SOME ERGODIC PROPERTIES OF HYPER $MV$–ALGEBRA DYNAMICAL SYSTEMS

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(Received: 17 January 2017, Accepted: 4 February 2017)

Abstract. This paper provides a review on major ergodic features of semi-independent hyper $MV$–algebra dynamical systems. Theorems are presented to make contribution to calculate the entropy. Particularly, it is proved that the total entropy of those semi-independent hyper $MV$–algebra dynamical systems that have a generator can be calculated with respect to their generator rather than considering all the partitions.

AMS Classification: 37A35, 37-XX, 03G99, 03G20, 03-XX.
Keywords: Hyper $MV$–algebra, Dynamical system, Uncertainty

1. Introduction

The concept of an $MV$–algebra was introduced by C.C. Chang in 1958 to prove the completeness theorem of infinite valued Lukasiewicz propositional calculus. Hyper structure theory was initiated by F. Marty at 8th congress of Scandinavian
Mathematicians in 1934. Since then, many researchers have worked in these areas, for example see [2, 4, 7]. For the first time, the notion of hyper $MV$–algebras was introduced in [5] as a generalization of $MV$–algebras. In [9], Rasouli and Davvaz studied several properties of hyper $MV$–algebras. Then in [10], they studied homomorphisms, dual homomorphisms and strong homomorphisms between hyper $MV$–algebras. Recently, L. Torkzadeh and Sh. Ghorbani found the conditions under which a hyper $MV$–algebra becomes an $MV$–algebra, and they characterized hyper $MV$–algebras of order 2 and order 3 [6]. P. Corsini and V. Leoreanu, in their book “Applications of hyperstructure theory” discussed applications of hyperstructures in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs [2], and B. Davvaz and V. Leoreanu-Fotea presented applications of hyperstructures in chemistry and physics [4], also see [3]. Thus, by extending the entropy function to hyperstructures, one can provide efficient criteria to measure the complexity of the systems in the categories mentioned above.

In this paper, essential ergodic characteristics of $HMV$–algebra dynamical systems are studied. In the next section that is the main section of this paper, the fundamental properties are studied, and the concept of generator for semi-independent hyper $MV$–algebra dynamical systems is defined. Then theorems that help calculate the entropy are given. The rest of this section is dedicated to a brief review of hyper $MV$–algebras and semi-independent systems over them.

Now, we recall the definition of hyper $MV$–algebra from [5, 8, 9].

**Definition. 1.1 ([7]).** A hyper $MV$–algebra is a non-empty set, '$M$', endowed with a hyperoperation '$\oplus : H \times H \rightarrow P^*(H)$', a unary operation '$\ast : H \rightarrow H$', and a constant '0' satisfying the following axioms for all $x, y, z \in M$:

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z;
2. x \oplus y = y \oplus x;
3. (x^\ast)^\ast = x;
4. (x^\ast \oplus y)^\ast \oplus y = (y^\ast \oplus x)^\ast \oplus x;
5. 0^\ast \in x \oplus 0^\ast;
6. 0^\ast \in x \oplus x^\ast;
7. if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined by $0^\ast \in x^\ast \oplus y$. 

Definition. 1.2 ([7]). For nonempty subsets $A$ and $B$ of $M$, we have the following definitions:

d1) $A \ll B$ if there exist $a \in A$ and $b \in B$ such that $a \ll b$;

d2) $A^* := \{a^* | a \in A\}$;

d3) $1 := 0^*$.

Example. 1.3 ([9]). There are two methods to obtain a hyper $MV$–algebra starting from an $MV$–algebra $(M, \oplus, *, 0, 1)$:

Method 1. Let $(M, \oplus, *, 0, 1)$ be an $MV$–algebra. Define $x \oplus' y := \{x \oplus y\}$ for each $x, y \in M$. One can see easily that $(M, \oplus', *, 0, 1)$ is a hyper $MV$–algebra.

Method 2. Define $x \oplus'' y := \{t \in M | 0 \ll t \ll x \oplus y\} = [0, x \oplus y]$ for each $x, y \in M$. Now, it will be shown that $(M, \oplus'', *, 0, 1)$ is a hyper $MV$–algebra. Clearly, $\oplus''$ and $*$ are well defined. For each $x, y, z \in M$, $1 \in x \oplus'' 1 = M$, $x \in x \oplus 0 = [0, x]$ and $x \oplus'' (y \oplus'' z) = \{t \in M | 0 \ll t \ll x \oplus (y \oplus z)\} = \{t \in M | 0 \ll t \ll (x \oplus y) \oplus z\} = (x \oplus'' y) \oplus'' z$.

Also,

$$
(x^* \oplus''' y)^* \oplus''' y = \bigcup_{s \in M, 0 \ll s \ll x^* \oplus y} s^* \oplus''' y = \bigcup_{s \in M, 0 \ll s \ll x^* \oplus y} \{t \in M | 0 \ll t \ll s^* \oplus y\} = \bigcup\{t \in M | 0 \ll t \ll (x^* \oplus y)^* \oplus y\} = \bigcup\{t \in M | 0 \ll t \ll (y^* \oplus x)^* \oplus x\} = \bigcup_{s \in M, 0 \ll s \ll y^* \oplus x} \{t \in M | 0 \ll t \ll s^* \oplus x\} = (y^* \oplus''' x)^* \oplus''' x.
$$

Now, let $x \ll'' y$, thus $1 \in [0, x^* \oplus y]$ and it implies that $1 = x^* \oplus y$, then by $MV$–algebra’s properties, it is obtained that $x \ll y$. Similarly, if $y \ll'' x$, then $y \ll x$. Now, $x \ll'' y$ and $y \ll'' x$ imply that $x \ll y$ and $y \ll x$, respectively which result that $x = y$. Also, it can be shown that $x \circ'' y = [x \circ y, 1]$. This shows that $(M, \circ'', *, 0, 1)$ is a hyper $MV$–algebra.

Example. 1.4 ([9]). Let $M = \{0, a, b, c, 1\}$. Consider Tables 1(a) and 1(b). Then, $(M, \circ, *, 0, 1)$ is a hyper $MV$–algebra.

Proposition. 1.5 ([7]). Every hyper $MV$–algebra satisfies the following statements for every $x, y, z \in M$ and for every subsets $A, B$ and $C$ of $M$:
Table 1. Cayley tables for the hyperoperation and the unary operation

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊕</td>
<td>{0}</td>
<td>{0, a}</td>
<td>{0, a, b}</td>
<td>{0, c}</td>
<td>M</td>
</tr>
<tr>
<td>a</td>
<td>{0, a}</td>
<td>{0, a}</td>
<td>M</td>
<td>{0, a, c}</td>
<td>M</td>
</tr>
<tr>
<td>b</td>
<td>{0, a, b}</td>
<td>M</td>
<td>{0, a, b, 1}</td>
<td>{0, a, b, c}</td>
<td>M</td>
</tr>
<tr>
<td>c</td>
<td>{0, c}</td>
<td>{0, a, c}</td>
<td>{0, a, b, c}</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>1</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
</tbody>
</table>

Definition. 1.6 ([8]). A partition of unity $U$ of 1 - partition for short - in $M$ is a $k$-tuple $(u_1, \ldots, u_k)$ of elements of $M$ such that $1 \in u_1 \oplus \cdots \oplus u_k$. Moreover, the index set of $U$ is denoted by $I_U$, i.e. $I_U = \{1, \ldots, k\}$. Also, $S(U) := \{u_1, \ldots, u_k\}$ and $P_M := \{U \mid U$ is a partition of unity of 1$\}$.

Definition. 1.7 ([8]). Let $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_n)$ be two partitions of unity. A common refinement - $c$-refinement for short - of $U$ and $V$ is defined as any matrix $C = \{c_{ij} \mid i \in I_U \text{ and } j \in I_V\}$ such that $u_i \in c_{i1} \oplus \cdots \oplus c_{in}$ for every $i \in I_U$ and $v_j \in c_{1j} \oplus \cdots \oplus c_{kj}$ for every $j \in I_V$.
Deﬁnition. 1.8 ([8]). Let $U, V \in P_M$. The partitions $U$ and $V$ are said to be relatively normal if there exists a $c$-reﬁnement for $U$ and $V$. The notation $\{U, V\} \in RN$ is used to show that $U$ and $V$ are relatively normal. Moreover, let $X$ and $Y$ are the sets that are composed of partitions of unity. $X$ and $Y$ are said to be relatively normal if for every $U \in X$ and every $V \in Y$, $\{U, V\} \in RN$. The notation $\{X, Y\} \in RN$ is used to show that $X$ and $Y$ are relatively normal.

Deﬁnition. 1.9 ([8]). Let $U_i \in P_M$ for $i \in \{1, \ldots, s\}$, and $s \geq 3$. The partitions $U_1, \ldots, U_s$ are said to be relatively normal if every way of computing $c$-reﬁnements of $U_1, \ldots, U_s$ leads to ﬁnd at least one $c$–reﬁnement. The notation $\{U_1, \ldots, U_s\} \in RN$ is used to show that $U_1, \ldots, U_s$ are relatively normal. Furthermore, the notation $\{U_1, \ldots, U_s\} \notin RN$ implies that the partitions $U_1, \ldots, U_s$ are not relatively normal. Moreover, the notation $C \in \lor_{i=1}^s U_i$ is applied to show that $C$ is a $c$–reﬁnement of $U_1, \ldots, U_s$.

Note that $\lor_{i=1}^s U_i = U_1$. In this case, by $C \in \lor_{i=1}^s U_i$ it is understood that $C = U_1$.

Deﬁnition. 1.10 ([8]). Let $M$ be a hyper $MV$–algebra, and $m : M \to [0, 1]$ be a mapping. Then, $m$ is called semi-independent if for every $U, V \in P_M$ and every $C \in U \lor V$,

$$\max_{i \in I_U} m(u_i) \cdot \max_{j \in I_V} m(v_j) \leq \max_{(i, j) \in I_C} m(c_{ij});$$

where $\{U, V\} \in RN$, and $I_C = \{(i, j) \mid i \in I_U \text{ and } j \in I_V\}$.

Deﬁnition. 1.11 ([8]). A semi-independent dynamical system on a hyper $MV$–algebra is a couple of mappings $\langle m : M \to [0, 1] \rangle$ and $\langle T : M \to M \rangle$ satisfying the following conditions:

f1) $m(t) = m(a) + m(b), \quad \forall t \in a \oplus b \smallsetminus \{a, b\}$;

f2) $T(a \oplus b) = T(a) \oplus T(b)$;

f3) $m(1) = 1$ and $T(1) = 1$;

f4) $m(T(a)) = m(a)$;

f5) $m$ is semi-independent;

for every $a, b \in M$.

Remark. 1.12 ([8]). If $a \ll b$, then $m(a) = m(b)$, where $a \neq 0, a \neq 1$ and $b \neq 1$. 
Remark. 1.13 ([8]). For every partition of unity $U = (u_1, \ldots, u_k)$, $\max_{i \in I_U} m(u_i) \neq 0$.

Definition. 1.14 ([8]). Let $U = (u_1, \ldots, u_k)$ be a partition of unity. Its entropy is defined by the formula $H(U) = -\log \max_{i \in I_U} m(u_i)$.

Definition. 1.15 ([8]). Let $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_n)$ be two partitions of unity, $\{U, V\} \in \mathcal{R}$, and $C \in U \cup V$. The conditional entropy of $U$ given $V$ with respect to $C$ is defined as

$$H_C(U|V) = \log \frac{\max_{(i,j) \in I_C} m(c_{i,j})}{\max_{j \in I_V} m(v_j)}.$$ 

Definition. 1.16 ([8]). Let $(M, m, T)$ be a semi-independent hyper $MV$-algebra dynamical system, and $U \in P_M$. Then, $U$ is said to be a perfect partition for $T$ if for every positive integer $n \geq 2$, $\{U, T(U), \ldots, T^{n-1}(U)\} \in \mathcal{R}$. The collection of all perfect partitions of $T$ is denoted by $P_T$.

Definition. 1.17 ([8]). For any partition $U \in P_T$ and any positive integer $n$, we define

$$H_n(T, U) = \inf \{H(C) \mid C \in U_n(T)\};$$

where $U_n(T) = \{C \mid C \in \mathcal{V}_{i=0}^{n-1} T^i(U)\}$. If there is no place for ambiguity, the notation $U_n$ is used rather than $U_n(T)$. If $U \in P_M \setminus P_T$, then we set $H_n(T, U) = 0$ for every positive integer $n \geq N$, where $N$ is the smallest positive integer for which $\{U, T(U), \ldots, T^N(U)\} \notin \mathcal{R}$.

Theorem. 1.18 ([8]). $\lim_{n \to \infty} \frac{1}{n} H_n(T, U)$ exists.

Definition. 1.19 ([8]). Entropy of a semi-independent hyper $MV$-algebra dynamical system $(M, m, T)$ is defined by the formula

$$h(T) = \sup \{h(T, U) \mid U \in P_M\}, \quad \text{where} \quad h(T, U) = \lim_{n \to \infty} \left(\frac{1}{n}\right) H_n(T, U).$$

2. Some ergodic properties

During this section, some characteristics semi-independent hyper $MV$-algebras dynamical systems and their entropy are studied. A couple of notions and theorems so as to help calculate the entropy are given.
**Definition. 2.1.** Let $U$ and $V$ be two relatively normal partitions, and $C$ be any $c$–refinement of $U$ and $V$. Then, $U$ is said to be $C$–dominated if for every $u_i$ there exists $c_{pj}$ such that $u_i \ll c_{pj}$, where $u_i \in S(U)$, and $c_{pj} \in S(C)$.

**Definition. 2.2.** Let $C = \{c_{ij} \mid i \in I_U \text{ and } j \in I_V\}$ be any $c$–refinement of the relatively normal partitions $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_n)$, and

$$S_i(C) := \{c_{i1}, \ldots, c_{im}\} \text{ for } i \in I_U \text{ and } S^j(C) := \{c_{ij}, \ldots, c_{kj}\} \text{ for } j \in I_V.$$

$U$ is said to be $C_i$–dominating if $u_i \notin S_i(C)$. By the statement ‘$U$ is $C$–dominating’, it is understood that $U$ is $C_i$–dominating for every $i \in I_U$. $C^j$–dominating and $V$ being $C$–dominating are defined similarly.

For any partition of unity $U = (u_1, \ldots, u_k)$ of a semi-independent hyper $MV$–algebra dynamical system $(M, m, T)$, by $u_m$ we mean the element of $U$ for which $m(u_m) = \max m(u_i)$. If $C$ is any $c$–refinement of the relatively normal partitions $U$ and $V$, by the statement ‘$U$ is $C_i$–dominating’, we mean that $U$ is $C_i$–dominating and $c_m \in S_i(C)$.

**Theorem. 2.3.** Let $(M, m, T)$ be a semi-independent hyper $MV$–algebra dynamical system. If $U = (u_1, \ldots, u_k)$, $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_q)$ are partitions of unity, then:

1. $H(C) \leq H(U) + H_C(V|U)$. Moreover, if $U$ is $C$–dominated, or $u_m \in S(C)$, then $H(C) = H(U) - H_C(V|U)$. If $U$ is $C$–dominating, or $U$ is $C_i$–dominating, then

$H(C) = H(U) + H_C(V|U)$;

2. if $S(U) \subseteq S(V)$, then $H(U) = H(V)$;

3. if $S(U) \subseteq S(V)$, then $H_C(U|W) \leq H_C(V|W)$, where $U$ is $C'$–dominating, $V$ is $C''$–dominated, and $W$ is $C'$–dominated;

4. if $S(U) \subseteq S(V)$, then $H_C(U|W) \geq H_C(V|W)$, where $U$ is $C'$–dominating, $V$ is $C''$–dominated, and $W$ is $C'$–dominated;

5. if $S(U) \subseteq S(V)$, then $H_C(W|U) \geq H_C(V|W)$, where $V$ is $C''$–dominated, $W$ is $C''$–dominating, and $W$ is $C'$–dominated;

6. if $S(U) \subseteq S(V)$, then $H_C(W|U) \leq H_C(V|W)$, where $U$ is $C'$–dominating, $W$ is $C''$–dominating, and $W$ is $C'$–dominated;

7. $H_C(U|V) \leq H(U)$, where $V$ is $C$–dominating;
\[ H(T(C)(T(U)|T(V)) = H_C(U|V); \]
\[ H_W(C|W) \leq H_C(U|W) + H_C(V|W), \text{ where } \{U, V\} \in RN, \ C \subseteq U \lor V, \]
\[ \{C, W\} \in RN, \ W' \subseteq C \lor W, \ C \text{ is } W'\text{-dominated}, \ W \text{ is } W'\text{-dominating}, \ U \]
\[ \text{is } C\text{-dominated, and } U \text{ is } C'\text{-dominating}; \]
\[ H_W(C|W) \leq H_C(U|W) + H_C(V|W), \text{ where } \{U, V\} \in RN, \ C \subseteq U \lor V, \]
\[ \{C, W\} \in RN, \ W' \subseteq C \lor W, \ C \text{ is } W'\text{-dominating}, \ W \text{ is } W'\text{-dominated}, \ U \]
\[ \text{is } C\text{-dominating, and } U \text{ is } C'\text{-dominated}; \]
\[ H_W(C|W) \leq H_C(U|W) + H_D(V|D), \text{ where } \{U, V\} \in RN, \ C \subseteq U \lor V, \]
\[ \{C, W\} \in RN, \ W' \subseteq C \lor W, \ D \subseteq U \lor W, \\{V, D\} \in RN, \ D' \subseteq V \lor D, \ C \]
\[ \text{is } W'\text{-dominated}, \ W \text{ is } W'\text{-dominating}, \ U \text{ is } C\text{-dominated, and } U \text{ is } C'\text{-dominating}; \]
\[ H_W(C|W) \leq H_C(U|W) + H_D(V|D), \text{ where } \{U, V\} \in RN, \ C \subseteq U \lor V, \]
\[ \{C, W\} \in RN, \ W' \subseteq C \lor W, \ D \subseteq U \lor W, \\{V, D\} \in RN, \ D' \subseteq V \lor D, \ C \]
\[ \text{is } W'\text{-dominating}, \ W \text{ is } W'\text{-dominated}, \ U \text{ is } C\text{-dominating, and } U \text{ is } C'\text{-dominated}. \]

**Proof.** g1) We have

\[ H(C) = -\log m(c_m) = -\log \frac{m(c_m)}{m(u_m)} \]
\[ = -\log \frac{m(c_m)}{m(u_m)} + (-\log m(u_m)) \]
\[ \leq |\log \frac{m(c_m)}{m(u_m)}| + H(U) \]
\[ = H_C(V|U) + H(U). \]

Notice that if \( U \) is \( C\)-dominated, or \( u_m \in S(C) \), then \( m(u_m) \leq m(c_m) \). Thus,

\[ H_C(V|U) = |\log \frac{m(c_m)}{m(u_m)}| = \log \frac{m(c_m)}{m(u_m)}. \]
Now, using Equation (1), the obtained result is
\[ H(C) = - \log m(c_m) = - \log(m(u_m) \frac{m(c_m)}{m(u_m)}) \]
\[ = \left( - \log m(u_m) \right) + \left( - \log \frac{m(c_m)}{m(u_m)} \right) \]
\[ = H(U) - | \log \frac{m(c_m)}{m(u_m)} | = H(U) - H_C(V|U). \]

The other part is proved similarly.

g2) It is straightforward from the definitions.

g3) Considering \( S(U) \subseteq S(V) \), the obtained result is
\[ (2) \quad m(u_m) = m(v_m). \]

Since \( U \) is \( C' \)-dominating, it follows that
\[ (3) \quad m(c'_m) \leq m(u_m). \]

Also,
\[ (4) \quad m(v_m) \leq m(c'_m); \]

since \( V \) is \( C'' \)-dominated. Now, considering Equations (2), (3) and (4), the obtained result is
\[ (5) \quad m(c'_m) \leq m(c''_m). \]

By using Equation (5), we have
\[ (6) \quad \log \frac{m(c'_m)}{m(w_m)} \leq \log \frac{m(c''_m)}{m(w_m)} \leq | \log \frac{m(c''_m)}{m(w_m)} | = H_{C''}(V|W). \]

Since \( W \) is \( C' \)-dominated, it follows that
\[ (7) \quad \log \frac{m(c'_m)}{m(w_m)} \geq 0. \]

Considering Equations (6) and (7), then we obtain \( H_{C'}(U|W) \leq H_{C''}(V|W). \)

g4) Similarly as the proof of (g3).

g5) Considering \( W \) is \( C'' \)-dominating, the obtained result is \( m(c''_m) \leq m(w_m). \)

Since \( W \) is \( C' \)-dominated, then \( m(w_m) \leq m(c'_m) \). Thus,
\[ (8) \quad m(c'_m) \leq m(c''_m). \]
Since $S(U) \subseteq S(V)$, it follows that
\[
\frac{1}{m(v_m)} = \frac{1}{m(u_m)}.
\]

By Equations (8) and (9), the obtained result is
\[
\log \frac{m(c'_m)}{m(v_m)} \leq \log \frac{m(c'_m)}{m(u_m)} \leq \log \frac{m(c'_m)}{m(u_m)} = HC(W|U).
\]

Since $V$ is $C''$-dominated, it follows that
\[
\log \frac{m(c'_m)}{m(v_m)} \geq 0.
\]

Now, Equations (10) and (11) imply that $HC(W|U) \geq HC'(W|V)$.

**g6)** Similarly as the proof of (g5).

**g7)** We have $m(u_m) \leq \frac{m(c_m)}{m(v_m)}$. Thus,
\[
H(U) = -\log m(u_m) \geq -\log \frac{m(c_m)}{m(v_m)}
\]

Since $V$ is $C$-dominating, it follows that
\[
HC(U|V) = -\log \frac{m(c_m)}{m(v_m)}.
\]

Considering Equations (12) and (13), the obtained result is $HC(U|V) \leq H(U)$.

**g8)** It is easy to check that $T(C) \in T(U) \lor T(V)$. Then,
\[
HC(T(U)|T(V)) = |\log \frac{m(T(c_m))}{m(T(v_m))}| = |\log \frac{m(c_m)}{m(v_m)}| = HC(U|V).
\]

**g9)** Considering $U$ is $C'$-dominating, the obtained result is
\[
m(c'_m) \leq m(u_m).
\]

Since $U$ is $C$-dominated, it follows that
\[
m(u_m) \leq m(c_m).
\]

In addition, considering $C$ is $W'$-dominated, it is obtained that
\[
m(c_m) \leq m(w'_m).
\]

Equations (14), (15) and (16) occur that $m(c'_m) \leq m(w'_m)$. Then,
\[
-\log \frac{m(w'_m)}{m(w_m)} \leq -\log \frac{m(c'_m)}{m(w_m)}.
\]
Since $W$ is $W'$-dominating, it follows that

\begin{equation}
H_{W'}(C|W) = - \log \frac{m(w'_m)}{m(w_m)}.
\end{equation}

Now, using Equations (17) and (18), the obtained result is

\begin{align*}
H_{W'}(C|W) &\leq - \log \frac{m(c'_m)}{m(w_m)} \\
&\leq \left| \log \frac{m(c'_m)}{m(w_m)} \right| + H_{C'}(V|W) \\
&= H_{C'}(U|W) + H_{C'}(V|W).
\end{align*}

**g10) Suppose $U$ is $C$-dominating.** This implies that

\begin{equation}
m(c_m) \leq m(u_m).
\end{equation}

Since $U$ is $C'$-dominated, it follows that

\begin{equation}
m(u_m) \leq m(c'_m).
\end{equation}

In addition, suppose that $C$ is $W'$-dominating. It is obtained that

\begin{equation}
m(w'_m) \leq m(c'_m).
\end{equation}

Equations (19), (20) and (21) imply that $m(w'_m) \leq m(c'_m)$. Then,

\begin{equation}
\log \frac{m(w'_m)}{m(w_m)} \leq \log \frac{m(c'_m)}{m(w_m)}.
\end{equation}

Since $W$ is $W'$-dominated, it follows that

\begin{equation}
H_{W'}(C|W) = \log \frac{m(w'_m)}{m(w_m)}.
\end{equation}

Now, using Equations (22) and (23), it is concluded that

\begin{align*}
H_{W'}(C|W) &\leq \log \frac{m(c'_m)}{m(w_m)} \\
&\leq \left| \log \frac{m(c'_m)}{m(w_m)} \right| + H_{C'}(V|W) \\
&= H_{C'}(U|W) + H_{C'}(V|W).
\end{align*}

**g11) The proof is similar to the proof of (g9).**

**g12) Similarly as the proof of (g10).**
Definition. 2.4. Let \((M, m, T)\) be a semi-independent hyper MV–algebra dynamical system, and \(U, V \in P_M\). Then \(U\) is said to be a refinement of \(V\) modulo \(m\) if \(m(u_m) \geq m(v_m)\). The notation \(V \preceq u \leq U\) is used to show that \(U\) is a refinement of \(V\) modulo \(m\). \(U\) and \(V\) are said to be \(m\)–equivalent if \(V \preceq u \leq U\) and \(U \preceq v \leq V\). The notation \(U \preceq V\) is used to show that \(U\) and \(V\) are \(m\)–equivalent.

Moreover, if \(U'\) and \(V'\) are the sets that are composed of partitions of unity of \((M, m, T)\), by saying \(U'\) is a refinement of \(V'\) modulo \(m\), \(V' \preceq u \leq U'\), it is understood that for every \(V \in V'\), \(V \preceq \leq U\) for some \(U \in U'\). The notation \(U' \preceq V'\) is used if \(V' \preceq \leq U'\) and \(U' \preceq \leq V'\).

Remark. 2.5. Let \((M, m, T)\) be a semi-independent hyper MV–algebra dynamical system, \(U, V \in P_T\), \(\{U, V\} \in RN\), and \(C \in U \vee V\). It is straightforward to check that

1) \(C \preceq U\) if and only if \(H(C) = H(U) + H_C(V|U)\);
2) \(U \preceq C\) if and only if \(H(C) = H(U) - H_C(V|U)\);
3) \(V \preceq U\) if and only if \(H(V) \geq H(U)\). In particular, \(U \preceq V\) if and only if \(H(U) = H(V)\);
4) \(H_C(U|W) = 0\) if and only if \(W \preceq C\).

Moreover, we have

5) if \(U \preceq V\), then \(U \preceq V\).

Definition. 2.6. Let \(U, V \in P_T\). We say that \(U\) is a generator of \(V\) of order \(K\) if the following conditions are satisfied:

\(
\begin{align*}
\text{m}1) \ & \text{there exists a positive integer } N \text{ such that for every } n \geq N, U_n \preceq V_n; \\
\text{m}2) \ & K = \min\{N \mid N \text{ satisfies Condition (m1)}\}.
\end{align*}
\)

In this case, the notation \(V \preceq_G K \ U\) is applied. If it is not important to emphasize on \(K\), it is just written as \(V \preceq_G U\). In addition, a perfect partition of unity, \(U\), of \(T\) is called a generator of the semi-independent hyper MV–algebra dynamical system \((M, m, T)\) if for every \(V \in P_T\), \(V \preceq_G U\). The notation \(U \leq_G T\) is used to show that \(U\) is a generator of \(T\).

Moreover, if \(U'\) and \(V'\) are the sets that are composed of partitions of unity of \((M, m, T)\), by saying \(U'\) is a generator of \(V'\), \(V' \preceq_G U'\), it is understood that for every \(V \in V'\), \(V \preceq_G U\) for some \(U \in U'\).
Theorem. 2.7. Let \((M, m, T)\) be a hyper \(MV\)-algebra dynamical system, and \(P_T \neq \emptyset\). If \(G = P_T \setminus U_0\), then \(h(T) = \sup\{h(T, W) \mid W \in G\}\), where \(U_0 = \{U \in P_T \mid U \ll_G V \text{ for some } V \in P_T\}\).

Proof. Suppose that \(U \ll_{G_K} V\) for some \(V \in P_T\). Let \(n \geq K\); then, for every \(C \in V_n\), \(C \ll D\) for some \(D \in U_n\). Thus, \(H_n(T, U) \leq H_n(T, V)\) for \(n \geq K\). It occurs that \(h(T, U) \leq h(T, V)\). \(\square\)

Theorem. 2.8. Let \((M, m, T)\) be a semi-independent hyper \(MV\)-algebra dynamical system, and \(k\) be any positive integer. If \(P_T \neq \emptyset\), and for every \(U \in P_T\), the following statements are satisfied:

1) there exists \(N > 0\) such that for every \(n \geq N\), and every \(C \in U_k(T)\), \(C \in P_{T^n}\) and \(C_n(T^k)^m \sim U_{nk}(T)\);

2) \(U \ll_G W\) for some \(W \in G_k\), where \(G_k = \bigcup_{V \in P_T} V_k(T)\);

then, \(h(T^k) = kh(T)\). If \(P_T = \emptyset\), then \(h(T^k) \geq kh(T)\).

Proof. Evidently, we have

\[
(24) \quad kh(T, U) = h(T^k, C).
\]

Now, considering (n2), there exists \(V \in P_T\) such that \(H_n(T^k, U) \leq H_n(T^k, W)\) for some \(W \in V_k(T)\); thus, \(h(T^k, U) \leq h(T^k, W)\). It follows that

\[
(25) \quad \sup_{W \in G_k} h(T^k, W) = \sup_{U \in P_T} h(T^k, U).
\]

Considering, Equations (24) and (25), it is obtained that

\[
kh(T) = k \sup_{U \in P_T} h(T, U) = \sup_{C \in G_k} h(T^k, C) = \sup_{W \in G_k} h(T^k, W) = \sup_{U \in P_T} h(T^k, U) = h(T^k).
\]

If \(P_T = \emptyset\), then \(h(T) = 0\). Therefore, \(h(T^k) \geq 0 = kh(T)\). \(\square\)

Theorem. 2.9. Let \((M, m, T)\) be a semi-independent hyper \(MV\)-algebra dynamical system, and \(U \in P_T\) for which there exists \(N > 0\) such that for every \(n \geq N\), \(\{U\} \leq U_n\). Then \(h(T, U) = 0\).
Proof. Let \( n \geq N \). Since \( \{U\} \leq U_n \), then \( H(C) \leq H(U) \) for some \( C \in U_n \). Thus, \( H_n(T;C) \leq H(U) \); then,

\[
h(T,U) = \lim_{n \to \infty} \frac{1}{n} H_n(T;C) \leq \lim_{n \to \infty} \frac{1}{n} H(U) = 0.
\]

\( \square \)

Corollary. 2.10. Let \((M,m,T)\) be a semi-independent hyper \(MV\)-algebra dynamical system. If for every \( U \in \mathcal{P}_T \), there exists \( N > 0 \) such that for every \( n \geq N \), \( \{U\} \leq U_n \), then \( h(T) = 0 \).

Proof. The proof is clear by using Theorem 2.9. \( \square \)

Theorem. 2.11. Let \((M,m,T)\) be a semi-independent hyper \(MV\)-algebra dynamical system, and \( U \ll_G T \). Then \( h(T) = h(T,U) \).

Proof. Let \( V \in \mathcal{P}_T \) be given. Since \( U \ll_G T \), it follows that \( V \ll_G U \). Considering the proof of Theorem 2.7, the obtained result is \( h(T,V) \leq h(T,U) \). Thus, \( \sup_{V \in \mathcal{P}_T} h(T,V) \leq h(T,U) \). \( \square \)

3. Conclusion

In this paper, the essential ergodic properties of semi-independent systems over hyper \(MV\)-algebras are discussed. Specifically, it is proved that the total entropy of a system that has a generator is calculated with respect to its generator. For the purpose of calculating the entropy, it is paramount to make use of sequences rather than tuples as partitions to define the idea of the entropy of a semi-independent hyper \(MV\)-algebra dynamical system for future research.

References


