STABILITY ANALYSIS OF FRACTIONAL-ORDER NONLINEAR SYSTEMS VIA LYAPUNOV METHOD

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(Received: 9 December 2015, Accepted: 18 December 2016)

Abstract. In this paper, we study stability of fractional-order nonlinear dynamic systems by means of Lyapunov method. To examine the obtained results, we employe the developed techniques on test examples.

AMS Classification: 93D05, 34A08, 26A33.
Keywords: Stability, Riemann-Liouville derivative, Caputo derivative, Lyapunov method.

1. Introduction

Classical calculus has provided an efficient tool for modeling and exploring the properties of the dynamical system problems concerning of physics, biology, engineering and applied sciences. However, experiments with a realistic approach reveal that there are a large class of complex systems where microscopic and macroscopic behaviors are not captured or properly explained using classical calculus. It has been found that these major classes of complex systems which contain non-local dynamics involving long-memory are captured using a more general class of operators known as fractional operators. These phenomena are always related to the

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JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER
VOL. 3, NUMBER 1 (2014) 61-73.

complexity and heredity of systems due to the fractional properties of system components, such as the fractional viscoelastic material, the fractional circuit element and the fractal structure [1, 2, 6]. In particular, the memristor (a contraction for memory resistor), which is said to be the missing circuit element [5], shows some hereditary properties. Allowing for the fact that the fractional calculus itself is a kind of convolution, the memristor is naturally likely to be linked to fractional calculus. Finally, it is possible that, in the future, there will be more fractional order dynamic systems in micro/nano scales.

Stability is the one of the most frequent terms used in literature whenever we deal with the dynamical systems and their behaviors. In mathematical terminology, stability theory addresses the convergence of solutions of differential and of trajectories of dynamical systems under small perturbations of initial conditions. Same as classical differential systems a lot of stress has been given to the stability and stabilization of the systems represented by fractional order differential equations.

An efficient tool to analyze the stability of a nonlinear system without explicitly solving the differential equations is the Lyapunov method. Infact, the Lyapunov method provides a sufficient condition for stability of nonlinear systems, which means a system may still be stable, even if one cannot find a Lyapunov function candidate to conclude the system stability property.

Integer-order methods of stability analysis were extended to stability of fractional order dynamic systems, for example [3, 4, 10, 11, 12, 13]. However, as pointed out in [4], the decay of generalized energy of a dynamic system does not have to be exponential for the system to be stable. The energy decay actually can be of any rate, including power law decay. For extending the application of fractional calculus in nonlinear systems, we propose the fractional Lyapunov method with a view to enrich the knowledge of both system theory and fractional calculus.

The paper is organized as follows. In section 2, we present some basic materials on fractional calculus. In section 3 we derive the Lyapunov stability theorem for fractional dynamical systems with Caputo derivative. In section 4, we present examples, in which we compute different orbits of the given system by means of numerical simulations.

2. **Preliminaries**

Two types of fractional derivatives of Riemann-Liouville and Caputo derivatives, have been often used in fractional differential systems. We briefly introduce these two definitions. definition The Riemann-Liouville integral $J^{\alpha}_{t_0,t}$ with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined as:

$$J^{\alpha}_{t_0,t}x(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau$$

where $\Gamma(.)$ is the Eulers gamma function, for $\alpha = 0$ we set $J^0_{t_0,t} := Id$, the identity operator. definition The Riemann-Liouville derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by:

$$_{RL}D^{\alpha}_{t_0,t}x(t) := \frac{d^m}{dt^m} J^{(m-\alpha)}_{t_0,t} x(t)$$

where $m = \lceil \alpha \rceil := \min\{k \in \mathbb{Z} : k \geq \alpha\}$, is the ceilling of α.

definition The Caputo derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by:

$$_{C}D^{\alpha}_{t_0,t}x(t) := J^{(m-\alpha)}_{t_0,t} \frac{d^m}{dt^m} x(t)$$

where $m = \lceil \alpha \rceil$.

We consider the following general type of fractional differential equations involving Caputo derivative

$$\begin{cases} {}_{C}D^{\alpha}_{0,t}x(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $0 < \alpha < 1$, and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. definition[8] (Lyapunov stability) (a) The equlibruim point $\overline{x}$ of the differential equation (1), is called stable if, for any $\varepsilon > 0$ there exists some $\delta > 0$ such that the solution of the initial value problem (1) satisfies $\|x(t) - \overline{x}\| < \varepsilon$ for all $t \geq 0$ whenever $\|x_0 - \overline{x}\| < \delta$. (b) The equlibruim point $\overline{x}$ of the differential equation (1), is called asymptotically stable if it is stable and there exists some $\delta > 0$ such that $\lim_{t\to+\infty} \|x(t) - \overline{x}\| = 0$ whenever $\|x_0 - \overline{x}\| < \delta$.

Theorem 2.1. [8] *Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Moreover let $x_0^{(0)}, \ldots, x_0^{(m-1)} \in \mathbb{R}, K > 0$, and $h^* > 0$. Define $G := \Big\{(t,x) : t \in [0,h^*], \|x - \sum_{k=0}^{m-1} \frac{t^k}{k!} x_0^{(k)}\| \leq K\Big\}$, and let the function $f : G \longrightarrow \mathbb{R}$ be continuous. Furthermore, define*
$M := \sup_{(t,z)\in G} \|f(t,z)\|$ and

$$h := \begin{cases} h^* \ if \ M = 0, \\ \min\{h^*, (\frac{K\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}\} \ else. \end{cases}$$

Then, there exists a function $x \in C[0,h]$ solving the initial value problem (1).

Theorem 2.2. [8] *With the assumptions of Theorem 2.1, the function $x \in C[0,h]$ is a solution of the initial value problem* (1) *if and only if it is a solution of the nonlinear Volterra integral equation of the second kind*

$$x(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} x_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \tag{2}$$

in which $m = \lceil \alpha \rceil$.

Theorem 2.3. [7]*(Dependence of solution on initial condition) Let $f_i : W \longrightarrow \mathbb{R}, i = 1, \ldots, n$, and*

$$W = [0, \chi^*] \times \prod_{j=1}^{n} [x_j(0) - l_j, x_j(0) + l_j], \ \chi^* > 0, l_j > 0,$$

be a bounded set. Let $0 < \alpha < 1$ and $f = (f_1, \ldots, f_n)$ be Lipschitz on the second variable with Lipschitz constant L, $x(t)$ and $y(t)$ be the solutions of the initial value problems

$${}_C D_{0,t}^{\alpha} x(t) = f(t, x(t)), \ x_0 = x(0),$$
$${}_C D_{0,t}^{\alpha} y(t) = f(t, y(t)), \ y_0 = y(0),$$

respectively. Then

$$\|x(t) - y(t)\|_\infty \leq \|x_0 - y_0\|_\infty E_\alpha(Lt^\alpha),$$

where E_α is the Mittag-Leffler function.

3. Lyapunov direct method for stability

Theorem 3.1. *Let x^* be an equilibrium point for the system* (1)*, $L : \mathcal{O} \longrightarrow \mathbb{R}$ defined on an open set $\mathcal{O}$ containing x^* and $L \in C^1(\mathcal{O})$. Suppose further that*

$$L(x^*) = 0 \text{ and } L(x) > 0 \; for \; x \neq x^*. \tag{3}$$

Then x^* is
(i) *stable if $\dot{L}(x) = \frac{d}{dt}L(x(t)) \leq 0$ on $\mathcal{O} \setminus \{x^*\}$,*
(ii) *unstable if $\dot{L}(x) = \frac{d}{dt}L(x(t)) > 0$ on $\mathcal{O} \setminus \{x^*\}$.*

A function L satisfying (3) *and (i) is called a Lyapunov function.*

Proof. Suppose $\delta > 0$ such that

$$B_\delta(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| \leq \delta\} \subseteq \mathcal{O}$$

and set

$$S_\delta(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| = \delta\}, \; N_\delta(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| < \delta\}.$$

Since $S_\delta(x^*)$ is compact, $\zeta = \min_{x \in S_\delta(x^*)} L(x)$ exists. Now we define

$$\mathcal{U} = \{x \in B_\delta(x^*) : L(x) < \zeta\}.$$

Since $L(x) > 0$ for $x \neq x^*$, thus we deduce

$$\mathcal{U} = B_\delta(x^*) \cap L^{-1}[0, \zeta).$$

Additionally

$$B_\delta(x^*) = N_\delta(x^*) \cup S_\delta(x^*), \; \mathcal{U} \cap S_\delta(x^*) = \emptyset.$$

It is easy to see that

$$\mathcal{U} = N_\delta(x^*) \cap L^{-1}(-\infty, \zeta)$$

so, $\mathcal{U} \subseteq B_\delta(x^*)$ is an open set. Since $L(x^*) = 0 < \zeta$ then $x^* \in \mathcal{U}$. Now suppose that $x(t)$ is a solution of (1) starting from $x_0 = x(0)$. Since L is non-increasing on $x(t)$ for each $t \geq 0$, we have

$$Lox(t) \leq Lox(0) = L(x_0) < \zeta. \tag{4}$$

If $x(t)$ leaves $B_\delta(x^*)$ then by continuity of $x(t)$, there exists $t_1 > 0$, such that $x(t_1) \in S_\delta(x^*)$ (see Fig (a), right). Thus

$$L(x(t_1)) \geq \zeta. \tag{5}$$

This contradicts to (4). Hence any solution starting in $\mathcal{U}$ never leaves $B_\delta(x^*)$, which means x^* is stable. This proves the part (i) of theorem.

Now suppose that assumption (ii) in the theorem holds. We prove x^* is unstable. First by the fact that L is continuous and positive, we have

$$\forall \varepsilon = L(x_0) > 0 \ \exists \delta_1 > 0 : \|x - x^*\| < \delta_1 \Rightarrow L(x) = |L(x)| = |L(x) - L(x^*)| < \varepsilon \tag{6}$$

also since L is increasing on the solution curves for all $t > t_0$, we get

$$Lox(t) > Lox(0) = L(x_0) = \varepsilon > 0. \tag{7}$$

So (6) and (7) show that the solution $x(t)$ never lies in $B_{\delta_1}(x^*)$. We next show that if the solution $x(t)$ never leaves $B_\delta(x^*)$ (see Fig1, left), then we get a contradiction. In this case, the solution forever remains in the compact set annulus $\{x \in \mathbb{R}^n : \delta_1 \leq \|x - x^*\| \leq \delta\}$ (Fig (a), left).

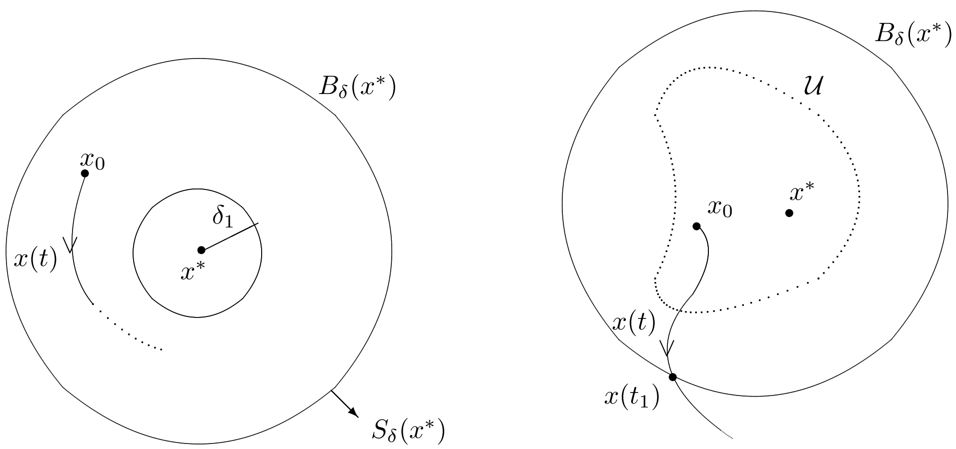

Figure (a): Neighborhood of x^*

On the other hand we have

$$\{x(t) : t \geq 0\} \subseteq B_\delta(x^*) \setminus B_{\delta_1}(x^*) \subseteq \{x \in \mathbb{R}^n : \delta_1 \leq \|x - x^*\| \leq \delta\} \tag{8}$$

Since $\dot{L}(x)$ is continuous, $\inf\limits_{B_\delta(x^*)\setminus B_{\delta_1}(x^*)} \dot{L}(x)$ exists which means $\inf\limits_{t\geq 0} \dot{L}(x) = m > 0$ also exists. Moreover there exist $t_0 \in [0, t]$ such that

$$\frac{L(x(t)) - L(x_0)}{t} = \dot{L}(x(t_0)) \geq \inf_{t>0} \dot{L}(x) = m > 0.$$

Thus

$$L(x(t)) \geq L(x_0) + mt \geq mt,$$

and

$$\sup_{t\geq 0} L(x(t)) \geq \sup_{t\geq 0} mt. \tag{9}$$

We also have

$$\sup_{t\geq 0} L(x(t)) \leq \sup_{B_\delta(x^*)\setminus B_{\delta_1}(x^*)} L(x(t)) \leq \sup_{B_\delta(x^*)} L(x(t)) = M < \infty. \tag{10}$$

By (9) and (10), we get

$$\sup_{t\geq 0} mt \leq \sup_{B_\delta(x^*)} L(x(t)) = M < \infty.$$

This is a contradiction since $m \neq 0$. □

Remark 1. The above theorem shows that every solution starting in $\mathcal{U}$ never leaves $B_\delta(x^*)$, i.e., for all $t \geq 0$ we have $x(t) \in B_\delta(x^*)$, otherwise by the equation (4), $L(x(t)) < \zeta$. Thus $x(t) \in \mathcal{U}$ for all $t \geq 0$, i.e., every solution starting in $\mathcal{U}$ remains in $\mathcal{U}$ forever.

Suppose that $K \subseteq \mathbb{R}^n$ is a compact set, $x_0 \in K$ and sequence (x_n) does not converge to x_0. Then it can be proved that there exists a subsequence (x_{n_k}) of (x_n) and a point $z_0 \in K \setminus \{x_0\}$ such that $x_{n_k} \longrightarrow z_0$. By this explanation we give the following theorem:

Theorem 3.2. *Let x^* be an equilibrium point for the system* (1)*, $\mathcal{O}$ be an open set containing x^*, and $L \in C^1(\mathcal{O})$. Suppose further that the following conditions hold:*
(i) $L(x^*) = 0$ *and* $L(x) > 0$ *for* $x \neq x^*$.

(ii) $\dot{L}(x) = \frac{d}{dt}L(x(t)) < 0$ *on* $\mathcal{O} \setminus \{x^*\}$.

Then x^* *is asymptotically stable.*

Proof. Similar to Theorem 3.1, there exists $\delta > 0$ such that $B_\delta(x^*) \subseteq \mathcal{O}$. So the stability holds by this theorem. If the solution $x(t)$ arrives in some time to x^*, we have $\lim_{t\to+\infty} x(t) = x^*$, then asymptotic stability holds. Otherwise, $x(t) \in \mathcal{U} \setminus \{x^*\}$. By the assumption L is strictly decreasing along the solutions in $\mathcal{U} \setminus \{x^*\}$. Now we claim that

$$\lim_{t\to+\infty} x(t) = x^*,$$

or equivalently

$$\forall (t_n)\ t_n \geq 0, t_n \uparrow +\infty \Longrightarrow \lim_{n\to+\infty} x(t_n) = x^*.$$

By contradiction, we assume $\lim_{n\to+\infty} x(t_n) \neq x^*$. Then there exists $t_n \geq 0$ with $t_n \uparrow +\infty$, such that the sequence $\big(x(t_{n_k})\big)$ does not convergence to x^*. Because $\{x(t_n); n \in \mathbb{N}\} \subseteq \mathcal{U} \subseteq B_\delta(x^*)$ and it does not converge to x^*, by explanation previous this theorem, we have

$$\exists\ z_0 \in B_\delta(x^*), z_0 \neq x_0, \exists\ (t_{n_k}) \subseteq (t_n) : x(t_{n_k}) \longrightarrow z_0$$

as $k \longrightarrow +\infty$. Replacing t_{n_k} by t_n, leads to

$$\lim_{n\to+\infty} x(t_n) = z_0. \tag{11}$$

Also if $t \geq 0$ then by the assumption that $t_n \uparrow +\infty$ we have

$$\exists\ N \in \mathbb{N} : \forall n \geq N : t_n > t + 1.$$

Thus $L(x(t_n)) < L(x(t+1))$ and if $n \longrightarrow +\infty$, we get

$$L(x(t_n)) < L(x(t+1)) < L(x(t)).$$

By using (11), we get

$$\forall t \geq 0 : L(z_0) < L(x(t)). \tag{12}$$

Now we suppose that $z(s)$ is a unique solution for $0 < \alpha < 1$, which starts at $z(0) = z_0$. By Theorem 2.3, if $y(s)$ is another solution starting at $y(0) = y_0$ and $s \in [0, 1]$, then there exists $M > 0$ such that

$$\|y(s) - z(s)\|_\infty \leq M\|y_0 - z_0\|_\infty,\ s \in [0, 1]. \tag{13}$$

Since L is a uniformly continuous on $B_\delta(x^*)$, for arbitrary $\varepsilon = L(z_0) - L(z(1)) > 0$, we have:

$$\exists\, \delta_1 > 0; \forall y, z \in B_\delta(x^*) : \|y - z\| < \delta_1 \Longrightarrow |L(y) - L(z)| < \varepsilon. \tag{14}$$

Since $x(t_n) \longrightarrow z_0$, we can choose $n_0 \in \mathbb{N}$ such that

$$\|x(t_{n_0}) - z_0\| < \frac{\delta_1}{M}. \tag{15}$$

Now by choosing $y(s) = x(t_{n_0} + s)$, and by using (13) and (15), we get

$$\|x(t_{n_0} + s) - z(s)\| = \|y(s) - z(s)\| \leq M\|y_0 - z_0\| = M\|x(t_{n_0}) - z_0\| < \delta_1.$$

for each $s \in [0, 1]$, also by using (14) and assuming $z = z(s)$ and $y(s) = x(t_{n_0} + s)$, and by the fact that this solution never leaves $B_\delta(x^*)$, we deduce that for each $s \in [0, 1]$

$$|L(x(t_{n_0} + s)) - L(z(s))| < \varepsilon = L(z_0) - L(z(1)). \tag{16}$$

Now by equation (12), we find

$$L(x(t_{n_0} + 1)) > L(z_0) > L(z(1)),$$

which implies

$$L(x(t_{n_0} + 1)) - L(z(1)) > L(z_0) - L(z(1)) = \varepsilon. \tag{17}$$

If we choose $s = 1$ in (16) and by help of (17), we get a contradiction. This completes the proof. □

Theorem 3.3. *Consider the fractional differential equation*

$${}_C D^{\alpha}_{0,t} x(t) = f(t, x(t))$$

with suitable initial values $x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 0, 1, ..., m-1)$ *where* $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $m - 1 < \alpha < m \in \mathbb{Z}_+$, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. *Then for* $\alpha > 1$ *the Lyapunov function does not exist.*

Proof. Suppose $p_0 \in \mathcal{U}$, then corresponding to every vector $u \in \mathbb{R}^n$ with $\|u\| = 1$, according to Theorem 2.1, we can find a unique solution $x(t)$ with $x(t_0) = p_0$ and $\dot{x}(t_0) = u$. We note that, this solution is unique only for $1 \leq \alpha < 2$. Suppose that $\dot{L} = \frac{d}{dt} L(x(t)) = \dot{x}(t_0).\nabla L(p_0) \leq 0$ then

$$\forall u \in \mathbb{R}^n, \|u\| = 1 : u.\nabla L(p_0) \leq 0.$$

We now claim that

$$\nabla L(p_0) = 0.$$

Otherwise if $\nabla L(p_0) \neq 0$ by assuming $u = \frac{\nabla L(p_0)}{\|\nabla L(p_0)\|}$, we get a contradiction. Thus all partial derivatives of L are zero and $L \in C^1(\mathcal{O})$. Without loss of generality we choice $\mathcal{U}$ to be a connected set then L is a constant function and by using $L(x^*) = 0$ we deduce $L \equiv 0$. This is a contradiction since $\forall x \neq x^* : L(x) > 0$. Thus the Lyapunov function can not exist. □

We introduce the symbol $(x_1, \cdots, x_n) \otimes (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$ and present the following theorem:

Theorem 3.4. *Consider the fractional differential equation*

$$_C D^{\alpha}_{0,t} x(t) = f(t, x(t))$$

with suitable initial values $x(0) = x_0 \in \mathbb{R}^n$, *where* $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $0 < \alpha < 1$ *and* $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. *If there exist positive numbers* $c_1, \ldots, c_n$ *such that the function*

$$h(x) = (c_1, \ldots, c_n) \otimes (x - x^*) ._C D^{1-\alpha}_{0,t} f(t, x(t))$$

is not positive in a neighborhood of x^*. *Then* x^* *is stable, and if* $h(x)$ *is negative in a neighborhood of* x^*, *then* x^* *is asymptotically stable. Further if* $h(x)$ *is positive in a neighborhood of* x^*, *then* x^* *is unstable.*

Proof. Consider the Lyapunov function

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$L(x) = \textstyle\sum_{i=1}^{n} c_i (x_i - x_i{}^*), \; x^* = (x_1{}^*, \ldots, x_n{}^*)$$

then

$$\dot{L}(x) = \sum_{i=1}^{n} 2c_i (x_i - x_i{}^*) \frac{dx_i}{dt} = 2[(c_1, \ldots, c_n) \otimes (x - x^*)].\dot{x} = 2h(x)$$

in which $\dot{x} = {}_C D^{1-\alpha}_{0,t} f(t, x(t))$. Thus by the assumption, there exists a neighborhood $\mathcal{O}$ such that for all $x \in \mathcal{O}, \dot{L}(x) \leq 0$ and the point x^* is stable by Theorem 3.1. By a similar argument the other statements can be proved. □

4. **Numerical approach and examples**

Example 4.1. Consider the following fractional order dynamic system

$$ {}_{C}D^{\alpha}_{0,t}x(t) = f(t, x(t)) = -\int_0^t (t-\tau)^{-\alpha}x(\tau)d\tau,\ x = (x_1, \dots, x_n) $$

where $x^* = 0$, is an equilibrium point, $\alpha \in (0, 1)$, then

$$ {}_{C}D^{1-\alpha}_{0,t}f(t, x(t)) = -\Gamma(1-\alpha)x(t). $$

By using Theorem 3.4,

$$ h(x) = (c_1, \dots, c_n)\otimes x.(-\Gamma(1-\alpha)x) = -\Gamma(1-\alpha)(c_1, \dots, c_n)\otimes x.x = -\Gamma(1-\alpha)(c_1x_1^2+\cdots+c_nx_n^2) $$

and by choosing $c_1 = c_2 = \ldots = c_n = 1$ we have

$$ h(x) = -\Gamma(1-\alpha)(x_1^2 + \cdots + x_n^2), $$

which is negative for $x = (x_1, \dots, x_n) \neq 0$. Thus according to Theorem 3.4, x^* is asymptotically stable.

Example 4.2. Consider the following fractional order dynamic system

$$ {}_{C}D^{\alpha}_{0,t}x(t) = f(t, x(t)) = -\int_0^t g(\tau, x)d\tau + h(t, x), $$

where $g(t, x) \geq 0$, $x(0) > 0$, $\alpha \in (0, 1)$ and

$$ h(t,x) = \begin{cases} 0 & x \neq 0 \\ \int_0^t g(\tau, x)d\tau & x = 0 \end{cases} $$

the relation $-\int_0^t g(\tau, 0)d\tau + h(t, 0) = 0$ follows that $x = 0$ is an equilibrium point. Similar to the method used in [10], let $L(x) = x^2$, a Lyapunov candidate, from $x(0) > 0$, $x = 0$ is an equilibrium point we have $\frac{dL(x)}{dx} = 2x \geq 0$, in which the equality holds if and only if $x = 0$. Moreover, $g(t, x) \geq 0$ follows that $\frac{d}{dt}(\int_0^t g(\tau, x)d\tau) \geq 0$. Therefore,

$$ \frac{dL(x)}{dt} = 2x\,{}_{C}D^{1-\alpha}_{0,t}\Big(-\int_0^t g(\tau, x)d\tau + h(t, x)\Big) \leq 0. $$

Here equality holds if and only if $x(t) = 0$. Therefore, the equilibrium point $x = 0$ is asymptotically stable.

Example 4.3. For $0 < \alpha < 1$ consider the Caputo fractional order system

$$ {}_{C}D^{\alpha}_{0,t} x(t) = -x^3(t), \tag{18} $$

with the initial condition $x(0) \neq 0$. Let $L(x) = \frac{1}{2}x^2$ be a Lyapunov candidate. Then we have $\dot{L}(x(t)) = x(t)\dot{x}(t)$, where $\dot{x}$ denotes the derivative of x with respect to t. First notice that $h(t) = x(0)x(t) > 0$ for all $t > 0$, otherwise there exists t_0 such that $x(t_0) = 0$, then for all $t \geq t_0$ we have $x(t) = 0$ and the stability holds. Also by using Theorem 2.2 and by the method which used in [10], we can show that $\dot{x}(t)x(0) < 0$. Thus $\dot{L}(x(t)) = x(t)\dot{x}(t) = \dfrac{x(0)x(t)\dot{x}(t)x(0)}{x^2(0)} < 0$, and the equilibrium point $x = 0$ is asymptotically stable.
Finally, the solution $x(t)$ of (18) is shown in Fig. 1, which is plotted by using Matlab.

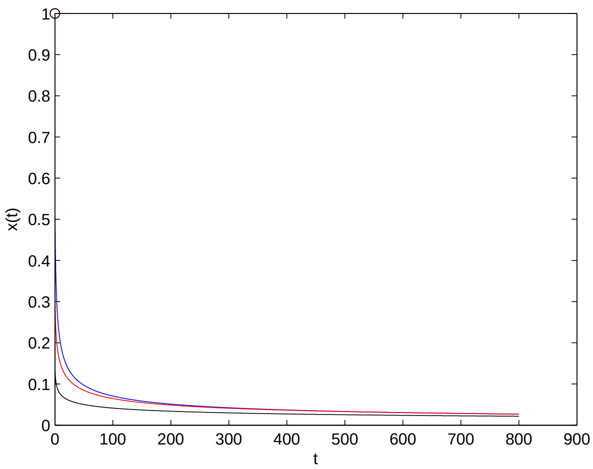

FIGURE 1. Numerical value of system (18), for the parameter value $\alpha = .095, .9, .85$

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