A NUMERICAL METHOD FOR SOLVING DELAY-FRACTIONAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

E. SOKHANVAR * AND A. ASKARI-HEMMAT **

*** DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND NEW TECHNOLOGIES, GRADUATE UNIVERSITY OF ADVANCED TECHNOLOGY, KERMAN, IRAN**

**** DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS AND COMPUTER, SHAHID BAHONAR UNIVERSITY OF KERMAN, KERMAN, IRAN**

E-MAILS: E.SOKHANVARMAHANI@STUDENT.KGUT.AC.IR, ASKARI@UK.AC.IR

(Received: 29 May 2016, Accepted: 10 May 2017)

Abstract. This article develops a direct method for solving numerically multi delay-fractional differential and integro-differential equations. A Galerkin method based on Legendre polynomials is implemented for solving linear and nonlinear of equations. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations. A convergence analysis and an error estimation are also given. Numerical results with comparisons are given to confirm the reliability of the proposed method.

AMS Classification: 34K37, 65L60.

Keywords: Delay-fractional differential and integro-differential equations, Galerkin method, Legendre polynomials.

** CORRESPONDING AUTHOR

JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER

VOL. 4, NUMBERS 1-2 (2015) 11-24.

1. Introduction

In the last few decades, many authors pointed out that fractional differential equations (FDEs) are very suitable for description of many problems in science and engineering such as, bioengineering [1], electromagnetism [2], economics [3], signal processing [4], medicine [5], continuum and statistical mechanics [6], etc. The fundamental existence and uniqueness theorems for solutions of FDEs have been presented by many authors [7], [8]. Most FDEs do not have analytical solutions, so numerical methods are required [9]-[17]. Recently, these equations have been solved by homotopy-perturbation method [18], variational iteration method [19], homotopy analysis method [20], Adomian decomposition method [21], finite difference approximation methods [22], Legendre, Bessel and Bernestein approximation methods [23]-[25], B-spline collocation methods [26], Legendre and Bernoulli wavelet methods [27], [28], and so on.

In recent years, solving delay FDEs draws increasing attention by scientists. In [29] Taylor collocation method was proposed to solve fractional pantograph equations. In [30] Bhalekar et al. investigated a fractional generalization of Bloch equation that includes both fractional derivatives and time delays. Ref. [31] presented modified Chebyshev wavelet methods and studied the convergence analysis for solving delay-fractional differential and integro-differential equations. Muthukumar and Priya [32] gave operational matrices to any interval for the differentation and integration of shifted Jacobi polynomials and applied them to solve the numerical solution of delay FDEs. Ref. [33] is devoted to the existence results for fractional neutral integro-differential evolution systems with infinite delay in Banach spaces. In [34], Baleanu et al. studied an initial value problem for a class of k-dimensional systems of fractional neutral functional differential equations with bounded delay by using Krasnoselskiis fixed point theorem. In [35], the authors proved the existence of solutions for delay FDEs at the neiborhood of its equilibrium point. Also, they obtained the birfurcation curves for a class of delay FDEs within a differential operator of Caputo type with the lower terminal at $-\infty$. Baleanu et al. [36] studied a numerical method and gave a stability analysis to solve the fractional Bloch equation with delay.

In [37]-[39], Doha et al. have presented spectral methods for solving boundary value problems. The main advantage of spectral methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In this study by means of a Galerkin method based on Legendre polynomials, we consider the approximate solution of multi delay-fractional differential and integro-differential equations in the form

$$D^{\alpha}u(x) = f(x, u(x), D^{\beta_1}u(x),..., D^{\beta_r}u(x), D^{\beta_1}u(ax-\tau), ..., D^{\beta_r}u(ax-\tau), \int_0^x g(x,t,u(at-\tau))dt, \int_{ax-\tau}^x h(x,t,u(t))dt). \tag{1}$$

The initial conditions are

$$u^{(i)}(0) = d_i, \quad i = 0, 1, ..., n, \tag{2}$$

where $n < \alpha \leq n+1$, $n \in \mathbb{N} \cup \{0\}$, $0 < \beta_1 < \beta_2 < ... < \beta_r < \alpha$, D^{α} denotes the caputo fractional derivative of order α and f, g, and h are continuous linear or nonlinear functions, τ is delay, $ax-\tau$ is called delay argumant, and d_i are constants. The fractional derivatives are defined in the caputo sense (see [7], page 79)

$$ {}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(n+1-\alpha)} \int_a^x \frac{f^{(n+1)}(t)}{(x-t)^{\alpha-n}} dt, \quad n < \alpha \leq n+1, \quad n \in \mathbb{N} \cup \{0\}.$$

Caputo's differential operator coincides with the usual differential operator of an integer order and has the property of linear operation as (see [7], page 90)

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x), \quad \forall\, \lambda, \mu \in \mathbb{R}, \tag{3}$$

where $D^{\alpha} = {}_a^C D_x^{\alpha}$. Also, Caputo fractional derivative of power function $f(x) = x^k$, $k \in \mathbb{N}$ is (see [40], page 36)

$$D^{\alpha}x^k = \begin{cases} 0 & k < \alpha \\ \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} & k \geq \alpha. \end{cases}$$

In this paper, the fractional derivatives are considered in the Caputo sense and we put ${}_a^C D_x^{\alpha} = D^{\alpha}$ in the next sections.

The paper is organized as follows: In Section 2, we give basic definitions and preliminaries. In Section 3, the convergence analysis is presented. Section 4 is devoted to the numerical method for solving multi delay-fractional differential and integro-differential equations. In Section 5, we present our method for selected examples and introduce an error estimation for proposed method. Finally a conclusion is given.

2. Preliminaries

2.1. **Shifted Legendre polynomials.** Legendre polynomials on the interval $[-1,1]$ can be determined with the following recursive formula (see [41], page 27):

$$L_0(z)=1, L_1(z)=z, L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z), i=1,2,....$$

By the change of variable $z=2x-1$, we will have the well-known shifted Legendre polynomials on $[0,1]$. These polynomials can be determined with the following recursive formula (see [41], page 27):

$$P_0(x)=1, P_1(x)=2x-1, P_{i+1}(x)=\frac{(2i+1)(2x-1)}{(i+1)}P_i(x)-\frac{i}{i+1}P_{i-1}(x), i=1,2,....$$

The analytical form of the shifted Legendre polynomials of degree i, $P_i(x)$, is as follows [23]:

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2},$$

where $P_i(0)=(-1)^i$ and $P_i(1)=1$. We also have the orthogonality conditions for $P_i(x)$ as

$$\int_0^1 P_i(x)P_j(x)dx=\begin{cases}\frac{1}{2i+1} & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$

2.2. **Function approximation in terms of shifted Legendre polynomials.** Suppose that $H=L^2([0,1])$, $m\in\mathbb{N}\cup\{0\}$, $\{P_0(x),P_1(x),...,P_m(x)\}\subset H$ be the set of shifted Legendre polynomials on $[0,1]$, $Y=span\{P_0(x),P_1(x),...,P_m(x)\}$, and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $f_1\in Y$ (see [42], page 328)

$$\|f(x)-f_1(x)\|\leq\|f(x)-g(x)\|,\ \ \forall g\in Y.$$

Since $f_1\in Y$, there exists unique coefficients $c_0,c_1,...,c_m$ such that

$$f(x)\cong f_1(x)=\sum_{j=0}^{m}c_jP_j(x).$$

By orthogonality condition of shifted Legendre polynomials, we have

$$c_j=\langle f,P_j\rangle=\int_0^1 f(x)(2j+1)P_j(x)dx,\ \ j=0,1,...,m. \tag{4}$$

3. Convergence analysis

Theorem. Let $\sum_{j=0}^{\infty} c_j P_j(x)$ be the Legendre series of $u(x) \in H = L^2([0,1])$, then $u_m(x) = \sum_{j=0}^{m} c_j P_j(x)$ convergences to $u(x)$ as $m \to \infty$.

Proof. By using relation (4), we have

$$c_j = \langle u(x), P_j(x) \rangle \,, \quad j = 0, 1, ..., m. \tag{5}$$

First we will show that the sequence of partial sums of $\sum_{j=0}^{\infty} c_j P_j(x)$, $u_m(x)$, is a Cauchy sequence in Hilbert space of H. Let $u_n(x)$ be an arbitrary partial sums of $\sum_{j=0}^{\infty} c_j P_j(x)$, i.e., $u_n(x) = \sum_{j=0}^{n} c_j P_j(x)$, and $m > n$, then we have

$$\begin{aligned}\|u_m(x) - u_n(x)\|^2 &= \| \sum_{j=n+1}^{m} c_j P_j(x) \|^2 = \left\langle \sum_{j=n+1}^{m} c_j P_j(x), \sum_{k=n+1}^{m} c_k P_k(x) \right\rangle \\ &= \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} c_j \bar{c_k} \langle P_j(x), P_k(x) \rangle = \sum_{j=n+1}^{m} \frac{1}{2j+1} \mid c_j \mid^2 < \sum_{j=n+1}^{m} \mid c_j \mid^2 .\end{aligned}$$

By Bessel' inequality, we have

$$\sum_{j=n+1}^{m} \mid c_j \mid^2 \leq \sum_{j=0}^{\infty} \mid c_j \mid^2 \leq \|u\|^2 < \infty.$$

Thus, $\|u_m(x) - u_n(x)\|^2 \to 0$ as $m, n \to \infty$, that is, $u_m(x)$ is a Cauchy sequence hence $u_m(x)$ converges to $g \in H$. Finally we show that $g(x) = u(x)$. By using relation (5) and property of continuity of inner product, we get

$$\begin{aligned}\langle g(x) - u(x), P_j(x) \rangle &= \langle g(x), P_j(x) \rangle - \langle u(x), P_j(x) \rangle \\ &= \lim_{m \to \infty} \langle u_m(x), P_j(x) \rangle - c_j \\ &= c_j - c_j = 0,\end{aligned}$$

hence $g(x) = u(x)$ and the proof is completed.

4. Numerical implementation

In this section, we consider the multi delay-fractional differential and integro-differential equation (1). We use a Galerkin method based on shifted Legendre polynomials on $[0, 1]$ to find an approximate solution of (1).

We first consider the solution $u(x)$ of equation (1) as

$$u(x) \cong u_m(x) = \sum_{j=0}^{m} c_j P_j(x), \quad 0 \leq x \leq 1, \tag{6}$$

where c_j, $j = 0, 1, ..., m$ are the unknown coefficients and $P_j(x)$, $j = 0, 1, ..., m$ are the shifted Legendre polynomials. By using the relations (3) and (6), we have

$$D^\beta u(x) \cong \sum_{j=0}^{m} c_j D^\beta P_j(x), \quad s < \beta \leq s+1, \quad s \in \mathbb{N} \cup \{0\}. \tag{7}$$

Now, we can find the approximation of $D^\beta u(ax - \tau)$ in terms of the series (7) at delay time as

$$D^\beta u(ax - \tau) \cong \sum_{j=0}^{m} c_j D^\beta P_j(ax - \tau). \tag{8}$$

Also the integral parts of the equation (1) at delay time are

$$\int_0^x g(x, t, u(ax - \tau))dt \cong \int_0^x g(x, t, \sum_{j=0}^{m} c_j P_j(at - \tau))dt, \tag{9}$$

and

$$\int_{ax-\tau}^x h(x, t, u(x))dt \cong \int_{ax-\tau}^x h(x, t, \sum_{j=0}^{m} c_j P_j(t))dt. \tag{10}$$

By substituting relations (7)-(10) in (1), we define the residual function $Res(x)$ as

$$\begin{aligned} Res(x) = &\sum_{j=0}^{m} c_j D^\alpha P_j(x) - f(x, \sum_{j=0}^{m} c_j P_j(x), \sum_{j=0}^{m} c_j D^{\beta_1} P_j(x), ..., \sum_{j=0}^{m} c_j D^{\beta_r} P_j(x), \\ &\sum_{j=0}^{m} c_j D^{\beta_1} P_j(ax - \tau), ..., \sum_{j=0}^{m} c_j D^{\beta_r} P_j(ax - \tau), \\ &\int_0^x g(x, t, \sum_{j=0}^{m} c_j P_j(at - \tau))dt, \int_{ax-\tau}^x h(x, t, \sum_{j=0}^{m} c_j P_j(t))dt). \end{aligned} \tag{11}$$

To employ the Galerkin algorithm, we choose $m > n$, then

$$\begin{cases} \int_0^1 (2j+1) Res(x) P_j(x) dx = 0, \ j = 0, 1, ..., m - n - 1, \\ \\ \sum_{j=0}^{m} c_j P_j^{(i)}(0) = d_i, \qquad i = 0, 1, ..., n. \end{cases} \tag{12}$$

Now, we have an algebraic system of $(m+1)$ equations with $(m+1)$ unknown coefficients $c_0, c_1, ..., c_m$. By solving this system of equations and using (6), we obtain the solution of (1). Note that the mentioned system of equations (12) can be linear or nonlinear.

5. Error estimation and numerical results

Now, we will obtain an error estimation for the proposed method. Let us consider $e_m(x) = u(x) - u_m(x)$ as the error function, where $u(x)$ is the exact solution of (1). Thus, $u_m(x)$ satisfies the following problem

$$D^{\alpha}u_m(x) = f(x, u_m(x), D^{\beta_1}u_m(x), ..., D^{\beta_r}u_m(x), D^{\beta_1}u_m(ax-\tau), ..., D^{\beta_r}u_m(ax-\tau), \int_0^x g(x,t,u_m(at-\tau))dt, \int_{ax-\tau}^x h(x,t,u_m(t))dt) + Res(x), \tag{13}$$

and

$$u_m^{(i)}(0) = d_i, \quad i = 0, 1, ..., n, \tag{14}$$

where $Res(x)$ is the residual function associated with $u_m(x)$ defined in (11). We proceed to find an approximation $e_{m,M}(x)$ to the $e_m(x)$ by $(M+1)$ elements of the Legendre basis, in a same way as we did before for the problem (1). Subtracting (13) and (14) from (1) and (2) respectively, the error function $e_m(x)$ satisfies in the equation

$$D^{\alpha}e_m(x) = F(x, e_m(x), D^{\beta_1}e_m(x), ..., D^{\beta_r}e_m(x), D^{\beta_1}e_m(ax-\tau), ..., D^{\beta_r}e_m(ax-\tau), \int_0^x G(x,t,e_m(ax-\tau))dt, \int_{at-\tau}^x H(x,t,e_m(t))dt) - Res(x),$$

and

$$e_m^{(i)}(0) = 0, \quad i = 0, 1, ..., n.$$

By solving this error problem in the similar way, presented in Section 4, we get the approximation $e_{m,M}(x)$. Also, in this section, we present some examples to illustrate the efficiency of proposed method in Section 4.

Example 5.1. Consider the linear delay FDE

$$\begin{cases} D^{\alpha}u(x) = -u(x) - u(x-0.3) + e^{-x+0.3}, & 0 \le x \le 1,\ 2 < \alpha \le 3, \\ u(x) = e^{-x}, & x < 0, \end{cases} \tag{15}$$

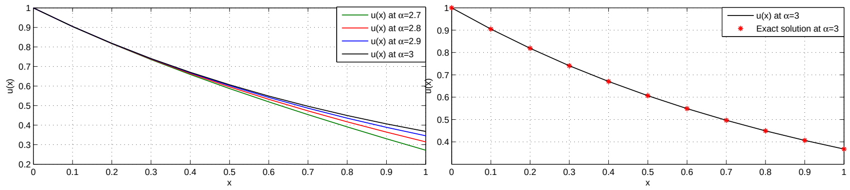

FIGURE 1. The comparison of $u(x)$ by the present method for $\alpha = 2.7$, 2.8, 2.9, 3, $m = 8$ and the exact solution at $\alpha = 3$ of Example 5.1.

with the initial conditions $u(0) = 1$, $u'(0) = -1$, $u''(0) = 1$ and the exact solution $u(x) = e^{-x}$ for $\alpha = 3$. Fig. 1 displays the comparison of $u(x)$ for $\alpha = 2.7, 2.8, 2.9, 3$, $m = 8$ and the exact solution at $\alpha = 3$. The computational results of $u(x)$ for different α and $m = 8$ together with the exact solution at $\alpha = 3$ are given in Table 1. Fig. 1 and Table 1 illustrate that the present method has a good convergence to the exact solution at $\alpha = 3$. The comparison of absolute errors by the present method with Hermit method [43] in the case of $\alpha = 3$ are shown in Table 2.

TABLE 1. The approximate solutions by the present method for $\alpha = 2.7$, 2.8, 2.9, 3 and $m = 8$ of Example 5.1.

x	Exact at $\alpha = 3$	$\alpha = 3$	$\alpha = 2.9$	$\alpha = 2.8$	$\alpha = 2.7$
0	1	1	1	1	1
0.2	0.81873075	0.81873075	0.81828725	0.81765402	0.81675161
0.4	0.67032005	0.67032005	0.66773158	0.66410535	0.65902566
0.6	0.54881164	0.54881164	0.54186684	0.53223045	0.51887618
0.8	0.44932896	0.44932896	0.43569767	0.41682932	0.39081390
1	0.36787944	0.36787944	0.34541921	0.31428085	0.27142778

TABLE 2. Comparison of the absolute errors for $\alpha = 3$ of Example 5.1.

	Present method		Hermite method [43]
x	$\lvert e_8(x)\rvert$	$\lvert e_{10}(x)\rvert$	$N = 8$
0	0	2.220×10^{-16}	0
0.2	2.211×10^{-11}	4.263×10^{-13}	6.200×10^{-9}
0.4	6.629×10^{-11}	1.301×10^{-13}	5.760×10^{-8}
0.6	1.482×10^{-10}	1.539×10^{-11}	1.796×10^{-7}
0.8	1.453×10^{-9}	4.343×10^{-11}	3.735×10^{-7}
1	3.046×10^{-9}	8.259×10^{-11}	6.368×10^{-7}

Example 5.2. Consider the linear delay-fractional intergro-differential equation

$$\begin{cases} D^{\alpha}u(x) = u(x-1) + \int_{x-1}^{x} u(t)dt, & x \geq 0,\ 0 < \alpha \leq 1, \\ u(x) = e^{x}, & x < 0, \end{cases} \tag{16}$$

with the initial condition $u(0) = 1$ and the exact solution $u(x) = e^x$ for $\alpha = 1$. The computational results of $u(x)$ for different α and $m = 6$ together with the exact solution at $\alpha = 1$ are given in Table 3. This table illustrates that the proposed method converges to the exact solution at $\alpha = 1$, when α approacges to 1. The comparison of absolute errors by the present method with Chebyshev wavelet method [31] are shown in Table 4.

TABLE 3. The approximate solutions by the present method for $\alpha = 0.7, 0.8, 0.9, 1$ and $m = 6$ of Example 5.2.

x	Exact at $\alpha = 1$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0	1	1	1	1	1
0.6	1.8221188	1.8223044	1.9076207	2.0255934	2.1664417
1	2.7182818	2.7180602	2.8001488	2.9341851	3.1128243
1.6	4.9530324	4.9535165	4.9437440	5.0520380	5.1660665
2	7.3890561	7.3897028	7.1709457	7.1587915	7.1832633
2.4	11.023176	11.023248	10.354205	10.055469	9.9027334
3	20.085537	20.086533	17.855939	16.550763	15.648826

TABLE 4. Comparison of the absolute errors for $\alpha = 1$ of Example 5.2.

	Present method		Cheby. wav. met. [31]	
x	$\lvert e_{10}(x)\rvert$	$\lvert e_{16}(x)\rvert$	$M = 10$	$M = 20$
0.3	1.7634×10^{-9}	2.2204×10^{-16}	5.8167×10^{-6}	5.6594×10^{-13}
0.9	8.2961×10^{-9}	4.4409×10^{-16}	1.9001×10^{-5}	7.1431×10^{-14}
1.5	7.8517×10^{-10}	7.7716×10^{-16}	3.0253×10^{-5}	2.2538×10^{-13}
2.1	1.7401×10^{-8}	2.7756×10^{-15}	5.6658×10^{-5}	3.2304×10^{-13}
2.4	1.4358×10^{-9}	1.8874×10^{-15}	7.6596×10^{-5}	4.2307×10^{-13}
2.7	9.1024×10^{-9}	4.5519×10^{-15}	1.0303×10^{-4}	5.9689×10^{-13}
3	4.6530×10^{-9}	6.4393×10^{-15}	1.4481×10^{-4}	8.1709×10^{-13}

Example 5.3. Consider houseflies model [32] in the form

$$(17)\qquad \begin{cases} D^\alpha u(x) = -du(x) + cu(x-\tau)(k - czu(x-\tau)), & x > 0,\ 0 < \alpha \le 1, \\ u(x) = 160, & x \in [-\tau, 0]. \end{cases}$$

In Eq. (17), we consider $\tau = 3$, $d = 0.147$, $k = 0.5107$, $c = 1.81$ and $z = 0.000226$. Fig. 2 displays the comparison of $u(x)$ by the present method for $\alpha = 0.5, 0.75, 0.9, 1$, $m = 3$ and the exact solution at $\alpha = 1$. This model is solved by Jacobi method [32] (Legendre basis: $a = 0$, $b = 0$) for $\alpha = 0.5, 0.75, 0.9, 1$, $m = 4$. Fig. 3 displays the comparison of $u(x)$ by Jacobi method for $\alpha = 0.5, 0.75, 0.9, 1$, $m = 4$ and the exact solution at $\alpha = 1$. We can see from Figs. 2 and 3 that the proposed method

TABLE 5. The approximate solutions by the present method for $\alpha = 0.5$, 0.75, 0.9, 1 and $m = 3$ of Example 5.3.

x	Exact at $\alpha = 1$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.5$
0	160.00	160.00	160.00	160.00	160.00
0.6	220.55	220.61	222.14	226.24	233.75
1.2	275.98	275.85	275.89	280.97	291.68
1.8	326.73	326.88	323.79	326.98	336.96
2.4	373.20	373.14	368.40	367.05	372.72
3	415.75	415.84	412.29	403.95	402.12
3.6	466.46	466.40	458.02	440.46	428.32
4.2	532.82	532.73	508.14	479.36	454.45
4.8	609.32	609.43	565.21	523.44	483.67
5.4	691.67	691.63	631.79	575.46	519.13
6	776.58	776.54	710.45	638.22	563.99

has better numerical results. Also, the computational results of $u(x)$ by the present method for different α and $m = 3$ together with the exact solution at $\alpha = 1$ are given in Table 5.

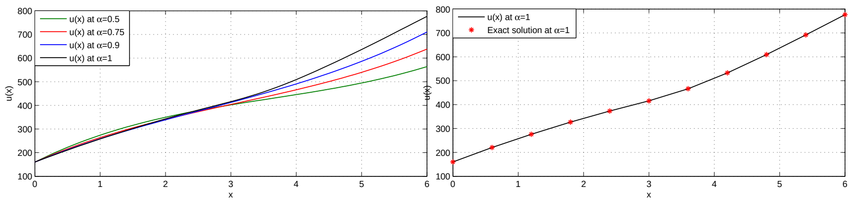

FIGURE 2. The comparison of $u(x)$ by the present method for $\alpha = 0.5$, 0.75, 0.9, 1, $m = 3$ and the exact solution at $\alpha = 1$ of Example 5.3.

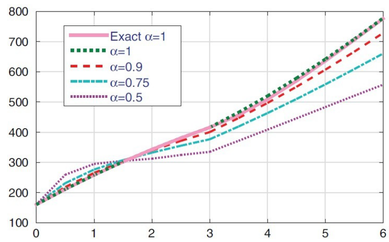

FIGURE 3. The comparison of $u(x)$ by Jacobi method [32] for $\alpha = 0.5$, 0.75, 0.9, 1, $m = 4$ and the exact solution at $\alpha = 1$ of Example 5.3.

TABLE 6. The approximate solutions by the present method for $\alpha = 0.3$, 0.6, 0.9, 1 and $m = 9$ of Example 5.4.

x	Exact at $\alpha = 1$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.6$	$\alpha = 0.3$
0	0	0	0	0	0
0.2	0.19866933	0.19866933	0.24493930	0.49616749	0.69498967
0.4	0.38941834	0.38941834	0.44437968	0.68633782	0.67100691
0.6	0.56464247	0.56464247	0.61226150	0.77695869	0.68988345
0.8	0.71735609	0.71735609	0.74656901	0.80518248	0.70040363
1	0.84147098	0.84147098	0.84101027	0.80351705	0.72725857

Example 5.4. Consider the nonlinear delay FDE

$$D^{\alpha}u(x) = 1 - 2u^2(\frac{x}{2}), \quad 0 \le x \le 1, \quad 0 < \alpha \le 1, \tag{18}$$

with the initial condition $u(0) = 0$ and the exact solution $u(x) = \sin(x)$ for $\alpha = 1$ [44]. In Table 6, a comparison between the numerical results of the approximate solutions obtained by the present method for $\alpha = 0.3$, 0.6, 0.9, 1 and $m = 9$ and the exact solution at $\alpha = 1$ are given. This table illustrates that the proposed method converges to the exact solution at $\alpha = 1$, when α approacges to 1.

6. CONCLUSION

In the present work, we developed a direct method for solving multi delay-fractional differential and integro-differential equations. By utilizing the Legendre basis and Galerkin method, we reduced the main problem to the problem of solving a system of algebraic equations. Comparing the present method with several other methods that have been advanced for solving our problems shows that the present technique is reliable and powerful.

REFERENCES

[1] R. L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng., 32 (1) (2004) 1–104.

[2] N. Engheta, On fractional calculus and fractional multipoles in electromagnetism, IEEE. T. Antenn. Propag., 44 (4) (1996) 554–566.

[3] R. T. Baillie, Long memory processes and fractional integration in econometrics, J. Econ., 73 (1996) 5–59.

[4] Y. Z. Povstenko, Signaling problem for time-fractional diffusion-wave equation in a half-space in the case of angular symmetry, Nonl. Dyn., 59 (4) (2010) 593–605.

[5] M. G. Hall, T. R. Barrick, From diffusion-weighted MRI to anomalous diffusion imaging, Magn. Reson. Med., 59 (3) (2008) 447–455.

[6] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics. In: A. Carpinteri, F. Mainardi, (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291– 348. Springer, New York, 1997.

[7] I. Podlubny, Fractional Differential Equations, Acadmic Press, New York, NY, USA, 1993.

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, San Diego, 2006.

[9] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer, New York, 2008.

[10] K. Diethelm, An algorithm for the numerical solution of differential equations of fractional order, Electron. Trans. Numer. Anal., 5 (1997), 1–6.

[11] J. A. Machado, And I say to myself: "What a fractional world !", Frac. Calc. Appl. Anal., 14 (4) (2011), 635–654.

[12] R. M. Evans, U. N. Katugampola, D. A. Edwards, Applications of fractional calculus in solving Abel-type integral equations: Surface-volume reaction problem, Comput. Math. Appl., 00 (2016) 1–19.

[13] M. Enelund, B. L. Josefson, Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations, AIAAJ., 35 (10) (1997), 1630–1637.

[14] I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPs, Commun. Nonlinear Sci. Numer. Simul., 14 (2009), 674–684.

[15] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering, 167 (1–2) (1998) 57–68.

[16] M. M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, J. Comput. Appl. Math., 172 (1) (2004) 65–77.

[17] N. H. Sweilam, M. M. Khader, A. M. Nagy, Numerical solution of two-sided space fractional wave equation using finite difference method, Comput. Appl. Math., 235 (2011), 2832–2841.

[18] O. Abdulaziz, I. Hashim, S. Momani, Solving systems of fractional differential equations by homotopy-perturbation method. Phys. Lett. A, 372 (4) (2008) 451–459.

[19] S. Momani, Z. Odibat, Numerical approach to differential equations of fractional order, J. Comput. Appl. Math., 207 (1) (2007) 96110.

[20] H. Jafari, K. Sayevand, H. Tajadodi, D. Baleanu, Homotopy analysis method for solving Abel differential equation of fractional order, Cen. Eur. J. Phys., 11 (10) (2013) 1523–1527.

[21] V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation using adomian decomposition, Appl. Math. Comput., 189 (1) (2007) 541–548.

[22] S. B. Yuste, Weighted average finite difference methods for fractional diffusion equations, J. Comput. Phys., 216 (1) (2006) 264–274.

[23] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, Comput. Math. Appl., 59 (3) (2010) 1326–1336.

[24] K. Parand, M. Nikarya, Application of Bessel functions for solving differential and integro-differential equations of the fractional order, Appl. Math. Model., 38 (15-16) (2014) 4137–4147.

[25] A. Saadatmandi, Bernstein operational matrix of fractional derivatives and its applications, Appl. Math. Model., 38 (4) (2014) 1365–1372.

[26] M. Lakestani, M. Dehghan, S. Irandoust-pakchin, The construction of operational matrix of fractional derivatives using B-spline functions, Commun. Nonlinear Sci. Numer. Simul., 17 (3) (2012) 1149–1162.

[27] M. ur Rehman, R. Ali Khan, The Legendre wavelet method for solving fractional differential equations, Commun. Nonl. Sci. Numer. Simul., 16 (11) (2011) 4163–4173.

[28] E. Keshavarz, Y. Ordokhani, M. Razzaghi, Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, Appl. Math. Model., 38 (24) (2014) 6038–6051.

[29] A. Anapali, Y. Öztürk, M. Gülsu, Numerical approach for solving fractional pantograph equations, Int. J. Comput. Appl., 113 (2015) 45–52.

[30] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, R. Magin, Fractional Bloch equation with delay, Comput. Math. Appl., 61 (2011) 1355–1365.

[31] U. Saeed, M. ur Rehman, M. A. Iqbal, Modified Chebyshev wavelet methods for fractional delay-type equations, Appl. Math. Comput., 264 (2015) 431–442.

[32] P. Muthukumar, B. Ganesh Priya, Numerical solution of fractional delay differential equation by shifted Jacobi polynomials, Int. J. Comput. Math., (2015) in press.

[33] C. Ravichandran, D. Baleanu, Existence results for fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces, Advances in Difference Equations, 215 (2013) 1687–1847.

[34] D. Baleanu, S. Z. Nazemi, S. Rezapour, A k-dimensional system of fractional neutral functional differential equations with bounded delay, Abstract and Applied Analysis, 2014 (2014).

[35] A. Babakhani, D. Baleanu, R. Khanbabaie, Hopf bifurcation for a class of fractional differential equations with delay, Nonlinear Dyn., 69 (2012) 721–729.

[36] D. Baleanu, R. L. Magin, S. Bhalekar, V. Daftardar-Gejji, Chaos in the fractional order nonlinear Bloch equation with delay, Commun. Nonlinear Sci. Numer. Simul., (2015).

[37] E. H. Doha, W. M. Abd-Elhameed, A. H. Bhrawy, Efficient spectral ultraspherical-Galerkin algorithms for the direct solution of 2nth-order linear differential equations, Appl. Math. Modell., 33 (2009) 1982–1996.

[38] E. H. Doha, A. H. Bhrawy, Efficient spectral-Galerkin algorithms for direct solution for second-order differential equations using Jacobi polynomials, Numer. Algor., 42 (2006) 137–164.

[39] E. H. Doha, A. H. Bhrawy, Efficient spectral-Galerkin Algorithms for direct solution of fourth-order differential equations using Jacobi polynomials, Appl. Numer. Math., 58 (2008) 1224–1244.

[40] K.S. Miller, B. Ross, An Introduction to The Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[41] W. Gautschi, Orthogonal polynomials: Computation and application, Oxford university press, New York, 2004.

[42] E. Kreyszig, Introductory functional analysis with applications, John Wiley, New York, 1978.

[43] S. Yalçinbaş, M. Aynigül, M. Sezer, A collocation method using Hermite polynomials for approximate solution of pantograph equations, Journal of the Franklin Institute, 348 (2011) 1128–1139.

[44] M. S. Hafshejani, S. K. Vanani, J. S. Hafshejani, Numerical solution of delay differential equations using Legendre wavelet method, World Appl. Sci. J., 13 (2011) 27–33.