COUNTEREXAMPLES IN CHAOTIC GENERALIZED SHIFTS

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(Received: 22 January 2017, Accepted: 20 February 2017)

Abstract. In the following text for arbitrary X with at least two elements, nonempty countable set Γ we make a comparative study on the collection of generalized shift dynamical systems like $(X^\Gamma, \sigma_\varphi)$ where $\varphi : \Gamma \to \Gamma$ is an arbitrary self-map. We pay attention to sub-systems and combinations of generalized shifts with counterexamples regarding Devaney, exact Devaney, Li-Yorke, e-chaoticity and P-chaoticity.

AMS Classification: 54H20, 37D99
Keywords: Devaney chaos, Exact Devaney chaos, Distributional chaos, e-chaos, Generalized shift, Li-Yorke chaos, P-chaos, $\omega-$chaos.

1. Introduction

By a *dynamical system* (Z, f) we mean a compact metric space Z and continuous map $f : Z \to Z$. Different definitions of chaos have been assigned to a dynamical system (Z, f), like Devaney chaos, Li-Yorke chaos, topological chaos etc. On the other hand one-sided shift $\underset{(x_n)_{n\geq 1}\mapsto (x_{n+1})_{n\geq 1}}{\{1,\dots,k\}^{\mathbb{N}} \to \{1,\dots,k\}^{\mathbb{N}}}$ is one of the most famous dynamical systems. For nonempty set Γ, arbitrary set X with at least two elements and self map $\varphi : \Gamma \to \Gamma$ the generalized shift $\underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto (x_{\varphi(\alpha)})_{\alpha\in\Gamma}}{\sigma_\varphi : X^\Gamma \to X^\Gamma}$ has been introduced for the first time in [3] as a generalization of one-sided (and two-sided) shift. For topological space X, equip X^Γ with product (pointwise convergence) topology, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous, also note to the fact X^Γ is compact metrizable if and only if X is compact metrizable and Γ is countable. So for finite discrete

*CORRESPONDING AUTHOR
SPECIAL ISSUE FOR SELECTED PAPERS OF CONFERENCE ON DYNALMICAL SYSTEMS AND GEOMETRIC THEORIES, 11-12 DECEMBER 2016, MAHANI MATHEMATICAL RESEARCH CENTER, SHAHID BAHONAR UNIVERSITY OF KERMAN
JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER
VOL. 5, NUMBERS 1-2 (2016) 85-94.

X and countable Γ we may consider generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$. Different chaos have been studied in generalized shifts (e,g., [4]), the main aim of this text is to study their interactions via diagram and counterexamples.
In details, in first section we have preliminaries and basic definitions, in section 2 we compare different mentioned entropies in generalized shift dynamical systems via a diagram, in section 3 we continue our study for product and factors of generalized shift dynamical systems and finally in section 4 we deal with composition of generalized shift dynamical systems.
Let's begin our investigations with recalling the definitions of Devaney chaos [11, 17], exact Devaney chaos [12], distributional chaos [13], e-chaos [14], Li-Yorke chaos [10], P-chaos [1], topological, and $\omega-$chaos [15].

Convention 1.1. In the following text suppose (Z, f) is a dynamical system with compact metric space (Z, d).

Convention 1.2. In the following text suppose X is a finite discrete space with at least two elements, Γ is an infinite countable set, and self-map $\varphi : \Gamma \to \Gamma$ is arbitrary. Equip X^Γ with product topology and consider generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$.

Devaney chaos. We say the dynamical system (Z, f) is *Devaney chaotic* if $f : Z \to Z$ is sensitive to initial conditions (SIC) and [11, 17]:

- TT. $f : Z \to Z$ is *topological transitive*, i.e. for all nonempty open subsets U, V of Z there exists $n \geq 1$ with $f^n(U) \cap V \neq \varnothing$,
- PP. the collection of all periodic points of f, $Per(f)$, is dense in Z (where $z \in Z$ is a *periodic point* of f if there exists $n \geq 1$ with $f^n(z) = z$).

However according to [9], if Z does not have any isolated point, SIC is redundant and TT+PP implies SIC.

Exact Devaney chaos. We say the dynamical system (Z, f) is *exact Devaney chaotic* if it is *locally eventually onto* or *leo* (i.e., for all nonempty open subset U of Z there exists $n \geq 1$ with $f^n(U) = Z$) and the collection of all periodic points of f, is dense in Z (hence exact Devaney chaotic means Devaney chaotic+leo) [12].

Li-Yorke chaos, distributional chaos, $\omega-$chaos and topological chaos. In the dynamical system (Z, f) we say $x, y \in Z$ are:

- *Li-Yorke scrambled* if
$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \text{ and } \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$
- *distributional scrambled* if:
$$\exists s > 0 \liminf_{n\to\infty} \frac{|\{i \in \{0, \dots, n-1\} : d(f^i(x), f^i(y)) < s\}|}{n} = 0\,,$$
and
$$\forall s > 0 \limsup_{n\to\infty} \frac{|\{i \in \{0, \dots, n-1\} : d(f^i(x), f^i(y)) < s\}|}{n} = 1\,,$$
- $\omega-$*scrambled pair* if (where $\omega_f(x) = \{z \in Z :$ there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ with $\lim_{k\to\infty} f^{n_k}(x) = z\}$):

– $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
– $\omega_f(x) \cap \omega_f(y) \neq \varnothing$,
– $\omega_f(x) \setminus Per(f) \neq \varnothing$.

We say $A \subseteq Z$ (with at least two elements) is an Li-Yorke scrambled set (resp. distributional scrambled set, $\omega-$scrambled set), if for all distinct points $x, y \in A$, x, y are Li-Yorke scrambled pair (resp. distributional scrambled pair, $\omega-$scrambled pair). We say (Z, f) is *Li-Yorke chaotic* (resp. *distributional chaotic*, $\omega-$*chaotic*) if it has an uncountable Li-Yorke scrambled (resp. distributional scrambled, $\omega-$scrambled) set [10, 13, 15].
For open covers $\mathcal{U}, \mathcal{V}$ of Z let $\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $N(\mathcal{U}) := \min\{|\mathcal{W}| : \mathcal{W}$ is a finite subcover of $\mathcal{V}\}$. In dynamical system (Z, f) the limit $\text{ent}_{\text{top}}(f, \mathcal{U}) := \lim\limits_{n\to\infty} \dfrac{\log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-(n-1)}(\mathcal{U}))}{n}$ exists [16] and we call $\text{ent}_{\text{top}}(f) := \sup\{\text{ent}_{\text{top}}(f, \mathcal{W}) : \mathcal{W}$ is a finite open cover of $Z\}$ the *topological entropy* of f. We say (Z, f) is *topological chaotic* if $\text{ent}_{\text{top}}(f) > 0$.

P-chaos. In the dynamical system (Z, f) for $\delta, \varepsilon > 0$ we say the sequence $(x_i)_{i\geq 0}$ is a $\delta-$*pseudo orbit*, if for all $i \geq 0$ we have $d(f(x_i), x_{i+1}) < \delta$ and we say x is an $\varepsilon-$*trace* of $(x_i)_{i\geq 0}$ if for all $i \geq 0$ we have $d(f^i(x), x_i) < \varepsilon$. We say (Z, f) has *pseudo orbit tracing property* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta-$pseudo orbit has an $\varepsilon-$trace. We say the system (Z, f) is *P-chaotic* if it has pseudo orbit tracing property and the collection of periodic points of f, is dense in Z [1].

e-chaos. In the dynamical system (Z, f) for homeomorphism $f : Z \to Z$ is *expansive* if there exists $\mu > 0$ such that for all distinct $x, y \in Z$ there exists $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) > \mu$. We say the dynamical system (Z, f) is *e-chaotic* if $f : Z \to Z$ is an expansive homeomorphism and the collection of all periodic points of f, is dense in Z [14].

Remark 1.3. The generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ is

- Devaney chaotic if and only if $\varphi : \Gamma \to \Gamma$ is one-to-one without periodic points [4, Theorem 2.13],
- exact Devaney chaotic if and only if $\varphi : \Gamma \to \Gamma$ is one-to-one moreover $\varphi : \Gamma \to \Gamma$ does not have periodic points nor infinite $\varphi-$anti orbit sequences [4, Corollarey 3.5] (where $(x_n)_{n\geq}$ infinite $\varphi-$anti orbit sequence if it is one-to-one and for all $n \geq 1$ we have $\varphi(x_{n+1}) = x_n$),
- Li-Yorke chaotic (resp. distributional chaotic, $\omega-$chaotic, topological chaotic) if and only if $\varphi : \Gamma \to \Gamma$ has at least a non-quasi-periodic point [2, 5, 6] (where we say $\alpha \in \Gamma$ is a *quasi-periodic point* of φ if there exist $n > m \geq 1$ with $\varphi^n(\alpha) = \varphi^m(\alpha)$),
- P-chaotic if and only if $\varphi : \Gamma \to \Gamma$ is one-to-one [7],
- e-chaotic if and only if $\varphi : \Gamma \to \Gamma$ is bijective and $\{\{\varphi^i(\alpha) : i \in \mathbb{Z}\} : \alpha \in \Gamma\}$ is a finite partition of Γ [8].

2. A diagram and counterexamples

In this section we compare different mentioned entropies (in the collection of generalized shift dynamical systems with phase space X^Γ) via a diagram.
Let's consider the following classes of generalized shifts:

- $\mathcal{C} := \{(X^\Gamma, \sigma_\eta) : \eta \in \Gamma^\Gamma\}$,
- $\mathcal{C}_{\text{Li}-\text{Yorke}} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is Li-Yorke chaotic$\}$,
- $\mathcal{C}_{\text{P}} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is P-chaotic$\}$,
- $\mathcal{C}_{\text{e}} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is e-chaotic$\}$,
- $\mathcal{C}_{\text{exact Devaney}} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is exact Devaney chaotic$\}$,
- $\mathcal{C}_{\text{Devaney}} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is Devaney chaotic$\}$.

Using Remark 1.3 it's evident that

$$\mathcal{C}_{\text{exact Devaney}} \subseteq \mathcal{C}_{\text{Devaney}} \subseteq \mathcal{C}_{\text{P}} \subseteq \mathcal{C}$$

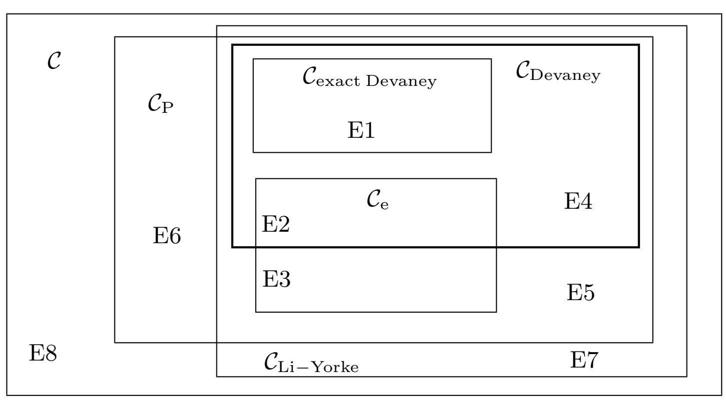
also

$$\mathcal{C}_{\text{Devaney}} \subseteq \mathcal{C}_{\text{Li}-\text{Yorke}} \subseteq \mathcal{C} ,$$

however if $(X^\Gamma, \sigma_\varphi)$ is e-chaotic, then $\varphi : \Gamma \to \Gamma$ is bijective and $\{\{\varphi^i(\alpha) : i \in \mathbb{Z}\} : \alpha \in \Gamma\}$ is a finite partition of Γ, since Γ is infinite, there exists $\alpha \in \Gamma$ such that $\{\varphi^i(\alpha) : i \in \mathbb{Z}\}$ is infinite and hence α is non-quasi-periodic. Thus $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic. Therefore

$$\mathcal{C}_{\text{e}} \subseteq \mathcal{C}_{\text{Li}-\text{Yorke}} .$$

Now we have the following diagram:



where "Ei" means counterexample $(X^\Gamma, \sigma_{\eta_i})$ (below).
Since Γ is infinite countable, we may suppose $\Gamma = \{x_n : n \in \mathbb{Z}\}$ with distinct x_ns. For $i = 1, \ldots, 8$ define $\eta_i : \Gamma \to \Gamma$ with $\eta_i(x_n) = x_{\lambda_i(n)}$ $(n \in \mathbb{Z})$ for $\lambda_i : \mathbb{Z} \to \mathbb{Z}$ with (p_m is the mth prime number, and $\{y_n : n \in \mathbb{Z}\} = \mathbb{Z} \setminus \{\pm p_n^k : n, k \geq 1\}$ with

$y_n < y_{n+1}$ for all $n \in \mathbb{Z}$):

$$\lambda_1(n) = \begin{cases} n+1 & n \geq 0\,, \\ n-1 & n < 0\,, \end{cases} \qquad \lambda_2(n) = n+1 \ (n \in \mathbb{Z})\,,$$

$$\lambda_3(n) = \begin{cases} n+1 & n \geq 1\,, \\ 0 & n = 0\,, \\ 1 & n = -1\,, \\ n+1 & n < -1\,, \end{cases} \qquad \lambda_4(n) = \begin{cases} -p_m^{k+1} & n = -p_m^k, k \geq 1\,, \\ p_m^{k+1} & n = p_m^k, k \geq 1\,, \\ y_{k+1} & n = y_k, k \in \mathbb{Z}\,, \end{cases}$$

$$\lambda_5(n) = \begin{cases} n+2 & 2|n\,, \\ n & \text{otherwise}\,, \end{cases} \qquad \lambda_6(n) = -n \ (n \in \mathbb{Z})\,,$$

$$\lambda_7(n) = n^2 \ (n \in \mathbb{Z})\,, \qquad \lambda_8(n) = |n| \ (n \in \mathbb{Z})\,.$$

3. Countable products and appropriate factors

In this section we study different entropies for product and (suitable) factors in generalized shift dynamical systems. Suppose $\{(Z_\alpha, f_\alpha) : \alpha \in \Lambda\}$ is a nonempty countable collection (Λ is countable) of dynamical systems, then $\prod_{\alpha\in\Lambda} Z_\alpha$ with product topology is a compact metrizable space and one may consider the dynamical system $(\prod_{\alpha\in\Lambda} Z_\alpha, \prod_{\alpha\in\Lambda} f_\alpha)$ with $\prod_{\alpha\in\Lambda} f_\alpha((x_\alpha)_{\alpha\in\Lambda}) = (f_\alpha(x_\alpha))_{\alpha\in\Lambda}$ (for $(x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} Z_\alpha$). Regarding product of generalized shifts we have the following table:

$\bigstar$	$\varrho(\bigstar)$
exact Devaney	for all $\alpha \in \Lambda$, $(X^{\Gamma_\alpha}, \sigma_{\varphi_\alpha})$ is exact Devaney chaotic
Devaney	for all $\alpha \in \Lambda$, $(X^{\Gamma_\alpha}, \sigma_{\varphi_\alpha})$ is Devaney chaotic
e-	Λ is finite and for all $\alpha \in \Lambda$, $(X^{\Gamma_\alpha}, \sigma_{\varphi_\alpha})$ is e-chaotic
P-	for all $\alpha \in \Lambda$, $(X^{\Gamma_\alpha}, \sigma_{\varphi_\alpha})$ is P-chaotic
Li-Yorke	there exists $\alpha \in \Lambda$, such that $(X^{\Gamma_\alpha}, \sigma_{\varphi_\alpha})$ is Li-Yorke chaotic

In the above table suppose Λ is a nonempty countable set and for each $\alpha \in \Lambda$, Γ_α is an infinite countable set and $\varphi_\alpha : \Gamma_\alpha \to \Gamma_\alpha$ is an arbitrary self-map, so for corresponding case we have "$(\prod_{\alpha\in\Lambda} X^{\Gamma_\alpha}, \prod_{\alpha\in\Lambda} \sigma_{\varphi_\alpha})$ is $\bigstar$ chaotic if and only if $\varrho(\bigstar)$".

First note that $(\prod_{\alpha\in\Lambda} X^{\Gamma_\alpha}, \prod_{\alpha\in\Lambda} \sigma_{\varphi_\alpha})$ is just $(X^{\bigsqcup_{\alpha\in\Lambda}\Gamma_\alpha}, \sigma_{\bigsqcup_{\alpha\in\Lambda}\varphi_\alpha})$ now use Remark 1.3 to establish the above table.

For self-map $\eta : \Gamma \to \Gamma$ let:

- $\mathcal{I}(\eta) := \{A \subseteq \Gamma : A \neq \varnothing, \eta(A) \subseteq A\}$,
- $\mathcal{J}(\eta) := \{A \in \mathcal{I} : \eta^{-1}(A) \subseteq A\}$,
- $\mathcal{O}(\eta) := \{\bigcup\{\eta^n(\alpha) : n \in \mathbb{Z}\} : \alpha \in \Gamma\}$,
- $\mathcal{O}_+(\eta) := \{\{\eta^n(\alpha) : n \geq 0\} : \alpha \in \Gamma\}$,

then it's evident that $\mathcal{O}(\eta) \subseteq \mathcal{J}(\eta) \subseteq \mathcal{I}(\eta)$ and $\mathcal{O}_+(\eta) \subseteq \mathcal{I}(\eta)$. Moreover for all $A \in \mathcal{I}(\eta)$ one may consider the generalized shift dynamical system $(X^A, \sigma_{\eta\restriction A})$ (note that $(X^A, \sigma_{\eta\restriction_A})$ is a factor of (X^Γ, σ_η), e.g. via conjugacy $\underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_\alpha)_{\alpha\in A}}{X^\Gamma \to X^A}$. Now we have the following table:

$\bigstar$	$\diamondsuit$	$\mathcal{K}$	counterexample
exact Devaney	$\forall$	$\mathcal{I}, \mathcal{J}, \mathcal{O}$	D1
Devaney	$\forall$	$\mathcal{I}, \mathcal{J}, \mathcal{O}$	D1
e-	$\forall$	$\mathcal{I}, \mathcal{J}$	D2, D3
P-	$\forall$	$\mathcal{I}, \mathcal{J}, \mathcal{O}$	D1
Li-Yorke	$\exists$	$\mathcal{I}, \mathcal{J}, \mathcal{O}, \mathcal{O}_+$	-

In the above table for the corresponding case, (X^Γ, σ_η) is $\bigstar$ chaotic if and only if
$\diamondsuit D \in \mathcal{K}(\eta)$ $((X^D, \sigma_{\eta\restriction_D})$ is $\bigstar$ chaotic)
moreover "Di" means counterexample "i" in the blow
(these counterexamples deal with column $\mathcal{K}$).

In the following items suppose $\Gamma = \{x_n : n \in \mathbb{Z}\}$ with distinct x_ns.

(1) Consider $\eta : \Gamma \to \Gamma$ with

$$\eta(x_n) = \begin{cases} x_{|n|} & n < 0\,, \\ x_{n+1} & n \geq 0\,, \end{cases}$$

then for all $D \in \mathcal{O}_+$, $\eta\restriction_D$ is one-to-one without periodic points and infinite anti orbit sequence thus $(X^D, \sigma_{\eta\restriction_D})$ is exact Devaney, Devaney, and P-chaotic, however η is not one-to-one and (X^Γ, σ_η) is not none of exact Devaney, Devaney, or P-chaotic.

(2) Consider $\eta : \Gamma \to \Gamma$ with $\eta(x_n) = x_{\lambda_4(n)}$ for $\lambda_4 : \mathbb{Z} \to \mathbb{Z}$ as in the previous section, then for all $D \in \mathcal{O}(\eta)$, $(X^D, \sigma_{\eta\restriction_D})$ is e-chaotic, however (X^Γ, σ_η) is not e-chaotic.

(3) Consider $\eta : \Gamma \to \Gamma$ with $\eta(x_n) = x_{-n}$ then $\mathcal{O}(\eta) = \mathcal{O}_+(\eta)$ and for all $D \in \mathcal{O}(\eta)$, D is finite. For all $D \in \mathcal{O}(\eta) = \mathcal{O}_+(\eta)$, $(X^D, \sigma_{\eta\restriction_D})$ is e-chaotic, however (X^Γ, σ_η) is not e-chaotic.

4. Iterations and compositions

In our last section we pay attention to different entropies in generalized shift dynamical systems. If (Z, f) and (Z, g) are dynamical systems, one may consider dynamical system $(Z, f \circ g)$. In particular for all $p \geq 1$, one may consider the dynamical system (Z, f^p), the following Note help us to improve our ideas on this matter.

Note 4.1. For $h : A \to A$, $t \in A$, $p \geq 2$, and sequence $(x_n)_{n\geq 1}$ in A we have:

1. $Per(h) = Per(h^p)$ (t is a periodic point of $h : A \to A$ if and only if it is a periodic point of $h^p : A \to A$),
2. t is a quasi-periodic point of $h : A \to A$ if and only if it is a quasi-periodic point of $h^p : A \to A$,

3. $h : A \to A$ is one-to-one (resp. onto) if and only if $h^p : A \to A$ is one-to-one (resp. onto),
4. if $(w_n)_{n\geq1}$ is an infinite $h-$anti orbit sequence, then $(w_{1+np})_{n\geq1}$ is an infinite h^p-anti orbit sequence,
5. if $(w_n)_{n\geq1}$ is an infinite h^p-anti orbit sequence, let

$$\begin{aligned}
&y_1 := w_1, y_2 := h^{p-1}(w_2), \dots, y_p = h(w_2),\\
&y_{p+1} := w_2, y_{p+2} := h^{p-1}(w_3), \dots, y_{2p} = h(w_3),\\
&\quad\vdots\\
&y_{mp+1} := w_m, y_{mp+2} := h^{p-1}(w_{m+1}), \dots, y_{(m+1)p} = h(w_{m+1}),\\
&\quad\vdots
\end{aligned}$$

then $(y_n)_{n\geq1}$ is an infinite $h-$anti orbit sequence,
6. consider (3) and suppose $h, h^p : A \to A$, using $\{h^i(t) : i \in \mathbb{Z}\} = \bigcup\{\{h^{ip}(y) : i \in \mathbb{Z}\} : y \in \{t, h(t), \dots, h^{p-1}(t)\}\}$, the set $\Xi_1 := \{\{h^i(x) : i \in \mathbb{Z}\} : x \in A\}$ is finite if and only if $\Xi_p := \{\{h^{ip}(x) : i \in \mathbb{Z}\} : x \in A\}$ is finite with $\text{card}(\Xi_p) \leq \text{card}(\Xi_1) \leq p\text{card}(\Xi_p)$.

Corollary 4.2. Using $\sigma_\varphi^p = \sigma_{\varphi^p}$, Remark 1.3 and Note 4.1, for $p \geq 1$, $(X^\Gamma, \sigma_\varphi)$ is chaotic (all kinds of chaos in Remark 1.3) if and only if $(X^\Gamma, \sigma_\varphi^p)$ is so.

Note 4.3. For one-to-one map $h : Z \to Z$, the following statements are equivalent:

- there is not any infinite $h-$anti orbit sequence,
- for all $\alpha \in Z$ there exists $n \geq 1$ such that $h^{-n}(\alpha) = \varnothing$ or α is a periodic point of h.

Lemma 4.4. For $\psi, \eta : \Gamma \to \Gamma$ we have:

1. If $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is P-chaotic, then (X^Γ, σ_η) is P-chaotic.
2. If (X^Γ, σ_η) and (X^Γ, σ_ψ) are P-chaotic, then $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is P-chaotic.
3. For $\psi \circ \eta = \eta \circ \psi$, (X^Γ, σ_η) and (X^Γ, σ_ψ) are P-chaotic if and only if $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is P-chaotic.
4. Suppose $\psi \circ \eta = \eta \circ \psi$, ψ is one-to-one and (X^Γ, σ_η) is exact Devaney chaotic, then $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is exact Devaney chaotic. In particular if $\psi \circ \eta = \eta \circ \psi$, and both dynamical systems (X^Γ, σ_η) and (X^Γ, σ_ψ) are exact Devaney chaotic, then $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is exact Devaney chaotic.
5. If $\psi \circ \eta = \eta \circ \psi$ and $(X^\Gamma, \sigma_\eta \circ \sigma_\psi)$ is Li-Yorke chaotic, then either (X^Γ, σ_η) is Li-Yorke chaotic or (X^Γ, σ_ψ) is Li-Yorke chaotic.

Proof. Note that $\sigma_\eta \circ \sigma_\psi = \sigma_{\psi \circ \eta}$.
4) Suppose $\psi \circ \eta = \eta \circ \psi$, ψ is one-to-one and (X^Γ, σ_η) is exact Devaney chaotic, then η is one-to-one without any periodic point, moreover there is not any infinite $\eta-$anti orbit sequence. Since η, ψ are one-to-one, $\psi \circ \eta$ is one-to-one too. By Note 4.3, since η is one-to-one without any periodic point and there is not any infinite $\eta-$anti orbit sequence, for all $\alpha \in \Gamma$ there exists $n \geq 1$ such that $\eta^{-n}(\alpha) = \varnothing$, thus $(\psi \circ \eta)^{-n}(\alpha) = \psi^{-n}(\eta^{-n}(\alpha)) = \varnothing$ and Γ does not contain any infinite $\psi \circ \eta-$anti orbit sequence.
If α is a periodic point of $\psi \circ \eta$, then for all $n \geq 1$ there exists $p \geq 1$ with $\alpha =$

$(\psi\circ\eta)^p(\alpha) = \eta^n(\eta^{p-n}(\psi^p(\alpha)))$ and $\eta^{p-n}(\psi^p(\alpha)) \in \eta^{-n}(\alpha)$ which leads to the contradiction $\eta^{-n}(\alpha) \neq \varnothing$ for all $n \geq 1$. Thus $\psi\circ\eta$ does not have any periodic point.

5) Suppose $\psi\circ\eta = \eta\circ\psi$ and both dynamical systems (X^Γ, σ_η) and (X^Γ, σ_ψ) are not Li-Yorke chaotic. Choose $\alpha \in \Gamma$, since (X^Γ, σ_η) is not Li-Yorke chaotic, $\{\eta^n(\alpha) : n \geq 0\}$ is finite, suppose $\{\eta^n(\alpha) : n \geq 0\} = \{\beta_1, \ldots, \beta_p\}$. Since (X^Γ, σ_ψ) is not Li-Yorke chaotic, for all $i = 1, \ldots, p$, $\{\psi^n(\beta_i) : n \geq 0\}$ is finite, hence $\{\psi^n(\beta_i) : n \geq 0, 1 \leq i \leq p\}$ is finite, using:

$$\begin{aligned}\{(\psi\circ\eta)^n(\alpha) : n \geq 0\} &= \{\psi^n(\eta^n(\alpha)) : n \geq 0\} \\ &\subseteq \{\psi^n(\beta_i) : n \geq 0, 1 \leq i \leq p\}\end{aligned}$$

the set $\{(\psi\circ\eta)^n(\alpha) : n \geq 0\}$ is finite too (for all $\alpha \in \Gamma$) and $(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$ is not Li-Yorke chaotic. □

Two tables. Regarding composition of generalized shifts we have the following tables (use Lemma 4.4):

	exact Devaney	Devaney	e-	P-	Li-Yorke	ρ
$(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$	C4, C5	C3, C2	C3, C2	⇒ C6	C3, C2	(X^Γ, σ_η) or (X^Γ, σ_ψ)
	C4, C5	C3, C2	C3, C2	⇒ C6	C3, C2	(X^Γ, σ_η)
	C1, C4	C3, C1	C3, C2	C6, C1	C3, C1	(X^Γ, σ_ψ)
	C1, C4	C2, C3	C2, C3	⇐ C1	C2, C3	(X^Γ, σ_η) and (X^Γ, σ_ψ)

For $\eta, \psi : \Gamma \to \Gamma$ we have the above table for studying
"if $(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$ is ⋇ chaotic, then ρ is ⋇ chaotic"
and
"if ρ is ⋇ chaotic, then $(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$ is ⋇ chaotic"
in the corresponding case, where "Ci" means counterexample "i" in the blow.

Also:

	exact Devaney	Devaney	e-	P-	Li-Yorke	
$(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$	C9, C5	C3, C5	C3, C8	⇒ C6	⇒ C3	(X^Γ, σ_η) or (X^Γ, σ_ψ)
	C9, C7	C3, C7	C3, C7	⇒ C6	C3, C7	(X^Γ, σ_η)
	⇐ C7	C3, C7	C3, C7	⇔	C3, C7	(X^Γ, σ_η) and (X^Γ, σ_ψ)

For $\eta, \psi : \Gamma \to \Gamma$ with $\psi\circ\eta = \eta\circ\psi$ we have the above table for studying
"if $(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$ is ⋇ chaotic, then ρ is ⋇ chaotic"
and
"if ρ is ⋇ chaotic, then $(X^\Gamma, \sigma_\eta\circ\sigma_\psi)$ is ⋇ chaotic"
in the corresponding case, where "Ci" means counterexample "i" in the blow.

Note that $\sigma_\eta\circ\sigma_\psi = \sigma_{\psi\circ\eta}$. In the following counterexamples once more since Γ is infinite countable, we suppose $\Gamma = \{x_n : n \in \mathbb{Z}\}$ with distinct x_ns.

(1) Consider $\theta, \mu : \Gamma \to \Gamma$ with $\theta(x_n) = x_{2n}$ and $\mu(x_{2n}) = x_{2n}$, $\mu(x_{2n+1}) = x_{|2n+1|})$ for $n \in \mathbb{Z}$, then $\mu\circ\theta = \theta : \Gamma \to \Gamma$ is one-to-one without periodic points and infinite anti orbit sequences, so $(X^\Gamma, \sigma_\theta)(= (X^\Gamma, \sigma_\theta\circ\sigma_\mu))$ is exact

Devaney, Devaney, Li-Yorke and P-chaotic but μ is not one-to-one and all points of Γ are quasi periodic points of μ, thus (X^Γ, σ_μ) is not chaotic in any of the above senses.

(2) Consider $\theta, \mu : \Gamma \to \Gamma$ with $\theta = \cdots (x_{-3}\, x_{-2})(x_{-1}\, x_0)(x_1\, x_2)(x_3\, x_4) \cdots$ and $\mu = \cdots (x_{-2}\, x_{-1})(x_0\, x_1)(x_2\, x_3) \cdots$, then neither $(X^\Gamma, \sigma_\theta)$ nor (X^Γ, σ_μ) is Li-Yorke chaotic (resp. Devaney chaotic, e-chaotic) but $(X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is.

(3) Consider $\theta, \mu : \Gamma \to \Gamma$ with $\theta(x_n) = x_{n+1}$ $(n \in \mathbb{Z})$ and $\mu = \theta^{-1}$, then $(X^\Gamma, \sigma_\theta)$ and (X^Γ, σ_μ) are Devaney, Li-Yorke and e-chaotic, but $(X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is not.

(4) Consider $\theta, \mu : \Gamma \to \Gamma$ with $\mu = (x_0\ x_1\ x_{-1}\ x_2\ x_{-2}\ x_3\ \cdots)$ and $\theta = (x_0\, x_{-1}\, x_1\, x_{-2}\, x_2\, x_{-3}\ \cdots)$, then $(X^\Gamma, \sigma_\theta)$ and (X^Γ, σ_μ) are exact Devaney, however -1 is a fix point of $\mu \circ \theta$, thus $(X^\Gamma, \sigma_{\mu\circ\theta}) = (X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is not exact Devaney chaotic.

(5) Consider $\theta, \mu : \Gamma \to \Gamma$ with

$$\theta(x_n) = \begin{cases} x_{n+1} & n \geq 0\,, \\ x_n & n < 0\,, \end{cases} \qquad \mu(x_n) = \begin{cases} x_n & n \geq 0\,, \\ x_{n-1} & n < 0\,, \end{cases}$$

$$\theta \circ \mu(x_n) = \mu \circ \theta(x_n) = \begin{cases} x_{n+1} & n \geq 0\,, \\ x_{n-1} & n < 0\,, \end{cases}$$

then θ and μ have fix points, so neither $(X^\Gamma, \sigma_\theta)$ nor (X^Γ, σ_μ) are exact Devaney chaotic, however $(X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is exact Devaney chaotic.

(6) Consider $\theta, \mu : \Gamma \to \Gamma$ with $\mu(x_n) = x_0$ for all $n \in \mathbb{Z}$ and

$$\theta(x_n) = \begin{cases} x_{n+1} & n \neq -1, 0\,, \\ x_0 & n = 0\,, \\ x_1 & n = -1\,, \end{cases}$$

then $\theta \circ \mu = \mu \circ \theta$ constant map. Hence $(X^\Gamma, \sigma_\theta)$ is Li-Yorke chaotic, e-chaotic and P-chaotic, however $(X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is not chaotic in any of the above senses.

(7) Consider $\eta : \Gamma \to \Gamma$ such that (X^Γ, σ_η) is exact Devaney (resp. Devaney, Li-Yorke) chaotic, then $\eta \circ id_\Gamma = id_\Gamma \circ \eta$, moreover $(X^\Gamma, \sigma_\eta) = (X^\Gamma, \sigma_\eta \circ \sigma_{id_\Gamma}) = (X^\Gamma, \sigma_{id_\Gamma} \circ \sigma_\eta)$ is exact Devaney (resp. Devaney, Li-Yorke, e-) chaotic, but $(X^\Gamma, \sigma_{id_\Gamma})$ is not exact Devaney (resp. Devaney, Li-Yorke, e-) chaotic.

(8) Consider $\theta, \mu : \Gamma \to \Gamma$ with:

$$\theta(x_n) = \begin{cases} x_n & n \text{ is odd}\,, \\ x_{n+2} & n \text{ is even}\,, \end{cases} \qquad \mu(x_n) = \begin{cases} x_{n+2} & n \text{ is odd}\,, \\ x_n & n \text{ is even}\,, \end{cases}$$

then neither $(X^\Gamma, \sigma_\theta)$ nor (X^Γ, σ_μ) are e-chaotic, however $(X^\Gamma, \sigma_{\theta\circ\mu})$ is e-chaotic.

(9) Consider $\theta, \mu : \Gamma \to \Gamma$ with:

$$\theta(x_n) = \begin{cases} x_{n+1} & n \geq 0\,, \\ x_{n-1} & n < 0, \end{cases} \qquad \mu(x_n) = \begin{cases} x_{-n-1} & n \geq 0, \\ x_n & n < 0, \end{cases}$$

then $(X^\Gamma, \sigma_\theta)$ is exact Devaney chaotic and $\theta \circ \mu = \mu \circ \theta$ however neither (X^Γ, σ_μ) nor $(X^\Gamma, \sigma_\theta \circ \sigma_\mu)$ is exact Devaney chaotic.

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