alpha)$ and $\mathbb{T}_k = \{(w, u) \in \mathbb{T}_\alpha : \ T_\alpha^{-k}(w, u) \in \mathbb{S}_\alpha\}$. Define

(1)
$$T_R(w, u) = \begin{cases} T_\alpha^{-k}(w, u) = (T_\alpha^{-k}(w), T_\alpha^{-k}(u)) = (w-k\alpha, u-k\alpha), & \text{on } \mathbb{T}_k \\ S(w, u) = (S(w), S(u)) = \left(\frac{-1}{w}, \frac{-1}{u}\right), & \text{on } \mathbb{S}_\alpha. \end{cases}$$

In [2], it is proved that there are 6 types of fundamental domains which are rectangular. These regions are due to the $\alpha = 1, \sqrt{2}, \frac{2\sqrt{3}}{3}$ and 2, but here, we will not consider the case $\alpha = \frac{2\sqrt{3}}{3}$ since it does not produce a Hecke group. Three types of these regions are associated to $\alpha = 1$. So to distinguish them, we use the letters G, A and H which are abbreviations for the names Gauss, Artin and Hurwitz respectively.

To any geodesic $\gamma = (w, u)$ we correspond a bi-infinite sequence of nonzero integers. To do this let $(w, u) \in \mathbb{T}_{n_0} \subset \mathbb{T}_\alpha$. Then by (1), $ST^{-k}(w, u) \in \mathbb{T}_{n_1}$. Continuing this procedure, we obtain a sequence $n_0, n_1, n_2, \cdots$ of nonzero integers. Also, $S(w, u) \in \mathbb{T}_{n_{-1}} \subseteq \mathbb{T}_\alpha$, $ST^{n_1}S(w, u) \in \mathbb{T}_{n_{-2}}$ and so on. For $c \in \{G, A, H, \sqrt{2}, 2\}$ and $\gamma \in \mathbb{T}_c$, the geodesic γ has the code $[\gamma]_c = [\cdots, n_{-2}, n_{-1}, n_0, n_1, \cdots]$ so that

(2)
$$n_0\alpha_c - \cfrac{1}{n_1\alpha_c - \cfrac{1}{\ddots}} \quad \text{and} \quad n_{-1}\alpha_c - \cfrac{1}{n_{-2}\alpha_c - \cfrac{1}{\ddots}}$$

converge to w and $\frac{1}{u}$ respectively.

Now for each $c \in \{G, A, H, \sqrt{2}, 2\}$, we can specify the set of all bi-infinite sequences denoted by Σ_c. Each of these sets can be characterized by a set of alphabets $\mathcal{A}_c$ showing which integers can be seen in the sequences and the set $\mathcal{F}_c$ which determines the forbidden blocks in sequences of Σ_c. For $c \in \{G, A, H, \sqrt{2}, 2\}$, theses sets are specified as follows [1].

- $\mathcal{A}_G = \mathbb{Z} \setminus \{0, \pm 1\}$ and $\mathcal{F}_G = \{[n, m], nm < 0\}$,
- $\mathcal{A}_A = \mathbb{Z} \setminus \{0\}$ and $\mathcal{F}_A = \{[n, m], nm > 0\}$,
- $\mathcal{A}_H = \mathbb{Z} \setminus \{0, \pm 1\}$ and $\mathcal{F}_H = \{[2, m], [-2, n],\ m < 0,\ n > 0\}$,
- $\mathcal{A}_{\sqrt{2}} = \mathbb{Z} \setminus \{0\}$ and $\mathcal{F}_{\sqrt{2}} = \{[1, m], [-1, n],\ m < 0,\ n > 0\}$,
- $\mathcal{A}_E = \mathbb{Z} \setminus \{0\}$ and $\mathcal{F}_E = \emptyset$,

2. Topological entropy

In this section we want to determine the entropy of the geodesic flows corresponding to the sets Σ_c, $\{G, A, H, \sqrt{2}, 2\}$. For all of the cases, the alphabet is an infinite set. There are not much routines to compute the topological entropy of geodesic flows. In [1], authors developed a formula to find the topological entropy of special flows. We denote the special flows by $\varphi^t_{f,\,\Sigma}$ where Σ is the set of bi-infinite sequences and f is the height function defined on Σ. Now let σ be the shift map on Σ. Then for $y \in \Sigma$ and $t \in \mathbb{R}$, the special flow is defined as $\varphi^t_{f,\,\Sigma}(y, s) = (y, s + t)$ through the condition that $(y, f(y)) = (\sigma(y), 0)$. In the following theorem we prove that the topological entropy of the special flow and the geodesic flow are related.

Theorem 2.1. *The topological entropy of geodesic flows with the code set Σ_c equals the topological entropy of special flow φ^t_{f,Σ_c} constructed on Σ_c with height function $f(y) = 2 \ln |w(y)|$. In here, $y \in \Sigma_c$ and $w(y)$ is as in (2).*

For a set Σ, let $\mathcal{A}$ and $\mathcal{F}$ be its alphabet and forbidden sets respectively. Let $V_v^+ = \{v' \in \mathcal{A} :\ vv' \notin \mathcal{F}\}$. Define an equivalent relation on $\mathcal{A}$ as $v \sim_\rho v'$ if and only if $V_v^+ = V_{v'}^+$. Let P be the associated partition. For an arbitrary $z \in \mathcal{A}$, let $P_{\{z\}}$ be the nonempty intersection of elements of P with $\{\{z\}, \mathcal{A} - \{z\}\}$. Denote the elements of $P_{\{z\}}$ by $V_0 = \{z\}$, V_1, $\cdots$ and V_m. For $x \in [0, 1]$, $v \in V_i$ and $v' \in V_j$, set

$$\alpha_i(x) = \alpha_{ij}(x) = \begin{cases} \Sigma_{v \in V_i} x^{f(v)} & \text{if } vv' \notin \mathcal{F} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let $\tau = n_i n_{i+1} \cdots n_{i+k}$ be a finite block of a sequence in Σ. Define $S_k f(\tau) = f(n_i) + f(n_{i+1}) + \cdots + f(n_{i+k})$. Set $C(z)$ be the collection of all blocks $\tau = n_i n_{i+1} \cdots n_{i+k}$

with $n_i = n_{i+k} = z$ and $n_j \neq z$ for $i+1 \leq j \leq i+k-1$. Now for $x \in [0,\, 1)$, let

$$\phi_{\Sigma_c,f,z}(x) = \Sigma_{\tau\in C(z)} x^{S_k f(\tau)}.$$

Denote by $r(F)$ and $r(\phi_{\Sigma,f,z})$ the radius of convergence of $F(x) = \Sigma_{v\in\mathcal{A}}$ and $\phi_{\Sigma,f,z}$ respectively. Using the arguments in [1] and [4], we can prove that

Theorem 2.2. *For each $c \in \{G, A, H, \sqrt{2}, 2\}$, there exists series $A_i^c(x)$ which are the solutions of the equations*

$$A_i^c(x) = \alpha_{i0}^c(x) + \alpha_{i0}^c(x)A_1^c(x) + \cdots + \alpha_{m0}^c(x)A_m^c(x) \tag{4}$$

so that

$$\varphi_{\Sigma_c,f,z}(x) = \alpha_{00}^c(x) + \alpha_{01}^c(x)A_1^c(x) + \alpha_{02}^c(x)A_2^c(x) + ... + \alpha_{0m}^c(x)A_m^c(x).$$

Now we can obtain the topological entropy of special flow using the following Theorems.

Theorem 2.3. [1] *The topological entropy of $\varphi_{\Sigma_c,f}^t$ equals $-\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\varphi_{\Sigma_c,f,z}(x) = 1$ or $\hat{x}_f = r(\phi_{\Sigma_c,f,z})$.*

In the next theorem, using the Theorems 2.1, 2.2 and 2.3, we can find the estimates for the topological entropy of the geodesic flows with code space Σ_c. This gives a good tool to compare the complexity of these subsystems.

Theorem 2.4. *For $c \in \{G, A, H, \sqrt{2}, 2\}$, let Σ_c and $\varphi_{\Sigma,\ell}^t$ be as before. The topological entropy $h_c(\varphi^t)$ of the geodesic flow on $\mathcal{H}_\alpha$ with the code space Σ_c is as follows.*

- *For $c = G$, $0.863992 < h_G(\varphi^t) < 0.864655$*
- *For $c = A$, $0.06661 < h_A(\varphi^t) < 0.06667$*
- *For $c = H$, $0.90835 < h_H(\varphi^t) < 1$*
- *For $c = \sqrt{2}$, $0.58035 < h_{\sqrt{2}}(\varphi^t) < 0.63747$*
- *For $c = 2$, $h_2(\varphi^t) = 1$.*

Sketch of proof. For all cases, the height function $f(y)$ equals $2\ln|w(y)|$ where $y \in \Sigma_c$.

case 1: . Let $c = G$. Then $\mathcal{A}_G = \mathbb{Z}\setminus\{0, \pm1\}$, $\mathcal{F}_G = \{[n,\, m],\, nm < 0\}$, $P_2 = \{V_0 = \{2\},\, V_1 = \{3, 4, \cdots\},\, V_2 = \{-2, -3, \cdots\}\}$. Since the system for positive and negative integers are disjoint, the entropy of the whole system is the maximum entropy of the subsystems. But both subsystems are exactly the same. Therefore we only compute the entropy for the subsystem with positive integers. So, $P_2 = \{V_0, V_1\}$. According to the set $\mathcal{F}_G$, for any $y \in \Sigma_G$ and $\epsilon \in (0, 1)$, we have

$$(1-\epsilon)n_0 < |w(y)| = \left| n_0 - \cfrac{1}{n_1 - \cfrac{1}{\ddots}} \right| < n_0 + 1 < (1+\epsilon)n_0.$$

So, for any $y \in \Sigma_G$, $2\ln(1-\epsilon)n_0 < f(y) < 2\ln(1+\epsilon)n_0$. First we let $f(y) = 2\ln(1-\epsilon)n_0$ and then we let $f(y) = 2\ln(1+\epsilon)n_0$.

Due to the Forbidden blocks set $\mathcal{F}_G$, and (3), we have $\alpha_0(x) = \alpha_{00}(x) = \alpha_{01}(x) = x^{f(2)}$ and $\alpha_1(x) = \alpha_{10}(x) = \alpha_{11}(x) = \sum_{v\in V_1} x^{f(v)}$. Using the Riemann Zeta function $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$, we get

$$\alpha_1(x) = \sum_{n=3}^{\infty} x^{f(n)} = \sum_{n=3}^{\infty} x^{2\ln(1-\epsilon)n} = \sum_{n=3}^{\infty}((1-\epsilon)n)^{2\ln x} = (1-\epsilon)^{2\ln x}\sum_{n=3}^{\infty} n^{2\ln x}.$$

Therefore, $A_1(x) = \frac{\alpha_1(x)}{1-\alpha_1(x)}$ and $\phi_{\Sigma_G,f,z}(x) = x^{f(2)}\left(1 + \frac{\alpha_1(x)}{1-\alpha_1(x)}\right)$ by Theorem 2.2. Using Theorem 2.3, the topological entropy of geodesic flow denoted by $h_G(\varphi^t)$ equals $-\ln(\hat{x}_f)$ where $\hat{x}_f$ the unique solution of $\phi_{\Sigma_G,f,z}(x) = 1$. By a similar procedure for $f(y) = 2\ln(1+\epsilon)n_0$ and for $\epsilon = 0.001$, we get

$$0.863992 < h_G(\varphi^t) < 0.864655.$$

case 2: . Let $c = A$. Then $\mathcal{A}_A = \mathbb{Z} \setminus \{0\}$, $\mathcal{F}_A = \{[n, m], nm > 0\}$, $P_z = \{V_0 = \{1\}, V_1 = \{2, 3, \cdots\}, V_2 = \{-1, -2, -3, \cdots\}\}$. Here, $\alpha_0(x) = \alpha_{02}(x) = x^{f(1)}$, $\alpha_1(x) = \alpha_{12}(x) = \sum_{v\in V_1} x^{f(v)}$, $\alpha_2(x) = \alpha_{20}(x) = \alpha_{21}(x) = \sum_{v\in V_1} x^{f(v)}$ and $\alpha_{00}(x) = \alpha_{01}(x) = \alpha_{10}(x) = \alpha_{11}(x) = \alpha_{22}(x) = 0$. By a calculation similar to Case 1, we have

$$(1-\epsilon)n_0 < |w(y)| < n_0 + 1 < (1+\epsilon)n_0,$$

$A_1(x) = \frac{\alpha_1(x)\alpha_2(x)}{1-\alpha_1(x)\alpha_2(x)}$, $A_2(x) = \frac{\alpha_2(x)}{1-\alpha_1(x)\alpha_2(x)}$ and $\phi_{\Sigma_A,f,z}(x) = x^{f(1)}A_2(x)$. In this case for $\epsilon = 0.001$, we get

$$0.06661 < h_A(\varphi^t) < 0.06667.$$

case 3: . Let $c = H$. Thus $\mathcal{A}_H = \mathbb{Z} \setminus \{0, \pm 1\}$, $\mathcal{F}_H = \{[2, m], [-2, n], n < 0, m > 0\}$ and $P_2 = \{V_0 = \{2\}, V_1 = \{3, 4, \cdots\}, V_2 = \{-2\}, V_3 = \{-3, -4, , \cdots\}\}$.

Here, $\frac{5}{6}n_0 < |w(y)| < 1.25n_0$ by Theorem 2.2. The series $A_i(x)$ and $\phi_{\Sigma_A,f,z}(x)$ satisfy

$$\begin{cases} A_1 = \alpha_1(1 + A_1 + A_2 + A_3) \\ A_2 = \alpha_2(1 + A_1) \\ A_3 = \alpha_3(1 + A_1 + A_2 + A_3) \end{cases}$$

and $\phi_{\Sigma_A,f,z}(x) = x^{f(2)}(A_2 + A_3)$. Now again by Theorem 2.3,

$$0.90835 < h_H(\varphi^t) < 1.$$

case 4: . For $c = \sqrt{2}$, $\mathcal{A}_{\sqrt{2}} = \mathbb{Z} \setminus \{0\}$, $\mathcal{F}_{\sqrt{2}} = \{[1,\, m],\, [-1,\, n],\ m < 0,\, n > 0\}$ and $P_1 = \{V_0 = \{1\},\, V_1 = \{2, 3, \cdots\},\, V_0 = \{-1\},\, V_3 = \{-2, -3, \cdots\}\}$.

According to $\mathcal{F}_{\sqrt{2}}$, $\alpha_0(x) = \alpha_{02}(x) = \alpha_{03}(x) = x^{f(1)}$, $\alpha_1(x) = \alpha_{10}(x) = \alpha_{11}(x) = \alpha_{12}(x) = \alpha_{13}(x) = \sum_{v\in V_1} x^{f(v)}$, $\alpha_2(x) = \alpha_{20}(x) = \alpha_{21}(x) = x^{f(1)}$, $\alpha_3(x) = \alpha_{30}(x) = \alpha_{31}(x) = \alpha_{32}(x) = \alpha_{33}(x) = \sum_{v\in V_1} x^{f(v)}$ and $\alpha_{00}(x) = \alpha_{01}(x) = \alpha_{22}(x) = \alpha_{23}(x) = 0$.

$$2\frac{\sqrt{2}}{4} = \sqrt{2}n_0 - \frac{1}{\sqrt{2}} < |w(y)| = \left| n_0 - \cfrac{1}{n_1 - \cfrac{1}{\ddots}} \right| < n_0\sqrt{2} + \frac{1}{\sqrt{2}} < \frac{5\sqrt{2}}{4}n_0.$$

By Theorem 2.2,

$$\begin{cases} A_1 = \alpha_1(A_1 + A_2 + A_3) \\ A_2 = \alpha_2(1 + A_1) \\ A_3 = \alpha_3(1 + A_1 + A_2 + A_3). \end{cases}$$

Also, $\phi_{\Sigma_{\sqrt{2}},f,z}(x) = x^{f(1)}(A_2 + A_3)$. Hence,

$$0.58035 < h_{\sqrt{2}}(\varphi^t) < 0.63747.$$

case 5: . According to a theorem of S. Katok, the topological entropy of the set of all geodesic flows on the quotient of $\mathcal{H}$ by a geometrically finite Fuchsian group of the first kind is equal to 1 [3].

□

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