PRECOMPACT TOPOLOGICAL GENERALIZED GROUPS

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Abstract. In this paper, we introduce and study the notion of precompact topological generalized groups and some new results are given.

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1. Introduction

Generalized groups or completely simple semigroups [3, 11] are an interesting generalization of groups. We recall that ([11]) a generalized group is a non-empty set G admitting an operation called multiplication, which satisfies the following conditions:

(1) $(xy)z = x(yz)$ for all $x, y, z \in G$.
(2) For each $x \in G$ there exists a unique element $z \in G$ such that $zx = xz = x$ (we denote z by $e(x)$).
(3) For each $x \in G$ there exists an element $y \in G$ such that $xy = yx = e(x)$.

It is well known that each x in G has a unique inverse in G. The inverse of x is denoted by x^{-1} [11]. Moreover, for a given $x \in G$, $e(e(x)) = e(x)$, $(x^{-1})^{-1} = x$ and $e(x^{-1}) = e(x)$.

Definition 1.1. ([9]) If G and H are two generalized groups, then a map $f : G \to H$ is called a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Theorem 1.2. *([9]) Let $f : G \to H$ be a homomorphism where G and H are two generalized groups. Then:*

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(1) $f(e(a)) = e(f(a))$,
(2) $f(a^{-1}) = (f(a))^{-1}$,

for all $a \in G$.

Definition 1.3. ([10, 14]) If G is a generalized group and $e(xy) = e(x)e(y)$ for all $x, y \in G$, then G is called a normal generalized group.

Recall that a nonempty subset H of a generalized group G is called a *generalized subgroup* if it is a generalized group under the operation of G [9]. We also recall that a *paratopological generalized group* is a generalized group G with a Hausdorff topology on G such that the generalized group operation $m : G \times G \to G$ defined by $(x, y) \mapsto x \cdot y$ is a continuous mapping [17]. A paratopological generalized group with continuous inversion is called a *topological generalized group* [12]. Moreover, if $a \in G$ then $G_{e(a)} = \{g \in G \mid e(g) = e(a)\}$ with the operations of G is a topological group, and G is disjoint union of such topological groups ($G = \dot{\bigcup}_{e(a)\in e(G)} G_{e(a)}$) [14].

Theorem 1.4. *([9]) Let H be a non-empty subset of a generalized group G. Then, H is a generalized subgroup of G if and only if $ab \in H$ and $a^{-1} \in H$ for all $a, b \in H$.*

2. Main results

In this section, some properties of paratopological generalized groups are investigated.

As shown in [14], if T is a topological generalized group which is normal, then the mapping $e : T \to T$ defined by $t \to e(t)$ is a continuous map, see also [8]. Recall that a family $\{A_s\}_{s\in S}$ of subsets of a topological space X is called *locally finite* if for every point $x \in X$ there exists a neighbourhood U of X such that the set $\{s \in S : U \cap A_s \neq \emptyset\}$ is finite. We refer to [5, 6, 13, 14, 16] for more theorems and more details about topological generalized groups and topological spaces. A resource about the importance of topological groups is [4].

Theorem 2.1. [2] *Let G be a paratopological generalized group such that the family $\mathcal{F} = \{G_{e(a)}\}_{a\in G}$ is locally finite. Then every $G_{e(a)}$ is closed and open in G. In particular, G can be represented as the sum of the family $\mathcal{F}$ of topological spaces, i.e., $G = \bigoplus_{a\in G} G_{e(a)}$.*

Proposition 2.2. *Let G and H be two topological generalized groups such that the family $\mathcal{F} = \{G_{e(a)}\}_{a\in G}$ is locally finite. If $f : G \to H$ is a homomorphism such that f is continuous at $e(a)$ for all $a \in G$, then f is continuous.*

Proof. Let $a \in G$ be given, then $G_{e(a)}$ is closed and open in G by Theorem 2.1 and $f|G_{e(a)} : G_{e(a)} \to G_{e(f(a))}$ is a group homomorphism by Theorem 1.2. Since $G_{e(a)}$ is a topological group and $f|G_{e(a)}$ is continuous at $e(a)$ by hypothesis, $f|G_{e(a)}$ is a continuous function [4]. Thus, f is continuous since the family $\mathcal{F}$ is a locally finite [5]. □

Proposition 2.3. *Let G be a generalized group and a topological space with continuous inversion such that the family $\mathcal{F} = \{G_{e(a)}\}_{a\in G}$ is locally finite and the function*

$e : G \to G$ is continuous. If the generalized group operation $m : G \times G \to G$ defined by $(x, y) \mapsto xy$ is a continuous mapping, then the following conditions are equivalent.

1. *G is a T_0-space,*
2. *G is a T_1-space,*
3. *G is a T_2-space, i.e., G is a topological generalized group,*
4. *G is a T_3-space,*
5. *G is a completely regular space.*

Proof. (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate. Let us check that (1) implies (5). We note that every $G_{e(a)}$ is closed in G since the function e is continuous. If $a_0 \in G$, then $\bigcup_{a_0 \neq a \in G} G_{e(a)}$ is closed since the family $\mathcal{F}$ is locally finite and so $G_{e(a_0)} = G \setminus \bigcup_{a_0 \neq a \in G} G_{e(a)}$ is open. Thus, G is the direct sum of the family $\mathcal{F}$, i.e., $G = \bigoplus_{a \in G} G_{e(a)}$. The topological group $G_{e(a)}$ is T_0 for every $a \in G$ since G is a T_0-space. Thus, $G_{e(a)}$ is completely regular [7] and so [[5], Theorem 2.2.7] implies that G is completely regular. □

Definition 2.4. ([4]) A left topological group G is called precompact if, for every open neighbourhood V of the neutral element in G, there exists a finite subset A of G such that $AV = G$ (similarly, A right topological group G is called precompact if, for every open neighborhood V of the neutral element in G, there exists a finite subset B of G such that $VB = G$).

Definition 2.5. A topological generalized group G is called precompact if G_a is a precompact topological group for all $a \in G$ and $card\ e(G) < \infty$.

A subset B of a topological generalized group G is called precompact in G if B intersects only finitely many G_a's and $B \cap G_a$ is a precompact subset of G_a for any $a \in e(G)$.

Proposition 2.6. *If f is a continuous homomorphism of a precompact topological generalized group G onto a topological generalized group H, then the generalized group H is also precompact.*

Proof. To prove, we must show that the following two conditions are satisfied for topological generalized group H.

1. $card\ e(H) < \infty$,
2. H_h is a precompact topological group for all $h \in e(H)$.

Since that f is homomorphism, we have $f(G_a) \subseteq H_{f(a)}$ for every $a \in e(G)$. On the other hand, f is onto. Therefore, $H_h = \cup_{x \in f^{-1}(h)} f(G_x)$ for any $h \in e(H)$. Hence, $card\ e(H) \leq card\ e(G) < \infty$. This proves condition (1).
To prove (2), let U be an open neighbourhood of $h \in e(H)$ in H_h and $x \in f^{-1}(h) \in e(G)$. Therefore, $f^{-1}(U)$ is an open neighbourhood of x in G and it follows that, $f^{-1}(U) \cap G_x$ is an open neighbourhood of x in the precompact topological group G_x. So, there exists a finite set $A_x \subseteq G_x$ such that $G_x = (f^{-1}(U) \cap G_x)A_x$. Since

$x \in f^{-1}(h)$ is arbitrary, we have

$$\begin{aligned} H_h = \cup_{x\in f^{-1}(h)} f(G_x) &= \cup_{x\in f^{-1}(h)} f((f^{-1}(U) \cap G_x)A_x) \\ &\subseteq \cup_{x\in f^{-1}(h)} (U \cap f(G_x)) f(A_x) \\ &\subseteq \cup_{x\in f^{-1}(h)} (U \cap H_h) f(A_x) \\ &= \cup_{x\in f^{-1}(h)} U f(A_x) \\ &= U \cup_{x\in f^{-1}(h)} f(A_x) \end{aligned}$$

Since $e(G)$ is finite, then $\cup_{x\in f^{-1}(h)} f(A_x)$ is finite too. Now, we define $A = \cup_{x\in f^{-1}(h)} f(A_x)$ that is a finite set in H_h. therefore H_h is a precompact topological group and H is a precompact topological generalized group. □

Definition 2.7. Let G be a topological generalized group. Then, an open neighbourhood U of G is called an e-neighbourhood of G if $e(G) \subseteq U$.

Proposition 2.8. *Let B be a subset of a topological generalized group G and S be dense in B such that card $e(S) < \infty$. Then $S_a = S \cap G_a$ is dense in $B_a = B \cap G_a$ for every $a \in e(G)$.*

Proof. Since $e(S)$ is a finite set, S intersects a finite numbers of G_a's. So, we can choose $\{G_{a_1}, G_{a_2}, \cdots, G_{a_n}\}$, as a subset of G_a's such that $S \subseteq \cup_{i=1}^n G_{a_i}$. Now it follows that,

$$\begin{aligned} B = \overline{S} &= \overline{S \cap (\cup_{i=1}^n G_{a_i})} = \overline{\cup_{i=1}^n (S \cap G_{a_i})} = \cup_{i=1}^n \overline{(S \cap G_{a_i})} \subseteq \cup_{i=1}^n \overline{G_{a_i}} \\ &= \cup_{i=1}^n G_{a_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} \cup_{i=1}^n B_i &= \cup_{i=1}^n (B \cap G_i) = B \cap (\cup_{i=1}^n G_{a_i}) = B \cap G = B = \overline{S} = \overline{S \cap (\cup_{i=1}^n G_{a_i})} \\ &= \overline{\cup_{i=1}^n S_i} = \cup_{i=1}^n \overline{S_i}. \end{aligned}$$

Moreover, $\overline{S_i} \subseteq \overline{B_i}$ and B_i's are closed and disjoint sets in B for every $i \in \{1, 2, \cdots, n\}$. Hence $\overline{S_i} = B_i$ and we can conclude that S_i is dense in B_i. Now it is clear that S_a is dense in B_a for every $a \in e(G)$. □

Proposition 2.9. *Let B be a precompact subset of a topological generalized group G and S be dense in B. Then, for any e-neighbourhood U of G, there exists a finite set $K \subseteq S$ such that $B \subseteq KU$ and $B \subseteq UK$.*

Proof. For every $a \in e(G)$, suppose that $B_a = B \cap G_a$. According to the definition of a precompact subset of G, B_a is a precompact subset of the topological group G_a and B intersects only finitely many G_a's. On the other hand if $S_a = S \cap G_a$, then S_a is dense in B_a by Proposition 2.8. Now, let U be an e-neighbourhood of G and $U_a = U \cap G_a$, then U_a is a non-empty open neighbourhood of a in G_a. Therefore we can apply Lemma 3.7.3 in ([4]) to find a finite set $F_a \subseteq S_a$ such that $B_a \subseteq F_a U_a \subset F_a U$ and $B_a \subseteq U_a F_a \subseteq U F_a$. Since the number of G_a's that intersects B is finite we have $B \subseteq \cup_{a\in e(G)} F_a U$ and $B \subseteq \cup_{a\in e(G)} U F_a = U \cup_{a\in e(G)} F_a$ and so $\cup F_a \subseteq S$ is finite. Let $K = \cup_{a\in e(G)} F_a$, then the finite set K is as required. □

Proposition 2.10. *Every generalized subgroup H of a precompact topological generalized group G is a precompact topological generalized group.*
Proof. *Since card $e(H) \leq$ card $e(G) < \infty$, it suffices to check that H_h is a precompact topological group for every $h \in e(H)$. We know that H_h is a subgroup of the precompact topological group G_h, then we can apply Proposition 3.7.4 in* [4] *to say that H_h is a precompact topological group. Therefore, H is a precompact topological generalized group.*

Proposition 2.11. *If a set B in a topological generalized group G contains a dense precompact subset, then B is also precompact in G. Hence, the closure of a precompact subset of G is precompact in G.*
Proof. *Let S be a dense precompact subset in B. For each $a \in e(G)$, define B_a and S_a by $B \cap G_a$ and $S \cap G_a$, repectively. S intersects only finitely many G_a's since S is a precompact subset in B. So S_a is dense in B_a by Proposition 2.8. Now, apply Proposition 3.7.5 in (*[4]*) to deduce that B_a is precompact in G_a. Moreover, similar to the proof of Proposition 2.8 we have B intersects only finitely many G_a's. Therefore, we can say that B is precompact in G.*

Corollary 2.12. *If a topological generalized group G contains a dense precompact generalized subgroup, then G is also precompact.*

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