DECOMPOSITION OF IDEALS INTO PSEUDO-IRREDUCIBLE IDEALS IN AMALGAMATED ALGEBRA ALONG AN IDEAL

ESMAEIL ROSTAMI
DEPARTMENT OF PURE MATHEMATICS, MAHANI MATHEMATICAL RESEARCH CENTER, FACULTY OF MATHEMATICS AND COMPUTER, SHAHID BAHONAR UNIVERSITY OF KERMAN, KERMAN, IRAN
E-MAIL: E_ROSTAMI@UK.AC.IR

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Abstract. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. In this paper, we give a necessary and sufficient condition for the amalgamated algebra along an ideal $A \bowtie^f J$ to be J-Noetherian. Then we give a characterization for pseudo-irreducible ideals of $A \bowtie^f J$, in special cases.

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1. Introduction

Throughout this paper all rings will be commutative with identity. We denote by Spec(R) and Max(R) the set of prime ideals and the set of maximal ideals of R, respectively.

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Let A and B be two rings and let J be an ideal of B. Then, for a ring homomorphism $f : A \to B$ consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f* which is a generalization of the amalgamated duplication of a ring along an ideal (See [3, 4, 5] for more details). Some classical constructions such as Nagata's idealization, the $A + xB[x]$ construction, the $A + xB[[x]]$ construction, and the $D + M$ construction can be considered as special cases of the amalgamation, see [3] for more details.

For a proper ideal I of a ring R, a *comaximal factorization* is a product $I = I_1 I_2 ... I_n$ of proper ideals with $I_i + I_j = R$ for $i \neq j$. A proper ideal I is called *pseudo-irreducible* if it has no comaximal factorizations except for $I = I$. If the factors of a comaximal factorization $I = I_1 I_2 ... I_n$ are pseudo-irreducible, then the comaximal factorization $I = I_1 I_2 ... I_n$ is called *complete*. McAdam and Swan [9, Section 5] began the study of comaximal factorization and Juett [8] expanded the comaximal factorization to ideal systems. In [7], the authors showed that the complete comaximal factorization for every proper ideal of a ring R exists if and only if R is J-Noetherian.

The rest of this paper is organized in three sections. Some preliminaries on pseudo-irreducible ideals and the max-spectrum are given in Section 2. In Section 3, we give a characterization for the amalgamated algebra along an ideal $A \bowtie^f J$ to be J-Noetherian. In Section 4, we give a characterization for pseudo-irreducible ideals of $A \bowtie^f J$, in special cases.

2. Preliminaries on Pseudo-irreducible Ideals and Max-Spectrum

Recall that a ring R is called *indecomposable* if it cannot be written as a direct product of two nonzero rings or, equivalently, if it has no nontrivial idempotents.

Definition 2.1. An ideal I of a ring R is called *pseudo-irreducible* if R/I is indecomposable.

In the following proposition we list some of the main properties of pseudo-irreducible ideals, see [6, 7, 9] for details and proofs.

Proposition 2.2. *For an ideal I of R, the following statements hold.*

(1) *I is a pseudo-irreducible ideal of R if and only if for all $r \in R$, $r(r-1) \in I \Rightarrow (r \in I$ or $r-1 \in I)$ if and only if for all ideals I_1 and I_2 of R, $(I = I_1 I_2$ and $I_1 + I_2 = R) \Rightarrow (I_1 = R$ or $I_2 = R)$.*
(2) *I is a pseudo-irreducible ideal of R if and only if $\sqrt{I}$ is a pseudo-irreducible ideal of R. In particular, every primary ideal is pseudo-irreducible.*
(3) *If I is a pseudo-irreducible ideal of R and J is an ideal of R such that $\sqrt{J} = \sqrt{I}$, then J is also a pseudo-irreducible ideal of R. In particular, any ideal that is between I and $\sqrt{I}$ is a pseudo-irreducible ideal.*
(4) *Every power of a prime ideal is a pseudo-irreducible ideal.*
(5) *For any two ideals $I \subseteq J$ of R, J/I is a pseudo-irreducible ideal of R/I if and only if J is a pseudo-irreducible ideal of R.*

Definition 2.3. A *comaximal factorization* of a proper ideal I of R is a product $I = \prod_{i=1}^{n} I_i$ of proper ideals with $I_i + I_j = R$ for $i \neq j$. A comaximal factorization is *complete* if its factors are pseudo-irreducible.

Theorem 2.4 (Uniqueness Theorem)**.** [9, Theorem 5.1] *If I has a complete comaximal factorization, it is unique.*

For a ring R, the *Zariski topology* on $\mathrm{Spec}(R)$ is the topology obtained by taking the collection of sets of the form $\mathcal{U}_R(I) := \{P \in \mathrm{Spec}(R) \mid I \nsubseteq P\}$ (resp. $\mathcal{V}_R(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$), for every ideal I of R, as the open (resp. closed) sets. When considered as a subspace of $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$ is called *max − spectrum* of R. So, its open and closed subsets are $U_R(I) := \mathcal{U}_R(I) \cap \mathrm{Max}(R) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \nsubseteq \mathfrak{m}\}$ and $V_R(I) := \mathcal{V}_R(I) \cap \mathrm{Max}(R) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\}$, respectively.

A topological space X is called *Noetherian* if every nonempty set of closed subsets of X, ordered by inclusion, has a minimal element. See [1, Chapter II, Section 4] for more details. An ideal I of R is called a *J-radical* ideal, if it is the intersection of all maximal ideals containing it. Clearly, J-radical ideals of R correspond to closed subsets of $\mathrm{Max}(R)$. Recall that a ring R is called J-Noetherian if it satisfies the ascending chain condition on J-radical ideals.

In the rest of this paper, we will frequently use the following theorem, which is the main result of [7].

Theorem 2.5. [7, Theorem 2.6] *Let R be a ring. The following are equivalent:*

(1) *Every ideal of R has a complete comaximal factorization.*

(2) *For every subset* $\{\mathfrak{m}_\alpha\}_{\alpha\in\Lambda}$ *of* $\mathrm{Max}(R)$*, and for all but finitely many* $\beta \in \Lambda$*,* $\bigcap_{\beta\neq\alpha\in\Lambda} \mathfrak{m}_\alpha \subseteq \mathfrak{m}_\beta$*.*

(3) *For every infinite subset* $\{\mathfrak{m}_\alpha\}_{\alpha\in\Lambda}$ *of* $\mathrm{Max}(R)$*, there exists some* $\beta \in \Lambda$ *such that* $\bigcap_{\beta\neq\alpha\in\Lambda} \mathfrak{m}_\alpha \subseteq \mathfrak{m}_\beta$*.*

(4) $\mathrm{Max}(R)$ *is a Noetherian space, i.e.,* R *is* J*-Noetherian.*

3. When $A \bowtie^f J$ is J-Noetherian

In this section, we give a characterization for the amalgamation of A with B along J with respect to f to be J-Noetherian. We begin with a result about the space of maximal ideals of the ring $A \bowtie^f J$.

Proposition 3.1. [2, Corollary 2.5 and Corollary 2.7] *Let* $f : A \to B$ *be a ring homomorphism and* J *an ideal of* B*. For the subring* $A \bowtie^f J := \{(a, f(a) + j)|a \in A, j \in J\}$ *of the ring* $A \times B$ *and for all* $\mathfrak{m} \in \mathrm{Max}(A)$ *and* $Q \in \mathrm{Max}(B)$*, set*

$$\mathfrak{m}'^f := \mathfrak{m} \bowtie^f J = \{(p, f(p) + j) \mid p \in \mathfrak{m}, j \in J\},$$
$$\overline{Q}' := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in Q\}.$$

Then, we have the following statements:

(1) $\mathfrak{m}'^f$ *and* $\overline{Q}'$ *are maximal ideals of* $A \bowtie^f J$ *for all* $\mathfrak{m} \in \mathrm{Max}(A)$ *and* $Q \in U_B(J) = \mathrm{Max}(B) \setminus V_B(J)$*.*

(2) $\mathrm{Max}(A \bowtie^f J) = \{\mathfrak{m}'^f \mid \mathfrak{m} \in \mathrm{Max}(A)\} \cup \{\overline{Q}' \mid Q \in \mathrm{Max}(B)$ *and* $J \not\subseteq Q\}$*.*

(3) *The map* $Q \mapsto \overline{Q}'$ *establishes a homeomorphism of* $U_B(J) = \mathrm{Max}(B)\setminus V_B(J)$ *onto* $U_{A\bowtie^f J}(\{0\} \times J) = \mathrm{Max}(A \bowtie^f J) \setminus V_{A\bowtie^f J}(\{0\} \times J)$*.*

Proposition 3.2. *Let* X *be a topological space and* $Y_1, Y_2, ..., Y_n$ *be* n *subsets of* X *such that* $X = \cup_{i=1}^n Y_i$*. Then* $Y_1, Y_2, ..., Y_n$ *are Noetherian subspaces of* X *if and only if* X *is Noetherian. In particular, with the notation of Proposition 3.1,* $A \bowtie^f J$ *is* J*-Noetherian if and only if* A *is* J*-Noetherian and* $U_B(J) = \{Q \in \mathrm{Max}(B) \mid J \not\subseteq Q\}$ *is a Noetherian subspace of* $\mathrm{Max}(B)$*.*

Proof. By [2, Corollary 2.5], Proposition 3.1, and Theorem 2.5.

□

Corollary 3.3. *With the notation of Proposition 3.1, if* $J \subseteq J(B)$*, where* $J(B)$ *is the Jacobson radical of* B*, then* $A \bowtie^f J$ *is* J*-Noetherian if and only if* A *is* J*-Noetherian.*

Proof. By Proposition 3.1 and Proposition 3.2. □

Corollary 3.4. *With the notation of Proposition 3.1, if $V_B(J)$ is a Noetherian subspace of* $\mathrm{Max}(B)$ *(e.g., $\mid V_B(J) \mid< \infty$), then $A \bowtie^f J$ is J-Noetherian if and only if A and B are J-Noetherian.*

Proof. Since $\mathrm{Max}(B) = V_B(J) \cup U_B(J)$, the corollary is obtained from Proposition 3.5 and Proposition 3.2. □

Proposition 3.5. *With the notation of Proposition 3.1, if A and B are J-Noetherian, then $A \bowtie^f J$ is J-Noetherian.*

Proof. By Proposition 3.2.

□

The converse of Proposition 3.5 is not true in general. See the following example.

Example 3.6. For a ring extension $A \subseteq B$, assume that x is an indeterminate over B. By [3, Example 2.5], the subring $A + xB[[x]] = \{f(x) \in B[[x]] \mid f(0) \in A\}$ of the ring of power series $B[[x]]$ is isomorphic to $A \bowtie^{i_1} J_1$, where $i_1 : A \hookrightarrow B[[x]]$ is the natural embedding and $J_1 := xB[[x]]$. By Corollary 3.3, A is J-Noetherian if and only if $A \bowtie^{i_1} J_1$ is J-Noetherian. As special case, assume that B is a ring such that $\mathrm{Max}(B)$ is not Noetherian (e.g., an almost Dedekind domain that is not a Dedekind domain, a polynomial ring with infinitely many variables) and A is the prime subring of B. Since A is Noetherian, it is J-Noetherian. Thus, $A \bowtie^{i_1} J_1$ is J-Noetherian, but B is not J-Noetherian.

Lemma 3.7. *Let R be a commutative ring. Then* $\mathrm{Max}(R)$ *is homeomorphic to the subspace $\mathbf{A}_r := \{\mathfrak{m} + (x - r)R[x] \mid \mathfrak{m} \in \mathrm{Max}(R)\}$ of* $\mathrm{Max}(R[x])$*, where $r \in R$.*

Proof. Define $\varphi : \mathrm{Max}(R) \to \mathbf{A}_r$ by

$$\varphi(\mathfrak{m}) = \mathfrak{m} + (x - r)R[x].$$

Clearly φ is well-defined and bijective. Now let C be an arbitrary closed subset of $\mathbf{A}_r$. Thus, there exists a subset $\{f_i\}_{i\in I}$ of $R[x]$ such that $C = V_{R[x]}(\langle f_i \rangle_{i\in I}) \cap \mathbf{A}_r$. If $\mathfrak{m} + (x - r)R[x] \in V_{R[x]}(\langle f_i \rangle_{i\in I}) \cap \mathbf{A}_r$, then $f_i \in \mathfrak{m} + (x - r)R[x]$ for all $i \in I$. Hence, $f_i(r) \in \mathfrak{m}$ for all $i \in I$. Now if $\mathfrak{n} \in \mathrm{Max}(R)$ such that $f_i(r) \in \mathfrak{n}$ for all

$i \in I$, then it is easily seen that $\mathfrak{n} + (x - r)R[x] \in V_{R[x]}(\langle f_i \rangle_{i \in I}) \cap \mathbf{A}_r$. Hence, $\varphi^{-1}(V_{R[x]}(\langle f_i \rangle_{i \in I}) \cap \mathbf{A}_r) = (V_R(\langle f_i(r) \rangle_{i \in I}))$. Therefore, ϕ is continuous.

Now let C' be an arbitrary closed subset of $\mathrm{Max}(R)$. Thus, there exists a subset $\{a_j\}_{j \in J}$ of R such that $C' = V_R(\langle a_j \rangle_{j \in J})$. For each $i \in J$, set $f_j(x) := a_j + x - r$. It is clearly that $\mathfrak{m} \in C' = V_R(\langle a_j \rangle_{j \in J})$ if and only if $\mathfrak{m} + (x-r)R[x] \in V_{R[x]}(\langle f_i \rangle_{i \in I}) \cap \mathbf{A}_r$. It follows that $\varphi(C') = \varphi(V_R(\langle a_j \rangle_{j \in J}) = V_{R[x]}(\langle f_j \rangle_{i \in I}) \cap \mathbf{A}_r$. Therefore, φ is a closed mapping and hence ϕ is a homeomorphism. □

For a ring extension $A \subseteq B$, assume that x is an indeterminate over B. By [3, Example 2.5], the subring $A + xB[x] = \{f(x) \in B[x] \mid f(0) \in A\}$ of the polynomial ring $B[x]$ is isomorphic to $A \bowtie^{i_2} J_2$, where $i_2 : A \hookrightarrow B[x]$ is the natural embedding and $J_2 := xB[x]$. In the following proposition we give a necessary and sufficient condition for the ring $A \bowtie^{i_2} J_2$ to be J-Noetherian.

Proposition 3.8. *Let $A \subseteq B$ be a ring extension and x an indeterminate over B. Then the ring $A \bowtie^{i_2} J_2 \cong A + xB[x]$ is J-Noetherian if and only if A and $B[x]$ are J-Noetherian.*

Proof. ($\Rightarrow$). Assume that $A \bowtie^{i_2} J_2 \cong A + xB[x]$ is J-Noetherian. By Proposition 3.2, A is J-Noetherian and $U_{B[x]}(xB[x])$ is a Noetherian space. By Lemma 3.7, $\mathrm{Max}(B)$ is homeomorphic to the subspace $\{\mathfrak{m} + (x-1)B[x] \mid \mathfrak{m} \in \mathrm{Max}(B)\}$ of $\mathrm{Max}(B[x])$. Now since $\{\mathfrak{m} + (x-1)B[x] \mid \mathfrak{m} \in \mathrm{Max}(B)\} \subseteq U_{B[x]}(xB[x])$, $\mathrm{Max}(B)$ is a Noetherian space. Now since $V_{B[x]}(xB[x])$ is homeomorphic to $\mathrm{Max}(B)$, $V_{B[x]}(xB[x])$ is Noetherian and hence $\mathrm{Max}(B[x]) = V_{B[x]}(xB[x]) \cup U_{B[x]}(xB[x])$ is a Noetherian space by Proposition 3.2. Thus, $B[x]$ is J-Noetherian.

($\Leftarrow$). By Proposition 3.5. □

Proposition 3.9. *Let $A \subseteq B$ be a ring extension and x an indeterminate over B. Then the ring $A \bowtie^{i_2} J_2 \cong A + xB[x]$ is J-Noetherian if and only if A is J-Noetherian and B satisfies the ascending chain condition on radical ideals.*

Proof. By Proposition 3.8 and [6, Theorem 4.4]. □

Remark 3.10. For a ring extension $A \subseteq B$ assume that x is an indeterminate over B. If $A + xB[x]$ is J-Noetherian, then $A + xB[[x]]$ is J-Noetherian, but the converse is not true in general.

Example 3.11. Let $D \subseteq T$ be a ring extension and J an ideal of T such that $J \cap D = \{0\}$. Then the ring $D + J := \{x + j \mid x \in D, j \in J\}$ is canonically isomorphic to $D \bowtie^i J$, where $i : D \hookrightarrow T$ is the natural embedding, see [3, Example 2.5]. By Corollary 3.4, if $V_T(J)$ is finite, then $D + J$ is J-Noetherian if and only if D and T are J-Noetherian. In particular, let $D \subseteq T$ be a ring extension and $\mathfrak{M}$ a maximal ideal of T such that $\mathfrak{M} \cap D = \{0\}$. Then $D + \mathfrak{M}$ is J-Noetherian ring if and only if D and T are J-Noetherian.

Lemma 3.12. *Let $R \subseteq S$ be a ring extension and S be J-Noetherian. Then R is J-Noetherian if and only if there exists a ring homomorphism $g : A \to S$ and an ideal K of S such that A is J-Noetherian and $R = g(A) + K$.*

Proof. ($\Rightarrow$). Let R be J-Noetherian. Set $A = R$, $K = \{0\}$ and assume that g is the natural embedding.

($\Leftarrow$). Assume that there exists a ring homomorphism $g : A \to S$ and an ideal K of S such that A is J-Noetherian and $R = g(A) + K$. Thus, by [3, Proposition 5.1 (3)], $R = g(A) + K \cong \frac{A \bowtie^g K}{g^{-1}(K) \times \{0\}}$. Now since $A \bowtie^g K$ is J-Noetherian, R is J-Noetherian. □

Lemma 3.13. *Let R and S be two rings. Then R and S are J-Noetherian if and only if $R \times S$ is J-Noetherian.*

Proof. Obviously. □

Now we are in a position to give a necessary and sufficient condition for the ring $A \bowtie^f J$ to be J-Noetherian.

Proposition 3.14. *With the notation of Proposition 3.1, $A \bowtie^f J$ is J-Noetherian if and only if A and $f(A) + J$ are J-Noetherian.*

Proof. ($\Rightarrow$). Let $A \bowtie^f J$ be J-Noetherian. By [3, Proposition 5.1], $A \cong \frac{A \bowtie^f J}{\{0\} \times J}$ and $f(A) + J \cong \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$. Thus, A and $f(A) + J$ are J-Noetherian.

($\Leftarrow$). Let A and $f(A) + J$ be J-Noetherian. Thus, the ring $A \times (f(A) + J)$ is J-Noetherian. Define $g : A \to A \times (f(A) + J)$ by

$$g(a) := (a, f(a)).$$

Clearly g is a ring homomorphism. Set $K := \{0\} \times J$. Hence, by Lemma 3.12, $g(A) + K = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\} = A \bowtie^f J$ is J-Noetherian. □

Example 3.15. Let A be a ring which is not J-Noetherian (e.g., an almost Dedekind domain that is not a Dedekind domain, a polynomial ring with infinitely many variables) and let $\mathfrak{m}$ be a maximal ideal of A. Then for the localization map $f : A \to A_{\mathfrak{m}}$ and $J := \mathfrak{m}A_{\mathfrak{m}}$, we have $f(A) + J = A_{\mathfrak{m}}$ which is J-Noetherian, but A and $A \bowtie^f J$ are not J-Noetherian.

Corollary 3.16. *With the notation of Proposition 3.1, if f is surjective, then $A \bowtie^f J$ is J-Noetherian if and only if A and B are J-Noetherian.*

Corollary 3.17. *With the notation of Proposition 3.1, if $J \subseteq f(A)$, then $A \bowtie^f J$ is J-Noetherian if and only if A is J-Noetherian.*

Proof. Since $J \subseteq f(A)$, we have $f(A) + J = f(A)$, and so $f(A) + J$ is a quotient of A. Hence, the result follows from Proposition 3.14. □

Proposition 3.18. *Let $f : A \to B$ be a ring homomorphism and J_1 and J_2 be two comaximal ideals of B. Then $A \bowtie^f J_1$ and $A \bowtie^f J_2$ are J-Noetherian if and only if A and B are J-Noetherian.*

Proof. Since J_1 and J_2 are comaximal ideals of B, we have $\mathrm{Max}(B) = U_B(J_1) \cup U_B(J_2)$. Hence, the proof completes by Theorem 2.5 and Proposition 3.2. □

4. PSEUDO-IRREDUCIBLE IDEALS OF $A \bowtie^f J$

In this section, we give a characterization for pseudo-irreducible ideals of $A \bowtie^f J$, in special cases.

Remark 4.1. With the notation of Proposition 3.1, we have the canonical isomorphism $A \cong \frac{A \bowtie^f J}{\{0\} \times J}$. Thus, every ideal of $A \bowtie^f J$ containing $\{0\} \times J$ is of the form $I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\}$ for some ideal I of A. Also, we have the following canonical isomorphism:

$$\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}.$$

Hence, an arbitrary ideal $I \bowtie^f J$ of $A \bowtie^f J$ containing $\{0\} \times J$ is pseudo-irreducible if and only if I is a pseudo-irreducible ideal of A.

With the notation of Proposition 3.1, for an ideal K of $f(A)+J$, set:

$$\overline{K}^f := \{(a, f(a)+j) \mid f(a)+j \in K\}.$$

Clearly, $\overline{K}^f$ is an ideal of $A \bowtie^f J$.

Proposition 4.2. *With the notation of Proposition 3.1, let T be an ideal of $A \bowtie^f J$ such that $f^{-1}(J) \times \{0\} \subseteq T$. Then $T = \overline{T_{f(A)+J}}^f$, where $T_{f(A)+J}$ is the ideal $\{f(a)+j \mid (a, f(a)+j) \in T\}$ of $f(A)+J$. Furthermore, T is a pseudo-irreducible ideal of $A \bowtie^f J$ if and only if $T_{f(A)+J}$ is a pseudo-irreducible ideal of $f(A)+J$.*

Proof. Let T be an ideal of $A \bowtie^f J$. Then, clearly $T_{f(A)+J}$ is an ideal of $f(A)+J$. Let $(a, f(a)+j) \in T$. Then $f(a)+j \in T_{f(A)+J}$, and so $(a, f(a)+j) \in \overline{T_{f(A)+J}}^f$. Conversely, let $(a, f(a)+j) \in \overline{T_{f(A)+J}}^f$. Thus, $f(a)+j \in T_{f(A)+J}$. Hence, there exist $a' \in A$ and $j' \in J$ such that $(a', f(a')+j') \in T$ and $f(a')+j' = f(a)+j$. Thus, $f(a-a') \in J$ and so $a-a' \in f^{-1}(J)$. Since $f^{-1}(J) \times \{0\} \subseteq T$, we have $(a-a', 0) \in T$. Now since $(a', f(a')+j')$ is also in T, we have $(a, f(a')+j') \in T$. Therefore, $(a, f(a)+j) \in T$. Thus, $T = \overline{T_{f(A)+J}}^f$.

Now let T be a pseudo-irreducible ideal of $A \bowtie^f J$, and $(f(a)+j)(f(a)+j-1) \in T_{f(A)+J}$ for some $f(a)+j \in f(A)+J$. Thus, $(a, f(a)+j)((a, f(a)+j)-(1,1)) \in T$. Since T is a pseudo-irreducible ideal of $A \bowtie^f J$, we have $(a, f(a)+j) \in T$ or $((a, f(a)+j)-(1,1)) \in T$. Now since $T = \overline{T_{f(A)+J}}^f$, we have $f(a)+j \in T_{f(A)+J}$ or $f(a)+j-1 \in T_{f(A)+J}$. It follows that $T_{f(A)+J}$ is a pseudo-irreducible ideal of $f(A)+J$. Conversely, assume that $T_{f(A)+J}$ is a pseudo-irreducible ideal of $f(A)+J$ and $(a, f(a)+j)((a, f(a)+j)-(1,1)) \in T$ for some $(a, f(a)+j)) \in A \bowtie^f J$. Thus, $(a(a-1), (f(a)+j)(f(a)+j-1)) \in T = \overline{T_{f(A)+J}}^f$. This implies that $(f(a)+j)(f(a)+j-1) \in T_{f(A)+J}$ and since $T_{f(A)+J}$ is a pseudo-irreducible ideal of $f(A)+J$, we have $f(a)+j \in T_{f(A)+J}$ or $f(a)+j-1 \in T_{f(A)+J}$. Therefore, $(a, f(a)+j) \in T$ or $((a, f(a)+j)-(1,1)) \in T$ and hence T is a pseudo-irreducible ideal of $A \bowtie^f J$. □

Proposition 4.3. *With the notation of Proposition 3.1 let $J \neq \{0\}$. Then $f^{-1}(J) = \{0\}$ if and only if every ideal of $A \bowtie^f J$ is of the form $\overline{K}^f$ for some ideal K of $f(A)+J$. In particular, if $f^{-1}(J) = \{0\}$, then every pseudo-irreducible ideal of $A \bowtie^f J$ is of the form $\overline{K}^f$ for some pseudo-irreducible ideal K of $f(A)+J$.*

Proof. (⇒). Since $f^{-1}(J) = \{0\}$, for every ideal T of $A \bowtie^f J$, we have $f^{-1}(J) \times \{0\} \subseteq T$. Hence, Proposition 4.2 completes the proof.

(⇐). Assume that every ideal of $A \bowtie^f J$ is of the form $\overline{K}^f$ for some ideal K of $f(A)+J$. Thus, for the ideal $\{(0,0)\}$ of $A \bowtie^f J$ there exists an ideal K of $f(A)+J$ such that $\{(0,0)\} = \overline{K}^f = \{(a, f(a)+j) \mid f(a)+j \in K\}$. Hence, $f^{-1}(J) = \{0\}$.

□

Proposition 4.4. *With the notation of Proposition 3.1, if $J \neq B$, then the ring $A \bowtie^f J$ is indecomposable, equivalently the ideal $\{(0,0)\}$ is pseudo-irreducible, if and only if A is indecomposable and the ideal J has no nonzero idempotents.*

Proof. (⇒). By contrapositive. If e is a nontrivial ideampotent element of A, then $(e, f(e))$ is a nontrivial idempotent element of $A \bowtie^f J$ or if j is a nonzero idempotent element of J, then $(0, j)$ is a nontrivial idempotent element of $A \bowtie^f J$.

(⇐). Suppose that A is indecomposable and the ideal J has no nonzero idempotents. If $(a, f(a)+j)$ is an idempotent element of $A \bowtie^f J$, then we have

$$a^2 = a \text{ and } (f(a)+j)^2 = f(a)+j.$$

Now since A is indecomposable, we have $a = 0$ or $a = 1$. If $a = 0$, then $j^2 = j$. Thus, $j = 0$. Hence, $(a, f(a)+j) = (0,0)$. If $a = 1$, then $(1+j)^2 = 1+j$. Thus, $j^2 = -j$. This implies that $-j$ is an idempotent element of J. Thus $j = 0$. Hence, $(a, f(a)+j) = (1,1)$. Therefore, $A \bowtie^f J$ has no nontrivial idempotent elements and so $A \bowtie^f J$ is indecomposable. □

Proposition 4.5. *With the notation of Proposition 3.1, if the ideal J has a generating set consisting of idempotents, then every pseudo-irreducible ideal of $A \bowtie^f J$ is of the form $I \bowtie^f J$ for some pseudo-irreducible ideal I of A or $\overline{K}^f$ for some pseudo-irreducible ideal K of $f(A)+J$.*

Proof. By Remark 4.1 and Proposition 4.2, it is sufficient to show that for a pseudo-irreducible ideal T of $A \bowtie^f J$, we have $\{0\} \times J \subseteq T$ or $f^{-1}(J) \times \{0\} \subseteq T$.

By assumption, there exists a subset $\{e_i\}_{i \in I}$ of idempotent elements of B such that $J = \langle e_i \rangle_{i \in I}$. Since for each $i \in I$, $(0, e_i) \in A \bowtie^f J$, we have $(0, e_i)((0, e_i) - (1,1)) = (0,0) \in T$. Now since T is a pseudo-irreducible ideal of $A \bowtie^f J$, we have $(0, e_i) \in T$ or $(0, e_i) - (1,1) = (-1, e_i - 1) \in T$. If $(-1, e_i - 1) \in T$ for some $i \in I$, then $f^{-1}(J) \times \{0\} = (-1, e_i - 1)(f^{-1}(J) \times \{0\}) \subseteq T$. Let us assume $(0, e_i) \in T$

for each $i \in I$. Now let $(0, j) \in \{0\} \times J$. Since $j \in J$, j has an expression of the form $j = \Sigma_{i \in I} b_i e_i$, where $b_i \in B$ and almost all $b_i = 0$. Since $e_i \in J$, we have $(0, b_i e_i) \in A \bowtie^f J$ for each $i \in I$. Thus, $(0, j) = \Sigma_{i \in I}(0, b_i e_i)(0, e_i) \in T$. Therefore, $\{0\} \times J \subseteq T$, which completes the proof. □

In Proposition 4.5, the assumption that J has a generating set consisting of idempotents is necessary.

Example 4.6. Let $A := \mathbb{Z}_4$ and $B := \mathbb{Z}_4[x]$. Since $\mathfrak{p} := \langle \overline{2} \rangle$ is a maximal ideal of A, by Proposition 3.1, $\mathfrak{p} \bowtie^f J$ is a maximal ideal of $A \bowtie^f J$, where J is the ideal $\langle \overline{2}, x \rangle$ of B and $f : \mathbb{Z}_4 \hookrightarrow \mathbb{Z}_4[x]$ is the natural embedding. Thus, by Proposition 2.2(4), $(\mathfrak{p} \bowtie^f J)^2$ is a pseudo-irreducible ideal of $A \bowtie^f J$, but $\{0\} \times J \nsubseteq (\mathfrak{p} \bowtie^f J)^2$ and $f^{-1}(J) \times \{0\} \nsubseteq (\mathfrak{p} \bowtie^f J)^2$. It follows that $(\mathfrak{p} \bowtie^f J)^2$ is not of the form $I \bowtie^f J$ for some pseudo-irreducible ideal I of A or $\overline{K}^f$ for some pseudo-irreducible ideal K of $f(A) + J$.

Remark 4.7. It is easily seen that if an ideal I of a ring R can be generated by a set of idempotents, then every element of I is a multiple of an idempotent of I.

Before proceeding, we need some notation. For an ideal I of a ring R let I' be the ideal of R generated by idempotent elements of I, that is, $I' = \langle \{e \in I \mid e^2 = e\} \rangle$.

Lemma 4.8. *Let I be an ideal of R. Then if I is a pseudo-irreducible ideal of R, then I' is a pseudo-irreducible ideal of R.*

Proof. Let $x^2 - x \in I'$ for some $x \in R$. Thus, $x^2 - x \in I$. Since I is a pseudo-irreducible ideal R, we have $x \in I$ or $x - 1 \in I$. Suppose that $x \in I$. Now since $x^2 - x \in I'$, by Remark 4.7, there exists $e^2 = e \in I'$ such that $x^2 - x = re$ for some $r \in R$. Thus, $x^2 - x = (x^2 - x)e$. Hence, $(1 - e)x^2 = (1 - e)x$. Thus, $((1 - e)x)^2 = (1 - e)^2 x^2 = (1 - e)x^2 = (1 - e)x$. This shows that $(1 - e)x$ is an idempotent in I, hence $(1 - e)x \in I'$. Now since $e \in I'$, we have $x \in I'$. A similar argument works when $x - 1 \in I$. Therefore, I' is a pseudo-irreducible ideal of R. □

Theorem 4.9. *With the notation of Proposition 3.1, let $f^{-1}(J) \neq \{0\}$ and A be an indecomposable ring (e.g., domains and local rings). Then every pseudo-irreducible ideal of $A \bowtie^f J$ is of the form $I \bowtie^f J$ for some pseudo-irreducible ideal I of A or $\overline{K}^f$ for some pseudo-irreducible ideal K of $f(A) + J$ if and only if the ideal J is generated by idempotent elements.*

Proof. ($\Rightarrow$). Since A is indecomposable, the ideal $\{0\}$ of A is pseudo-irreducible. Thus, $\{0\} \bowtie^f J$ is a pseudo-irreducible ideal of $A \bowtie^f J$, and so $(\{0\} \bowtie^f J)'$ is a pseudo-irreducible of $A \bowtie^f J$, by Lemma 4.8. By assumption, there are two cases:

*Case*1. There exists an ideal I of A such that $(\{0\} \bowtie^f J)' = I \bowtie^f J$. In this case, since A is indecomposable, we have $(\{0\} \bowtie^f J)' = \{0\} \times J'$. Hence, $I \bowtie^f J = \{0\} \times J'$, and so $J = J'$. Therefore, the ideal J is generated by its idempotent elements.

*Case*2. There exists an ideal K of $f(A) + J$ such that $(\{0\} \bowtie^f J)' = \overline{K}^f$. In this case, since $f^{-1}(J) \times \{0\} \subseteq \overline{K}^f$, we have $f^{-1}(J) = \{0\}$, a contradiction.

($\Leftarrow$). By Proposition 4.5. □

In Theorem 4.9, the assumption that $f^{-1}(J) \neq \{0\}$ is necessary.

Example 4.10. Let x be an indeterminate over the ring of integers $\mathbb{Z}$. Set $A := \mathbb{Z}$, $B := \mathbb{Z}[x]$ and $J := xB$. Then, for the canonical embedding $i : A \hookrightarrow B$ we have $i^{-1}(J) = \{0\}$. Thus, by Proposition 4.3, every pseudo-irreducible ideal of $A \bowtie^i J$ is of the form $\overline{K}^i$ for some pseudo-irreducible ideal K of $i(A) + J = A + J$, but J is not generated by idempotents.

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