# ON THE FINE SPECTRA OF THE GENERALIZED DIFFERENCE OPERATOR $\Delta_{uv}$ OVER THE SEQUENCE SPACE $c_0$

**JAVAD FATHI AND RAHMATOLLAH LASHKARIPOUR**
DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS, UNIVERSITY OF SISTAN AND BALUCHESTAN, ZAHEDAN, IRAN
E-MAILS: FATHI756@GMAIL.COM, LASHKARI@HAMOON.USB.AC.IR

Abstract. The main purpose of this paper is to determine the fine spectrum of the generalized difference operator $\Delta_{uv}$ over the sequence space $c_0$. These results are more general than the fine spectrum of the generalized difference operator $\Delta_v$ of Srivastava and Kumar [17].

**AMS Classification:** 47A10, 47B37.
**Keywords:** Spectrum of an operator; Matrix mapping; Difference operator; Sequence space.

## 1. Preliminaries, Background and Notations

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

*JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER*
*VOL. 1, NUMBER 1 (2012) 1-12.*

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space $\ell_p$ for $(1 < p < \infty)$ has been studied by Gonzalez [11]. Also, Wenger [18] examined the fine spectrum of the integer power of the Cesaro operator over $c$, and Rhoades [16] generalized this result to the weighted mean methods. Reade [15] worked the spectrum of the Cesaro operator over the sequence space $c_0$. Okutoyi [14] computed the spectrum of the Cesaro operator over the sequence space $bv$. The fine spectrum of the Rhally operators on the sequence spaces $c_0$ and $c$ is studied by Yildirim [20]. The fine spectra of the Cesaro operator over the sequence spaces $c_0$ and $bv_p$ have determined by Akhmedov and Basar [1, 4]. Akhmedov and Basar [2, 3] have studied the fine spectrum of the difference operator $\Delta$ over the sequence spaces $\ell_p$, and $bv_p$, where $(1 \leq p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces $\ell_1$ and $bv_1$ have been examined by Altay and Karakus [6]. Altay and Basar [5, 9] have determined the fine spectrum of the difference operator $\Delta$ over the sequence spaces $c_0$, $c$ and $\ell_p$, where $(0 < p < 1)$. The fine spectrum of the difference operator $\Delta$ over the sequence spaces $\ell_1$ and $bv$ is investigated by Kayaduman and Furkan [12]. Altun and Karakaya [7, 8] has been studied the fine spectra of Lacunary matrices and fine spectra of upper triangular double-band matrices. recently, Srivastava and Kumar [17] has been examined the fine spectrum of the generalized difference operator $\Delta_v$ over the sequence space $c_0$.

In this work, our purpose is to determine the fine spectra of the generalized forward difference operator $\Delta_{uv}$ as an operator over the sequence space $c_0$.

By $w$, we denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called a sequence space. Let $\mu$ and $\nu$ be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. We say that $A$ defines a matrix mapping from $\mu$ into $\nu$ and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = \big(x_k\big) \in \mu$ the sequence $Ax = \big((Ax)_n\big)$, the $A$-transform of $x$, is in $\nu$, where $(Ax)_n = \sum\limits_{k=0}^{\infty} a_{n,k}x_k$. We say that $A \in (\mu, \nu)$ if and only if $Ax \in \nu$ whenever $x \in \mu$.

Let $X$ and $Y$ be Banach spaces and $T : X \longrightarrow Y$, also be a bounded linear operator. By $R(T)$, we denote the range of $T$, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operator on $X$ into itself. If $X$ is any Banach space and $T \in B(X)$ then the *adjoint* $T^*$ of $T$ is a bounded linear operator on the dual $X^*$ of $X$ defined by $(T^*\psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.
Let $X \neq \emptyset$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, also be a bounded linear operator with domain $\mathcal{D} \subseteq X$. With $T$, we associate the operator $T_\lambda = T - \lambda I$, where $\lambda$ is a complex number and $I$ is the identity operator on $\mathcal{D}(T)$, if $T_\lambda$ has an inverse, which is linear, we denote it by $T_\lambda^{-1}$, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and call it the *resolvent* operator of $T$.

The name resolvent is appropriate, since $T_\lambda^{-1}$ helps to solve the equation $T_\lambda x = y$. Thus, $T_\lambda^{-1}$ exists provided the solution $x = T_\lambda^{-1} y$ . More important, the investigation of properties of $T_\lambda^{-1}$ will be basic for an understanding of the operator $T$ itself. Naturally, many properties of $T_\lambda$ and $T_\lambda^{-1}$ (when it exists) depend on $\lambda$, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all $\lambda$ in the complex plane such that $T_\lambda^{-1}$ exists. Boundedness of $T_\lambda^{-1}$ is another property that will be essential. Also we shall ask for what $\lambda$ the domain of $T_\lambda^{-1}$ is dense in $X$, to name just a few aspects. For our investigation of $T$, $T_\lambda$ and $T_\lambda^{-1}$, we shall need some basic concepts in spectral theory which are given as follows(see [10, pp. 370-371]):

**Definition 1.1.** *Let $X \neq \emptyset$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, be a linear operator with domain $\mathcal{D} \subseteq X$. A regular value of $T$ is a complex number $\lambda$ such that*
$(R1)$ $T_\lambda^{-1}$ *exists,*
$(R2)$ $T_\lambda^{-1}$ *is bounded,*

$(R3)$ $T_\lambda^{-1}$ *is defined on a set which is dense in* $X$.

The *resolvent* set $\rho(T,X)$ of $T$ is the set of all *regular* value $\lambda$ of $T$. Its complement $\sigma(T,X)=\mathbb{C}-\rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of $T$. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:
The *point spectrum* $\sigma_p(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that $T_\lambda^{-1}$ dose not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of $T$.
The *continuous spectrum* $\sigma_c(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that $T_\lambda^{-1}$ exists and satisfies $(R3)$ but not $(R2)$, that is, $T_\lambda^{-1}$ is unbounded.
The *residual spectrum* $\sigma_r(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that $T_\lambda^{-1}$ exists but do not satisfy $(R3)$, that is, the domain of $T_\lambda^{-1}$ is not dense in $X$. The condition $(R2)$ may or may not holds good.

**Goldberg's classification of operator** $T_\lambda=(T-\lambda I)$ (see $[10], PP.\ 58-71$): Let $X$ be a Banach space and $T_\lambda=(T-\lambda I)\in B(X)$, where $\lambda$ is a complex number. Again let $R(T_\lambda)$ and $T_\lambda^{-1}$ be denote the range and inverse of the operator $T_\lambda$, respectively. Then following possibilities may occur:
$(A)$ $R(T_\lambda)=X$,
$(B)$ $R(T_\lambda)\neq\overline{R(T_\lambda)}=X$,
$(C)$ $\overline{R(T_\lambda)}\neq X$,
and
(1) $T_\lambda$ is injective and $T_\lambda^{-1}$ is continuous,
(2) $T_\lambda$ is injective and $T_\lambda^{-1}$ is discontinuous,
(3) $T_\lambda$ is not injective.
If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1,A_2,A_3,B_1,B_2,B_3,C_1,C_2$ and $C_3$. If $\lambda$ is a complex number such that $T_\lambda\in A_1$ or $T_\lambda\in B_1$, then $\lambda$ is in the resolvent set $\rho(T,X)$ of $T$ on $X$. The other classifications give rise to the fine spectrum of $T$. We use $\lambda\in\sigma_{B_2}(T,X)$ means the operator $T_\lambda\in B_2$, i.e. $R(T_\lambda)\neq\overline{R(T_\lambda)}=X$ and $T_\lambda$ is injective but $T_\lambda^{-1}$ is discontinuous. Similarly others.

**Lemma 1.2.** *([10], p.59). A linear operator $T$ has a dense range if and only if the adjoint $T^*$ is one to one.*

**Lemma 1.3.** *([10], p.60). The adjoint operator $T^*$ is onto if and and only if $T$ has a bounded inverse.*

**Lemma 1.4.** *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from $c_0$ to itself if and only if*
(1) *the rows of $A$ in $\ell_1$ and their $\ell_1$ norms are bounded.*
(2) *the columns of $A$ are in $c_0$.*

**Note**: The operator norm of $T$ is the supremum of the $\ell_1$ norms of rows.

In this paper, we introduce a class of a generalized difference operator $\Delta_{uv}$ over space $c_0$.
Let $u = (u_k)$ be a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $U = \lim_{k\to\infty} u_k \neq 0$ and $v = (v_k)$ is either constant or strictly decreasing sequence of positive real numbers with $V = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k < U + V$. We define the operator $\Delta_{uv}$ on sequence space $c_0$ as follows:

$$\Delta_{uv}x = \Delta_{uv}(x_n) = (u_{n-1}x_{n-1} + v_n x_n)_{n=0}^{\infty}. \text{ with } x_{-1} = 0$$

It is easy to verify that the operator $\Delta_{uv}$ can be represented by the matrix,

$$\Delta_{uv} = \begin{bmatrix} v_0 & 0 & 0 & 0 & 0 & \cdots \\ u_0 & v_1 & 0 & 0 & 0 & \cdots \\ 0 & u_1 & v_2 & 0 & 0 & \cdots \\ 0 & 0 & u_2 & v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

## 2. Main Results

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized forward difference operator $\Delta_{uv}$

over the sequence space $c_0$.

**Theorem 2.1.** *The operator* $\Delta_{uv} : c_0 \longrightarrow c_0$ *is a bounded linear operator and*

$$\|\Delta_{uv}\|_(c_0, c_0) = \sup_k (|u_{k-1}| + |v_k|).$$

*P*roof: It is elementary.

**Theorem 2.2.** $\sigma_p(\Delta_{uv}, c_0) = \emptyset$.

*P*roof: The proof of this theorem is divided into two cases.
Case(i): Suppose $(v_k)$ is a constant sequence, say $v_k = V$ for all $k$. Consider $\Delta_{uv}x = \lambda x$, for $x \neq \mathbf{0} = (0, 0, 0, \ldots)$ in $c_0$, which gives

$$\begin{gathered} v_0 x_0 = \lambda x_o \\ u_0 x_0 + v_1 x_1 = \lambda x_1 \\ u_1 x_1 + v_2 x_2 = \lambda x_2 \\ \vdots \\ u_k x_k + v_{k+1} x_{k+1} = \lambda x_{k+1} \\ \vdots \end{gathered}$$

Let $x_m$ be the first non-zero entry of the sequence $x = (x_n)$. So we get $u_{m-1}x_{m-1} + v_m x_m = \lambda x_m$ which implies $\lambda = v_m$ and from the equation $u_m x_m + v_{m+1} x_{m+1} = \lambda x_{m+1}$ we get $x_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p(\Delta^v, c_0) = \emptyset.$$

Case(ii): Suppose $(v_k)$ is a strictly decreasing sequence. Consider $\Delta_{uv}x = \lambda x$, for $x \neq \mathbf{0} = (0, 0, 0, \ldots)$ in $c_0$, which gives system of equations, above. Hence, for all $\lambda \notin \{v_0, v_1, v_2, \ldots\}$, we have $x_k = 0$ for all $k$, which is a contradiction. So $\lambda \notin \sigma_p(\Delta_{uv}, c_0)$. This shows that

$$\sigma_p(\Delta_{uv}, c_0) \subseteq \{v_0, v_1, v_2, \ldots\}.$$

Let $\lambda = v_m$ for some $m$. Then $x_0 = x_1 = \ldots = x_{m-1} = 0$. Now if $x_m = 0$, then $x_k = 0$ for all $k$, which is a contradiction. Also if $x_m \neq 0$, then

$$x_{k+1} = \frac{u_k}{v_{k+1} - v_m} x_k, \ \textit{for all } k \geq m,$$

and hence,

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \lim_{k\to\infty} \left| \frac{u_k}{v_{k+1} - v_m} \right| = \left| \frac{U}{v_m - V} \right| > 1,$$

since $\sup_k v_k < U + V$. Then, $x \notin c_0$. Thus

$$\sigma_p(\Delta_{uv}, c_0) = \emptyset.$$

If $T : c_0 \longrightarrow c_0$ is a bounded linear operator with matrix $A$, then it is known that the adjoint operator $T^* : c_0^* \longrightarrow c_0^*$ is defined by the transpose of the matrix $A$. The dual space of $c_0$ is isomorphic to $\ell_1$, the space of all absolutely summable sequences, with the norm $\|x\| = \sum\limits_{k=0}^{\infty} |x_k|$.
We now obtain spectrum of the dual operator $\Delta_{uv}^*$ of $\Delta_{uv}$ over the space $c_0^*$.

**Theorem 2.3.** $\{\lambda \in \mathbb{C} : |\lambda - V| < U\} \subseteq \sigma_p(\Delta_{uv}^*, c_0^*)$.

*P*roof: Suppose $\Delta_{uv}^* y = \lambda y$, for $y \neq \mathbf{0} = (0, 0, 0, \ldots)$ in $\ell_1$, where

$$\Delta_{uv}^* = \begin{bmatrix} v_0 & u_0 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_1 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_2 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \ \textit{and } y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

this gives

$$v_0 y_0 + u_o y_1 = \lambda y_o$$

$$v_1 y_1 + u_1 y_2 = \lambda y_1$$

$$v_2 y_2 + u_2 y_3 = \lambda y_2$$

$$\vdots$$

$$v_k y_k + u_k y_{k+1} = \lambda y_k$$

$$\vdots$$

If $y_0 = 0$, then $y_k = 0$ for all $k$. Hence $y_0 \neq 0$ and solving the equation above, we get

$$y_{k+1} = \left(\frac{\lambda - v_k}{u_k}\right) y_k \ for \ all \ k \geq 0.$$

And consequently

$$\lim_{k\to\infty} \left|\frac{y_{k+1}}{y_k}\right| = \left|\frac{V - \lambda}{U}\right| < 1 \ provided \ |V - \lambda| < U.$$

Hence

$$|V - \lambda| < U \Rightarrow y = (y_k) \in \ell_1$$

this shows that

$$\{\lambda \in \mathbb{C} : |\lambda - V| < U\} \subseteq \sigma_p(\Delta_{uv}^*, c_0^*).$$

The following example shows that the equality in Theorem 2.3, do not hold, in general.

**Example 2.1** Suppose that $v_k = \left(\frac{k+1}{k+3}\right)^2$ and $u_k = \left(\frac{k+1}{k+2}\right)^2$, $k \in \mathbb{N}$. Then $\lim_{k\to\infty} u_k = U = 1$ and $\lim_{k\to\infty} v_k = V = 1$. Clearly, $0 \notin \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$. But $0 \in \sigma_p(\Delta_{uv}^*, c_0^*)$ since there exists $y = (y_0.y_1, y_2, ...)$ such that $y_0 \neq 0$, $y_1 \neq 0$ and $y_{k+1} = \frac{-v_{k-1}}{u_{k-1}} y_k$, $k \geq 1$ and

$$\sum |y_k| = |y_0| + |y_1| + 4|y_1| \sum_{k=2}^{\infty} \left(\frac{1}{k+2}\right)^2 < \infty.$$

**Theorem 2.4.** $\sigma_r(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$.

*P*roof: We show that the operator $\Delta_{uv} - \lambda I$ has an inverse and $\overline{R(\Delta_{uv} - \lambda I)} \neq c_0$ for $\lambda$ satisfying $|\lambda - V| < U$. If $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$, then the operator $\Delta_{uv} - \lambda I$ is a triangle except for $\lambda = V$ (when $(v_k)$ is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and consequently the operator $\Delta_{uv} - \lambda I$ has an inverse. Further by Theorem 2.2, the operator $\Delta_{uv} - \lambda I$ is one to one for $\lambda = V$ (when $(v_k)$ is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and hence has an inverse. But $\Delta_{uv}^* - \lambda I$ is not one to one by Theorem 2.3. Now Lemma 1.2 yields the fact that the range of the operator $\Delta_{uv} - \lambda I$ is not dense in $c_0$ and this step completes the proof.

**Theorem 2.5.** $\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| \leq U\}$.

*Proof*: Let $\lambda \in \mathbb{C}$ with $|\lambda - V| > U$. Clearly, $\lambda = V$ as well as $\lambda = v_k$, for all $k$ do not satisfied. So, $\lambda \neq V$ and $\lambda \neq v_k$, for all $k$. We get the operator $\Delta_{uv} - \lambda I = (a_{nk})$ is a triangle and hence has an inverse. Thus, $(\Delta_{uv} - \lambda I)^{-1} = (b_{nk})$ where

$$b_{nk} = \begin{cases} \frac{(-1)^{n+k}}{(v_n - \lambda)} \prod_{i=k}^{n-1} \left(\frac{u_i}{v_i - \lambda}\right) & n > k \\ \frac{1}{v_n - \lambda} & n = k \\ 0 & n < k. \end{cases}$$

Now we show that $(\Delta_{uv} - \lambda I)^{-1} \in B(c_0)$. Let $R_n = \sum\limits_{k=0}^{\infty} |b_{nk}|$ then,

$$R_n = \frac{1}{|v_n - \lambda|} \left(1 + \sum_{k=0}^{n-1} \prod_{i=k}^{n-1} \left|\frac{u_i}{v_i - \lambda}\right|\right).$$

Clearly, for each $n \in \mathbb{N}$, the series $\sum\limits_{k=0}^{\infty} |b_{nk}|$ is convergent.

Next, we show that $\sup\limits_n R_n < \infty$. Let $\alpha = \lim_{n\to\infty} \left|\frac{u_{n-1}}{v_n - \lambda}\right|$ then $\alpha = \left|\frac{U}{V - \lambda}\right|$ which shows that $0 < \alpha < 1$ and so

$$\lim_{n\to\infty} \frac{1}{|v_n - \lambda|} = \lim_{n\to\infty} \left(\left|\frac{u_{n-1}}{v_n - \lambda}\right| \left|\frac{1}{u_{n-1}}\right|\right) = \frac{\alpha}{U}.$$

We have,

$$R_n = \left|\frac{u_{n-1}}{v_n - \lambda}\right| R_{n-1} + \left|\frac{1}{v_n - \lambda}\right|.$$

Then $\lim_{n\to\infty} R_n = \alpha \lim_{n\to\infty} R_{n-1} + \frac{\alpha}{U}$, consequently

$$\lim_{n\to\infty} R_n = \frac{\alpha}{(1-\alpha)U} < \infty.$$

Since $(R_n)$ is a convergent sequence of positive real numbers, we have $\sup\limits_n R_n < \infty$. Again since $\alpha = \lim_{n\to\infty} \left|\frac{u_{n-1}}{v_n - \lambda}\right| < 1$, therefore $\left|\frac{u_{n-1}}{v_n - \lambda}\right| < 1$, for large $n$ and consequently

$$\lim_{n\to\infty} |b_{(n0)}| = \lim_{n\to\infty} \left|\frac{(-1)^n}{(v_0 - \lambda)} \prod_{i=1}^{n} \left(\frac{u_{i-1}}{v_i - \lambda}\right)\right| = 0.$$

similarly, we can show that $\lim_{n\to\infty} |b_{(nk)}| = 0$, for all $k = 1, 2, 3, \ldots$.

Thus

$$\lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| > U\} \Rightarrow (\Delta_{uv} - \lambda I)^{-1} \in B(c_0).$$

On the other hand since $(\Delta_{uv} - \lambda I)^{-1} \in (c_0, c_0)$, we have equivalently $\overline{R(\Delta_{uv} - \lambda I)} = c_0$ which means that $\overline{D((\Delta_{uv} - \lambda I)^{-1})} = c_0$. This shows that

$$\sigma(\Delta_{uv}, c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - V| \leq U\}.$$

Combining this with Theorem 2.4, we get

$$\{\lambda \in \mathbb{C} : |\lambda - V| < U\} \subseteq \sigma(\Delta_{uv}, c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - V| \leq U\}.$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| \leq U\}.$$

**Theorem 2.6.** $\sigma_c(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| = U\}$.

*P*roof: Since $\sigma_r(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$, $|\lambda - V| < U\}$, $\sigma_p(\Delta_{uv}, c_0) = \emptyset$ and $\sigma(\Delta_{uv}, c_0)$ is the disjoint union of the parts $\sigma_p(\Delta_{uv}, c_0)$, $\sigma_r(\Delta_{uv}, c_0)$ and $\sigma_c(\Delta_{uv}, c_0)$, we deduce that

$$\sigma_c(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - V| = U\}.$$

**Theorem 2.7.** *If $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| > U\}$, then $\Delta_{uv} - \lambda I \in A_1$.*

*P*roof: Since $\lambda \neq v_k$, therefore the operator $\Delta_{uv} - \lambda I$ is triangle. Hence it has inverse and in the proof of Theorem 2.5 we show that

$$\lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| > U\} \Rightarrow (\Delta_{uv} - \lambda I)^{-1} \in B(c_0).$$

this is equivalent to the fact that the operator $(\Delta_{uv} - \lambda I)^{-1}$ is continues for $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| > U\}$. Also, since $(\Delta_{uv} - \lambda I)^{-1} \in (c_0, c_0)$, therefore for every $y \in c_0$, we can find $c_0$, we can find $x \in c_0$ such that $(\Delta_{uv} - \lambda I)x = y$, i.e.

$\Delta_{uv} - \lambda I \in A_1$.

**Theorem 2.8.** *Let $v_k$ be a constant sequence, say $v_k = V$ and $\lambda \neq V, \lambda \in \sigma_r(\Delta_{uv}, c_0)$. Then $\lambda \in C_2$.*

$P$roof: Since $\lambda \neq V$, therefore the operator $(\Delta_{uv} - \lambda I)$ is triangle. Hence it has an inverse. Now suppose $V \neq \lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$. Then $\alpha = \lim_{n\to\infty} \left|\frac{u_{n-1}}{v_n - \lambda}\right| > 1$ this means that $\left|\frac{u_{n-1}}{v_n - \lambda}\right| > 1$, for large $n$ and so

$$\lim_{n\to\infty} |b_{(n0)}| = \lim_{n\to\infty} \left| \frac{(-1)^n}{(v_0 - \lambda)} \prod_{i=1}^{n} \left( \frac{u_{i-1}}{v_i - \lambda} \right) \right| \neq 0.$$

Hence

$$V \neq \lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| < U\} \Rightarrow (\Delta_{uv} - \lambda I)^{-1} \notin B(c_0)$$

this is equivalent to the operator $(\Delta_{uv} - \lambda I)^{-1}$ is discontinues for $V \neq \lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$. Also by Theorem 2.3 and Lemma 1.2 for $V \neq \lambda \in \{\lambda \in \mathbb{C} : |\lambda - V| < U\}$ we have $\overline{R(\Delta_{uv} - \lambda I)} \neq c_0$ and hence $\lambda \in C_2$.

Now, we may give the consequence on the non-compactness of the operator $\Delta_{uv}$.

**Definition 2.9.** *Let $X$ and $Y$ be the normed spaces. An operator $T : X \to Y$ is called a compact linear operator if $T$ is linear and and if for every bounded subset $M$ of $X$, the image $T(M)$ is relatively compact.*

**Lemma 2.10** (13, p.432)**.** . *Let $T : X \to X$ be a compact linear operator on a Banach spaces $X$. Then spectral value $\lambda$ of $T$, if exists, is a eigenvalue of $T$.*

combining the consequences obtained in Theorems 2.2-2.6 with Lemma 2.10 we have:

**Corollary 2.11.** *The operator $\Delta_{uv}$ is not compact.*

**Acknowledgement(s) :**
The authors thank the referees for their careful reading of the manuscript and insightful comments.

## References

[1] Ali M. Akhmedov, Feyzi Başar, On the spectrum of the cesaro operator in $c_0$. Math. J. Ibaraki Univ., 36 (2004), 25-32.

[2] Ali M. Akhmedov, Feyzi Başar, The fine spectra of the difference operator $\Delta$ over the sequence space $\ell_p$, $(1 \leq p < \infty)$. Demonstratio Math., 39 (2006), 586-595.

[3] Ali M. Akhmedov, Feyzi Başar, On the fine spectra of the difference operator $\Delta$ over the sequence space $(1 \leq p < \infty)$. Acta. Math. Sin. Eng. Ser. Oct., Vol. 23 ser. Oct. Vol. 23 No.10 (2007), 1757-1768.

[4] Ali M. Akhmedov, Feyzi Başar, The fine spectra of cesaro operator $C_1$ over the sequence space $bv_p$, $(1 \leq p < \infty)$. Math. J. Okayama Univ. 50 (2008), 135-147. 50(2008), 135-147.

[5] Bilal Altay , Feyzi Başar, The fine spectrum and the matrix domain of the difference operator $\Delta$ on the sequence space $\ell_p$ $(0 < p < 1)$, Commun. Math. Anal. Volume 2, Number 2 (2007), 1-11.

[6] Bilal Altay , Mahmut Karakuş, On the spectrum and fine spectrum of the Zweier matrix as an operator on some sequence spaces, Thai J. Math. Vol. 3 No.2 (2005), 153-162.

[7] Mumammed Altun, Vatan Karakaya, Fine spectra of Lacunary Matrices, commun. Math. Anal. Volume 7, Number 1 (2009), 1-10.

[8] Mumammed Altun, Vatan Karakaya, Fine spectra of upper triangular double-band matrices, J. comput. Appl. Math. 234 (2010), 1387-1394.

[9] Feyzi Başar, Bilal. Altay, On the fine spectrum of the difference operator on $c_0$ and $c$, Inform. Sci. 168 (2004), 217-224.

[10] Seymour Goldberg, Unbounded Linear Operator, Dover publications, Inc. New York, 1985.

[11] Manuel Gonzalez, The fine spectrum of the cesaro operator in $\ell_p$, $(1 < p < \infty)$. Arch. Math.(Basel) 44 (1985), 355-358.

[12] Kuddusi Kayaduman, Hasan Furkan, The fine spectra of the difference operator $\Delta$ over the sequence spaces $\ell_1$ and $bv$. Int. Math. Forum, 1, 24 (2006), 1153-1160.

[13] Ervin Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons Inc. New York- chichester- Brisbane-Toronto, 1978.

[14] J.T. Okutoyi, On the spectrum of $c_1$ as an operator on $bv$. commun. Fac. Sci. Univ. Ank. Ser. $A_1$, math. stat 41 (1992), 197-207.

[15] J.B. Reade, On the spectrum of the cesaro operator. Bull. Lond. Math. Soc. 17 (1985), 263-267.

[16] Billy. E. Rhoades, The fine spectra for weighted mean operators, Pacific J. Math. 104, 1 (1983), 263-267.

[17] Parmeshwary. D. Srivastava, Sudhanshu Kumar, On the fine spectrum of the generalized difference operator $\Delta_v$ over the sequence space $c_0$, Commun. Math. Anal. Volume 6, Number 1 (2009), 8-21.

[18] R.B. Wenger, The fine spectra of Hölder summability operators. Indian J. Pure Appl. Math., 6 (1975), 695-712.

[19] Albert Wilansky, Summability through Functional Analysis, North Holland, 1984.

[20] Mustafa Yildirim, On the spectrum and fine spectrum of the compact Rhally operators. Indian J. Pure Appl. Math. 27, 8 (1996), 779-784.