GENERALIZATION OF GENERAL HELICES AND SLANT HELICES

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Abstract. In this work, we use the formal definition of k-slant helix [5] to obtain the intrinsic equations as well as the position vector for *slant-slant helices* which a generalization of *general helices* and *slant helices*. Also, we present some characterizations theorems for k-slant helices and derived, in general form, the intrinsic equations for such curves. Thereafter, from a Salkowski curve, anti-salkowski curve, a curve of constant precession and spherical slant helix, as examples of slant helices, we apply this method to find the parametric representation of some *slant-slant* helices by means of intrinsic equations. Finally, the parametric representation and the intrinsic equations of *Slakowski slant-slant* and *Anti-Slakowski slant-slant* helices have been given.

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1. Introduction

Curves theory is an important branch in the differential geometry studies. We have a lot of special curves such as geodesics, circles, Bertrand curves, circular helices, general helices, slant helices, k-slant helices etc. Characterizations of these special curves are heavily studied for a long time and are still studied. We can see the applications of helical structures in nature and mechanic tools. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or design of highways. Also we can see the helix curve or helical structure in fractal geometry, for instance hyperhelices [1, 8].

A *straight line* is a geometric curve with the curvature $\kappa(s) = 0$. A *plane curve* is a family of geometric curves with torsion $\tau(s) = 0$. *Helix* (*circular helix*) is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ which is the simplest example of three-dimensional spirals [2, 6].

General helix (a curve of constant slope) is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result was stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [18] for details) says that: *A necessary and sufficient condition that a curve to be a general helix is that the function*

$$\sigma_0 = \frac{\tau(s)}{\kappa(s)} \tag{1}$$

is constant along the curve, where κ and τ denote the curvature and the torsion, respectively.

Izumiya and Takeuchi [11] introduced the concept of *slant helix* by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the *geodesic curvature* of the principal image of the principal normal indicatrix

$$\sigma_1 = \frac{\sigma_0'(s)}{\kappa(s)\Big(1+\sigma_0^2(s)\Big)^{3/2}} \tag{2}$$

is a constant function.

A family of slant helices with constant curvature but non-constant torsion are called *Salkowski curves*; a family of slant helices with constant torsion but non-constant curvature are called *anti-Salkowski curves* which introduced in [16, 17]; a family of curves of *constant precession* [14, 15] which characterized by having

$$\kappa = \frac{\mu}{m}\sin[\mu\, s], \qquad \tau = \frac{\mu}{m}\cos[\mu\, s],$$

where μ and m are constants and a family of spherical slant helices are the important examples of a slant helices (see [4, 7] for details).

Recently, Ali [5] defined a new kind of curves which called it a $k-$*slant helices.* He proved that the curve is a k-slant helix if and only if the *geodesic curvature* of the spherical image of ψ_k indicatrix of the curve ψ

$$\sigma_k = \frac{\sigma'_{k-1}(s)}{\kappa(s)\sqrt{1+\sigma_0^2(s)}\sqrt{1+\sigma_1^2(s)}\ldots\Big(1+\sigma_{k-1}^2(s)\Big)^{3/2}}, \tag{3}$$

is a constant function, where $\psi_{\kappa+1} = \dfrac{\psi'_k(s)}{\|\psi'_k(s)\|}$, $\psi_0(s) = \psi(s)$, $\sigma_0(s) = \dfrac{\tau(s)}{\kappa(s)}$ and $k \in \{0, 1, 2, ...\}$. It is worth noting that, the straight lines, plane curves, general helices and slant helices are a special subclasses of curves from the family of k-slant helices.

Here, we give the following important definition:

Definition 1.1. *A family of k-slant helices with constant curvature but non-constant torsion are called Salkowski k-slant helices and a family of k-slant helices with constant torsion but non-constant curvature are called Anti-Salkowski k-slant helices.*

In this paper, we will use the definition of k-slant helix [5] and the notation of the principal-direction curve curve [9] of a Frenet curve to obtain the intrinsic equations and position vector of *slant-slant helices* as a generalization of *general helices* and *slant helices* in Euclidean 3-space. Also, we will give the parametric representation and the intrinsic equations of Slakowski slant-slant and Anti-Slakowski slant-slant helices.

2. Preliminaries

In Euclidean space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ which are respectively called, the tangent, the principal normal and the binormal vector fields. We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle,\rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\psi : I \subset \mathbb{R} \to \mathbf{E}^3$, $\psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. In particular, if $\psi(s) \neq 0$ for any s, then it is possible to re-parameterize ψ, that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. A unit speed curve ψ is called a *Frenet curve* if $\psi''(s) \neq 0$, that is, it has non-zero curvature. Let $\psi : I \to \mathbf{E}^3$ be a Frenet curve and $\{\psi'(s) = \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the Frenet frame along ψ, where the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors satisfying $\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1$. The Frenet equations for ψ are given by ([10])

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{4}$$

where κ and τ are smooth functions on I called the curvature and the torsion of ψ, respectively. If $\tau(s) = 0$ for any $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean 2-space $\mathbf{E}^2$.

Firstly, we will introduce a conclusion of the definition of a k-slant helix and some important results as follows:

Definition 2.1. [5] *Let $\psi = \psi(s)$ a natural representation of a unit speed regular curve in Euclidean 3-space with Frenet apparatus $\{\kappa, \tau, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. A curve ψ is called a k-slant helix if the unit vector*

$$\psi_{\kappa+1} = \frac{\psi_k'(s)}{\|\psi_k'(s)\|}, \quad k \in \{0, 1, 2, ...\}, \tag{5}$$

makes a constant angle with a fixed direction, where $\psi_0 = \psi(s)$.

From the above definition, we can deduce some important remarks:

(1): The general helix is a 0-*slant helix* which the curve whose the unit vector $\mathbf{T}(s)$ (which is the tangent vector of the curve ψ) makes a constant angle with a fixed direction.

(2): The slant helix is a 1-*slant helix* such that the curve whose the unit vector $\mathbf{N}(s)$ (which is the principal normal vector of the curve ψ) makes a constant angle with a fixed direction.

(3): The *slant-slant helix* or 2-*slant helix* is the curve whose the unit vector $\dfrac{-\mathbf{T}+\sigma_0\mathbf{B}}{\sqrt{1+\sigma_0^2}}$ makes a constant angle with a fixed direction. On other hand, the curve is a slant-slant helix if and only if the *geodesic curvature* of the spherical image of ψ_2 indicatrix of the curve ψ

$$\sigma_2 = \frac{\sigma_1'(s)}{\kappa(s)\sqrt{1+\sigma_0^2(s)}\Big(1+\sigma_1^2(s)\Big)^{3/2}}, \tag{6}$$

is a constant function.

Now, Choi and Kim [9] defined some associated curve $\tilde{\psi}$ of a Frenet curve ψ in $\mathbf{E}^3$. We will introduce the conclusion of the notation of notation of the the principal-direction curve and principal-donor curve and some important results as follows:

Let $V : I \to \mathbf{E}^3$ be a continuous vector valued function defined on an open interval I. Then, it is easily seen that an integral curve $\psi :\to \mathbf{E}^3$ of V is a unique up to translation of $\mathbf{E}^3$. If V is a unit vector field, i.e., $V : I \to S^2$, then we may assume that the parameter s of ψ is an arc-length parameter.

Definition 2.2. [9] *An integral curve ψ of $\tilde{\mathbf{N}}(s)$ is called the principal-direction curve of $\tilde{\psi}$ and the curve $\tilde{\psi}$ is called the principal-donor curve of ψ.*

Theorem 2.3. [9] *Let $\tilde{\psi}$ be a Frenet curve in $\boldsymbol{E}^3$ with the curvature $\tilde{\kappa}$ and the torsion $\tilde{\tau}$ and ψ the principal-direction curve of $\tilde{\psi}$. Then the curvature κ and torsion τ are given by*

$$\kappa = \sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}, \ \text{ and } \ \tau = \frac{\tilde{\kappa}^2}{\tilde{\kappa}^2+\tilde{\tau}^2}\big(\frac{\tilde{\tau}}{\tilde{\kappa}}\big)'. \tag{7}$$

Theorem 2.4. [9] *If a curve $\tilde{\psi}$ in $\boldsymbol{E}^3$ is a principal-donor curve of a curve ψ with the curvature κ and the torsion τ, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ of the curve $\tilde{\psi}$ are given by*

$$\tilde{\kappa}(s) = \kappa(s)|\cos\Big(\int\tau(s)ds\Big)| \ \text{ and } \ \tilde{\tau}(s) = \kappa(s)\sin\Big(\int\tau(s)ds\Big). \tag{8}$$

Corollary 2.5. [9] *let $\tilde{\psi}$ be a Frenet curve in $\boldsymbol{E}^3$ with the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ and ψ the principal-direction curve of $\tilde{\psi}$. Then it satisfies*

$$\frac{\tau}{\kappa} = \frac{\tilde{\kappa}^2}{(\tilde{\kappa}^2+\tilde{\tau}^2)^{3/2}}\big(\frac{\tilde{\tau}}{\tilde{\kappa}}\big)'. \tag{9}$$

Definition 2.6. [9] *Let ψ be a principal-direction curve of a Frenet curve $\tilde{\psi}$ and $\tilde{\psi}$ a principal-direction curve of $\tilde{\tilde{\psi}}$ in $\boldsymbol{E}^3$. Then, ψ is called a second-principal-direction curve of $\tilde{\tilde{\psi}}$ and $\tilde{\tilde{\psi}}$ a second principal-donor curve of ψ.*

By the definition of the principal-direction of $\tilde{\psi}$, $\dfrac{d\psi}{ds} = \mathbf{T} = \tilde{\mathbf{N}}$ and $\mathbf{T}' = \tilde{\mathbf{N}}' = -\tilde{\kappa}\tilde{\mathbf{T}} + \tilde{\tau}\tilde{\mathbf{N}}$. Also,, the principal normal vector field $\mathbf{N}$ and the binormal vector field $\mathbf{B}$ of ψ are given by

$$\begin{aligned} \mathbf{N} &= -\frac{\tilde{\kappa}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\tilde{\mathbf{T}} + \frac{\tilde{\tau}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\tilde{\mathbf{B}}, \\ \mathbf{B} = \mathbf{T}\times\mathbf{N} &= \frac{\tilde{\tau}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\tilde{\mathbf{T}} + \frac{\tilde{\kappa}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\tilde{\mathbf{B}}. \end{aligned} \tag{10}$$

By solving the above equation, we have

$$\tilde{\mathbf{T}} = -\frac{\tilde{\kappa}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\mathbf{N} + \frac{\tilde{\tau}}{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}\mathbf{B}. \tag{11}$$

Then, we have $\tilde{\psi} = \int \tilde{\mathbf{T}} ds$.

3. Position vector of some special curves

In this section we will deduce the position vector of some special curves such as: general helix, slant helix and slant-slant helix. Firstly, we give the following remark:

Remark 3.1. *We will refer to the position vector of the k-slant helix by $\Psi_k = \Psi_k(s)$ and the Frenet apparatus by $\{\kappa_k, \tau_k, \boldsymbol{T}_k, \boldsymbol{N}_k, \boldsymbol{B}_k\}$. It is worth noting that $\psi_0 = \Psi_k$, $\psi_1 = \dfrac{\Psi_k'(s)}{\|\Psi_k'(s)\|}$ and so on.*

3.1. General helices.

In this subsection, we study a general helices as principal-donor curve. A Frenet curve ψ is said to be a general helix if $\sigma_0 = \dfrac{\tau_0}{\kappa_0} = m$ is a constant. It is well known that a general helix in $\mathbf{E}^3$ is a geodesic on a general cylinder and it can be constructed by a plane curve [3, 12, 18].

The following characterization of general helices in $\mathbf{E}^3$ is obtained from corollary 2.5.

Theorem 3.2. [9] *The following are equivalent:*

(1): *A Frenet curve ψ is a general helix in $\boldsymbol{E}^3$.*

(2): *ψ is a principal-donor curve of a plane curve.*

(3): *A principal-direction curve of ψ is a plane curve.*

Theorem 3.2(2) gives a canonical method to construct a general helix from a plane curve. Let ψ be a unit speed plane curve. Then, ψ can be expressed by [1, 13]:

$$\psi(s) = \int \Big(\cos\Big[\int \kappa(s)ds\Big], \sin\Big[\int \kappa(s)ds\Big], 0\Big) ds, \tag{12}$$

where $\kappa(s)$ is the curvature and the torsion $\tau(s) = 0$.

Then the curve ψ has the following Frenet vectors as the following:

$$\begin{cases} \mathbf{T}(s) = \Big(\cos\Big[\int \kappa(s)ds\Big], \sin\Big[\int \kappa(s)ds\Big], 0\Big) = \mathbf{N}_0(s), \\ \mathbf{N}(s) = \Big(-\sin\Big[\int \kappa(s)ds\Big], \cos\Big[\int \kappa(s)ds\Big], 0\Big), \\ \mathbf{B} = (0, 0, 1). \end{cases} \tag{13}$$

From (7) and (11), we have $\kappa(s) = \sqrt{1+m^2}\kappa_0(s) = \dfrac{m}{n}\kappa_0(s)$ and

$$\mathbf{T}_0(s) = \frac{n}{m}\Big(\sin\Big[\frac{m}{n}\int \kappa_0(s)ds\Big], -\cos\Big[\frac{m}{n}\int \kappa_0(s)ds\Big], m\Big). \tag{14}$$

If we integrate the tangent vector $\mathbf{T}_0(s)$, we get the following theorem:

Theorem 3.3. *The position vector Ψ_0 of a general helix whose tangent vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:*

$$\Psi_0(s) = \frac{n}{m}\int \Big(\sin\Big[\frac{m}{n}\int \kappa_0(s)ds\Big], -\cos\Big[\frac{m}{n}\int \kappa_0(s)ds\Big], m\Big) ds, \tag{15}$$

where $\kappa_0(s)$ is the curvature of the curve Ψ_0, the quantity $m\,\kappa_0(s) = \tau_0(s)$ is the torsion, $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a general helix) and the tangent vector $\boldsymbol{T}_0(s) = \psi_1(s)$ of the curve Ψ_0.

3.2. **Slant helices.** In this subsection, we study a slant helix in $\mathbf{E}^3$ as a second principal-donor curve of a plane curve or a principal-donor of a general helix. Theorem 3.2 and Corollary 2.5 give a characterization of the slant helices in $\mathbf{E}^3$ as follows:

Theorem 3.4. [9] *The following are equivalent:*

- ***(1):*** *A Frenet curve ψ is a slant helix in $\boldsymbol{E}^3$.*
- ***(2):*** *ψ is a principal-donor curve of a general helix in $\boldsymbol{E}^3$.*
- ***(3):*** *ψ is a second principal-donor curve of a plane curve.*
- ***(4):*** *A principal-direction curve of ψ is a general helix in $\boldsymbol{E}^3$.*
- ***(5):*** *A second principal-direction curve of ψ is a plane curve.*

From the construction (19) of a general helix Ψ_0, we can construct a slant helix Ψ_1 in $\mathbf{E}^3$. In fact, from (19), the Frenet frame of $\psi 0$ is given by:

$$(16)\qquad \begin{cases} \mathbf{T}_0(s) = \dfrac{n}{m}\Big(\sin\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], -\cos\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], m\Big) = \mathbf{N}_1(s), \\ \mathbf{N}_0(s) = \Big(\cos\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], \sin\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], 0\Big), \\ \mathbf{B}_0(s) = n\Big(-\sin\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], \cos\Big[\dfrac{m}{n}\int \kappa_0(s)ds\Big], \dfrac{1}{m}\Big). \end{cases}$$

From (7), (11) and (16), we have

$$(17)\qquad \mathbf{T}_1(s) = \begin{cases} \mathbf{T}_{11} = -n\,\theta \sin\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big] - \sqrt{1-\theta^2}\cos\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big], \\ \mathbf{T}_{12} = n\,\theta\cos\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big] - \sqrt{1-\theta^2}\sin\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big], \\ \mathbf{T}_{13} = \dfrac{n}{m}\,\theta, \end{cases}$$

where $\kappa_0(s) = \dfrac{\kappa_1(s)}{\sqrt{1-\theta^2}}$ and $\theta = m\int \kappa_1(s)ds$. The integration of the tangent vector $\mathbf{T}_1(s)$ leads to the following theorem:

Theorem 3.5. *The position vector $\Psi_1 = (\Psi_{11}, \Psi_{12}, \Psi_{13})$ of a slant helix whose normal vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:*

$$(18)\qquad \begin{cases} \Psi_{11} = -\int \Big[n\,\theta\sin\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big] + \sqrt{1-\theta^2}\cos\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big]\Big]ds, \\ \Psi_{12} = \int \Big[n\,\theta\cos\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big] - \sqrt{1-\theta^2}\sin\Big[\dfrac{1}{n}\sin^{-1}[\theta]\Big]\Big]ds, \\ \Psi_{13} = \dfrac{n}{m}\int \theta\, ds, \end{cases}$$

where $\theta = m\int \kappa_1(s)ds$, $\kappa_1(s)$ is the curvature of the curve Ψ_1, the quantity $\tau_1(s) = \dfrac{m\kappa_1(s)\int \kappa_1(s)ds}{\sqrt{1-m^2\big(\int \kappa_1(s)ds\big)^2}}$ is the torsion, $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a slant helix) and the normal vector $\boldsymbol{N}_1(s) = \psi_2(s)$ of the curve Ψ_1.

3.3. **Slant-slant helices.** In this subsection, we study a slant-slant helix in $\mathbf{E}^3$ as a third principal-donor curve of a plane curve or a second principal-donor of a general helix or a principal-donor slant helix. Theorem 2.4 and Corollary 2.5 give a characterization of the slant-slant helices in $\mathbf{E}^3$ as follows:

Theorem 3.6. *The following are equivalent:*

(1): *A Frenet curve ψ is a slant-slant helix in $\boldsymbol{E}^3$.*

(2): *ψ is a principal-donor curve of a slant helix in $\boldsymbol{E}^3$.*

(3): *ψ is a second principal-donor curve of a general helix in $\boldsymbol{E}^3$.*

(4): *ψ is a third principal-donor curve of a plane curve.*

(5): *A principal-direction curve of ψ is a slant helix in $\boldsymbol{E}^3$.*

(6): *A second principal-direction curve of ψ is a general helix in $\boldsymbol{E}^3$.*

(7): *A third principal-direction curve of ψ is a plane curve.*

From the construction (18) of a slant helix Ψ_1, we can construct a slant-slant helix Ψ_2 in $\mathbf{E}^3$. In fact, from (18), the Frenet frame of Ψ_1 is given by:

$$\begin{cases} \mathbf{T}_1(s) = \big(\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{13}\big) = \mathbf{N}_2(s), \\ \mathbf{N}_1(s) = \mathbf{T}_0(s), \\ \mathbf{B}_1(s) = \mathbf{T}_1(s) \times \mathbf{N}_1(s). \end{cases} \tag{19}$$

From (7) and (11), we have $\kappa_1(s) = \sqrt{\kappa_2^2(s) + \tau_2^2(s)}$ and

$$\mathbf{T}_2(s) = \begin{cases} \mathbf{T}_{21} = -\frac{1}{\kappa_1}\Big[\theta\tau_2 \sin[\Theta] + \big(\frac{n\kappa_2}{m} + n\tau_2\sqrt{1-\theta^2}\big)\cos[\Theta]\Big], \\ \mathbf{T}_{22} = \frac{1}{\kappa_1}\Big[\theta\tau_2 \cos[\Theta] - \big(\frac{n\kappa_2}{m} + n\tau_2\sqrt{1-\theta^2}\big)\sin[\Theta]\Big], \\ \mathbf{T}_{23} = \frac{n}{m\kappa_1}\big(\tau_2\sqrt{1-\theta^2} - m\kappa_2\big). \end{cases} \tag{20}$$

Then the position vector of a slant-slant helix can be written as

$$\Psi_2(s) = \begin{cases} \Psi_{21} = -\int \frac{1}{\kappa_1}\Big[\theta\tau_2 \sin[\Theta] + \big(\frac{n\kappa_2}{m} + n\tau_2\sqrt{1-\theta^2}\big)\cos[\Theta]\Big]ds, \\ \Psi_{22} = \int \frac{1}{\kappa_1}\Big[\theta\tau_2 \cos[\Theta] - \big(\frac{n\kappa_2}{m} + n\tau_2\sqrt{1-\theta^2}\big)\sin[\Theta]\Big]ds, \\ \Psi_{23} = \int \frac{n}{m\kappa_1}\big(\tau_2\sqrt{1-\theta^2} - m\kappa_2\big)\,ds, \end{cases} \tag{21}$$

where $\Theta = \frac{1}{n}\sin^{-1}[\theta]$, $\theta = m\int \kappa_1(s)ds$, $\kappa_1(s) = \sqrt{\kappa_2^2(s) + \tau_2^2(s)}$ and $\kappa_2(s)$ is the curvature of the curve Ψ_2, the quantity $\tau_2(s)$ is the torsion, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a slant-slant helix) and the vector

$$\frac{-\kappa_2(s)\mathbf{T}_2(s) + \tau_2(s)\mathbf{B}_2(s)}{\sqrt{\kappa_2^2(s) + \tau_2^2(s)}} = \psi_3(s)$$

of the curve Ψ_2.

4. Some characterizations of k-slant helices

In this subsection, we will generalize the results in the above sections as follows:

Theorem 4.1. *The following are equivalent:*

(1): *A Frenet curve ψ is a k-slant helix in $\boldsymbol{E}^3$.*

(2): *ψ is a principal-donor curve of a $(k-1)$-slant helix in $\boldsymbol{E}^3$.*

(3): *ψ is a second principal-donor curve of a $(k-2)$-slant helix in $\boldsymbol{E}^3$.*

(4): *ψ is a $(k-1)$-principal-donor curve of a slant helix in $\boldsymbol{E}^3$.*

(5): *ψ is a k-principal-donor curve of a general helix in $\boldsymbol{E}^3$.*

(6): *ψ is a $(k+1)$-principal-donor curve of a plane curve.*

(7): *A principal-direction curve of ψ is a $(k-1)$-slant helix in $\boldsymbol{E}^3$.*

(8): *A second principal-direction curve of ψ is a $(k-2)$-slant helix in $\boldsymbol{E}^3$.*

(9): *A $(k-1)$-principal-direction curve of ψ is a slant helix in $\boldsymbol{E}^3$.*

(10): *A k-principal-direction curve of ψ is a general helix in $\boldsymbol{E}^3$.*

(11): *A $(k+1)$-principal-direction curve of ψ is a plane curve.*

As a similar methods as above, we can deduce the position vector of a k-slant helices in Euclidean 3-space. Here we can write the following result:

From theorem 2.4, we can write the following result:

Theorem 4.2. *The intrinsic equations of a $(k+1)$-slant helix $\Psi_{(k+1)}$ takes the form*

$$\kappa_{k+1}(s)=\kappa_k(s)\Big|\cos\Big(\int\tau_k(s)ds\Big)\Big| \text{ and } \tau_{k+1}(s)=\kappa_k(s)\sin\Big(\int\tau_k(s)ds\Big), \tag{22}$$

where $\kappa_k=\kappa_k(s)$ and $\tau_k=\tau_k(s)$ are the intrinsic equations of a k-slant helix Ψ_k.

From the above theorem we can deduce the intrinsic equations of some special curves.

(1): When the curve ψ is a plane curve, then we know that the intrinsic equations of this curve is $\kappa=\kappa(s)$ and $\tau=0$. So that the intrinsic equation of a 0-slant helix (general helix) Ψ_0 is

$$\kappa_0(s)=a\,\kappa(s) \text{ and } \tau_0(s)=b\,\kappa(s), \tag{23}$$

or in the standard form

$$\kappa_0(s)=\kappa(s) \text{ and } \tau_0(s)=m\,\kappa(s), \tag{24}$$

In this case, it is easy to prove that:

$$\sigma_0(s)=\frac{\tau_0(s)}{\kappa_0(s)}=m,$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line (axis of a general helix) and the tangent vector of the general helix.

(2): When the curve $\psi = \Psi_0$ is a general helix, then the intrinsic equation of a 1-slant helix (slant helix) Ψ_1 is

$$\kappa_1(s) = \kappa(s)|\cos\Big(m\int\kappa(s)ds\Big)| \text{ and } \tau_1(s) = \kappa(s)\sin\Big(m\int\kappa(s)ds\Big), \tag{25}$$

In this case, we have

$$\sigma_0(s) = \frac{\tau_1(s)}{\kappa_1(s)} = \tan\Big(m\int\kappa(s)ds\Big),$$

$$\sigma_1(s) = \frac{\sigma_0'(s)}{\kappa_1(s)\big(1+\sigma_0^2(s)\big)^{3/2}} = m.$$

We can solved the above ordinary differential equation and obtained the explicit relation between the torsion and curvature (intrinsic equations) for a slant helix as follows:

$$\kappa_1 = \kappa_1(s), \quad \tau_1 = \pm\frac{m\,\kappa_1(s)\int\kappa_1(s)ds}{\sqrt{1-m^2\Big(\int\kappa_1(s)ds\Big)^2}}, \tag{26}$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector of the slant helix.

(3): When the curve $\psi = \Psi_1$ is a slant helix, then the intrinsic equation of a 2-slant helix (slant-slant helix) Ψ_2 is

$$\kappa_2(s) = \kappa_1(s)|\cos\Big(\int\tau_1(s)ds\Big)| \text{ and } \tau_2(s) = \kappa_1(s)\sin\Big(\int\tau_1(s)ds\Big), \tag{27}$$

From (40), we have $\int\tau_1(s)ds = -\dfrac{\sqrt{1-m^2\big(\int\kappa_1(s)ds\big)^2}}{m}$. Now, from equation (27) we can write the following lemma:

Lemma 4.3. *The intrinsic equations of a slant-slant helix takes the form*

$$\begin{cases} \kappa_2(s) = \kappa_1(s)|\cos\Big(\dfrac{\sqrt{1-m^2\big(\int\kappa_1(s)ds\big)^2}}{m}\Big)|, \\ \tau_2(s) = -\kappa_1(s)\sin\Big(\dfrac{\sqrt{1-m^2\big(\int\kappa_1(s)ds\big)^2}}{m}\Big). \end{cases} \tag{28}$$

where $\kappa_1(s)$ is an arbitrary function of s.

Proof: For a slant-slant helix with intrinsic equations (28), we can prove the quantity $\sigma_2(s)$ is constant as the following:

$$\sigma_0(s) = \frac{\tau_2(s)}{\kappa_2(s)} = -\tan\Big(\frac{\sqrt{1-m^2\big(\int\kappa_1(s)ds\big)^2}}{m}\Big),$$

$$\sigma_1(s) = \frac{\sigma_0'(s)}{\kappa_2(s)\big(1+\sigma_0^2(s)\big)^{3/2}} = \frac{m\int\kappa_1(s)ds}{\sqrt{1-m^2\big(\int\kappa_1(s)ds\big)^2}},$$

$$\sigma_2(s) = \frac{\sigma_1'(s)}{\kappa_2(s)\sqrt{1+\sigma_0^2}\big(1+\sigma_1^2(s)\big)^{3/2}} = m.$$

Then the proof is completed.

If we substitute (28) in (21), we can write the following important theorem:

Theorem 4.4. *The position vector $\Psi_2 = (\Psi_{21}, \Psi_{22}, \Psi_{23})$ of a slant-slant helix whose the vector ψ_3 makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:*

(29)

$$\Psi_2(s) = \begin{cases} \Psi_{21} = \frac{n}{m}\int\Big[m\sin[\Phi]\big(\sqrt{1-\theta^2}\cos[\Theta] + \frac{\theta}{n}\sin[\Theta]\big) - \cos[\Phi]\cos[\Theta]\Big]ds, \\ \Psi_{22} = \frac{n}{m}\int\Big[m\sin[\Phi]\big(\sqrt{1-\theta^2}\sin[\Theta] - \frac{\theta}{n}\cos[\Theta]\big) - \cos[\Phi]\sin[\Theta]\Big]ds, \\ \Psi_{23} = -\frac{n}{m}\int\big(m\cos[\Phi] + \sqrt{1-\theta^2}\sin[\Phi]\big)ds, \end{cases}$$

where $\Phi = \frac{\sqrt{1-\theta^2}}{m}$, $\Theta = \frac{1}{n}\sin^{-1}[\theta]$, $\theta = m\int\kappa_1(s)ds$, $\kappa_1(s)$ is an arbitrary function of s, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a slant-slant helix) and the vector $\psi_3(s)$ of the curve Ψ_2.

It is worth noting that: the curvature $\kappa_2(s)$ and the torsion $\tau_2(s)$ of a slant-slant helix $\Psi_2(s)$ takes the form (28).

5. Applications

In this section, we introduce the position vector of some slant-slant helices for several choices for the curvature κ_2 and torsion τ_2 from the curvature κ_1 and torsion τ_1 of the slant helices.

Example 5.1. *If we take the case of a slant helix with*

$$\kappa_1 = 1, \quad \tau_1 = \frac{m\,s}{\sqrt{1-m^2\,s^2}}, \tag{30}$$

which are the intrinsic equations of a Salkowski curve [16] *and substituting (30) in (28), we can deduce the intrinsic equations of a slant-slant helix as the following:*

$$\kappa_2 = \cos\big[\frac{\sqrt{1-m^2 s^2}}{m}\big], \quad \tau_2 = -\sin\big[\frac{\sqrt{1-m^2 s^2}}{m}\big]. \tag{31}$$

Substituting (30) and (31) in (29) we have the explicit parametric representation of such curve as follows:

$$\Psi_2(s) = \begin{cases} \Psi_{21}(s) = \dfrac{n}{2m}\int\Bigg(\big(\dfrac{m^2\, s}{n}-1\big)\cos[\Phi_-]+\big(\dfrac{m^2\, s}{n}+1\big)\cos[\Phi_+] \\ \qquad +m\sqrt{1-m^2s^2}\big(\sin[\Phi_+]+\sin[\Phi_-]\big)\Bigg)ds, \\ \Psi_{22}(s) = \dfrac{n}{2m}\int\Bigg(\big(1-\dfrac{m^2\, s}{n}\big)\sin[\Phi_-]-\big(1+\dfrac{m^2\, s}{n}\big)\sin[\Phi_+] \\ \qquad +m\sqrt{1-m^2s^2}\big(\cos[\Phi_-]-\cos[\Phi_+]\big)\Bigg)ds, \\ \Psi_{23}(s) = -\dfrac{n}{m}\int\Bigg(m\cos\big[\Omega\big]+\sqrt{1-m^2s^2}\sin\big[\Omega\big]\Bigg)ds, \end{cases} \tag{32}$$

where $\Phi_\pm = \Omega \pm \dfrac{1}{n}\sin^{-1}[ms]$ *and* $\Omega = \dfrac{\sqrt{1-m^2s^2}}{m}$.

Example 5.2. *If we take the case of a slant helix with*

$$\kappa_1 = \frac{m\, s}{\sqrt{1-m^2\, s^2}}, \quad \tau_1 = -1, \tag{33}$$

which are the intrinsic equations of anti-Salkowski curve [16] *and substituting (33) in (28), we can deduce the intrinsic equations of a slant-slant helix as the following:*

$$\kappa_2 = \frac{m\, s\cos[s]}{\sqrt{1-m^2s^2}}, \quad \tau_2 = -\frac{m\, s\sin[s]}{\sqrt{1-m^2s^2}}. \tag{34}$$

Substituting (33) and (34) in (29) we have the explicit parametric representation of such curve as follows:

(35)

$$\Psi_2(s) = \begin{cases} \Psi_{21}(s) = \dfrac{1}{2}\int\Big[\big(\sqrt{1-m^2s^2}-\dfrac{n}{m}\big)\cos[\Phi_-]-\big(\sqrt{1-m^2s^2}+\dfrac{n}{m}\big)\cos[\Phi_+] \\ \qquad +n\, m\, s\big(\sin[\Phi_-]+\sin[\Phi_+]\big)\Big]ds, \\ \Psi_{22}(s) = \dfrac{1}{2}\int\Big[\big(\sqrt{1-m^2s^2}-\dfrac{n}{m}\big)\sin[\Phi_-]+\big(\sqrt{1-m^2s^2}+\dfrac{n}{m}\big)\sin[\Phi_+] \\ \qquad +n\, m\, s\big(\cos[\Phi_+]-\cos[\Phi_-]\big)\Big]ds, \\ \Psi_{23}(s) = n\big(s\cos[s]-2\sin[s]\big), \end{cases}$$

where $\Phi_\pm = s \pm \dfrac{1}{n}\sin^{-1}\left[\sqrt{1-m^2 s^2}\right]$.

Example 5.3. *If we take the case of a slant helix with*

$$\kappa_1 = \frac{\mu}{m}\cos[\mu\, s], \quad \tau_1 = -\frac{\mu}{m}\sin[\mu\, s], \tag{36}$$

where μ is an arbitrary constant, and substituting (36) in (28), we can deduce the intrinsic equations of a slant-slant helix as the following:

$$\kappa_2 = \frac{\mu}{m}\cos[\mu\, s]\cos\Big[\frac{1}{m}\cos[\mu\, s]\Big], \quad \tau_2 = -\frac{\mu}{m}\cos[\mu\, s]\sin\Big[\frac{1}{m}\cos[\mu\, s]\Big]. \tag{37}$$

Substituting (36) and (37) in (29) we have the explicit parametric representation of such curve as follows:

(38)

$$\Psi_2(s) = \begin{cases} \Psi_{21}(s) = \int \Bigg(\sin[\mu\, s]\sin\big[\frac{\mu\, s}{n}\big]\sin\Big[\frac{1}{m}\cos[\mu\, s]\Big] \\ \quad +\frac{n}{m}\cos\big[\frac{\mu\, s}{n}\big]\Big(m\cos[\mu\, s]\sin\Big[\frac{1}{m}\cos[\mu\, s]\Big] - \cos\Big[\frac{1}{m}\cos[\mu\, s]\Big]\Big)\Bigg)ds, \\ \Psi_{22}(s) = \int \Bigg(-\frac{n}{m}\sin\big[\frac{\mu\, s}{n}\big]\cos\Big[\frac{1}{m}\cos[\mu\, s]\Big] \\ \quad +\Big(n\cos[\mu\, s]\sin\big[\frac{\mu\, s}{n}\big] - \sin[\mu\, s]\cos\big[\frac{\mu\, s}{n}\big]\Big)\sin\Big[\frac{1}{m}\cos[\mu\, s]\Big]\Bigg)ds, \\ \Psi_{23}(s) = -\frac{n}{m}\int\Big(m\cos\Big[\frac{1}{m}\cos[\mu\, s]\Big] + \cos[\mu\, s]\sin\Big[\frac{1}{m}\cos[\mu\, s]\Big]\Big)ds. \end{cases}$$

Example 5.4. *If we take the case of a spherical slant helix* [7] *with*

$$\begin{aligned} \kappa_1 &= \frac{1}{c\,\cos\Big[\frac{\sin[\theta]}{m}\Big] + \sqrt{1-c^2}\,\sin\Big[\frac{\sin[\theta]}{m}\Big]}, \\ \tau_1 &= \frac{\cot[\theta]}{c\,\cos\Big[\frac{\sin[\theta]}{m}\Big] + \sqrt{1-c^2}\,\sin\Big[\frac{\sin[\theta]}{m}\Big]}, \end{aligned} \tag{39}$$

where c is an arbitrary constant and the arc-length parameter is

$$s = \int \frac{\sin[\theta]}{m}\left(c\,\cos\Big[\frac{\sin[\theta]}{m}\Big] + \sqrt{1-c^2}\,\sin\Big[\frac{\sin[\theta]}{m}\Big]\right)d\theta.$$

Substituting (39) in (28), we can deduce the intrinsic equations of a slant-slant helix as the following:

$$\kappa_2 = \frac{\cos\left[\frac{\sin[\theta]}{m}\right]}{c\,\cos\left[\frac{\sin[\theta]}{m}\right] + \sqrt{1-c^2}\,\sin\left[\frac{\sin[\theta]}{m}\right]},$$

(40)

$$\tau_2 = -\frac{\sin\left[\frac{\sin[\theta]}{m}\right]}{c\,\cos\left[\frac{\sin[\theta]}{m}\right] + \sqrt{1-c^2}\,\sin\left[\frac{\sin[\theta]}{m}\right]}.$$

Substituting (39) and (40) in (29) we have the explicit parametric representation of such curve as follows:

(41)

$$\Psi_2(\theta) = \begin{cases} \Psi_{21}(\theta) = \int \left[\cos[\theta]\sin\left[\frac{\sin[\theta]}{m}\right]\sin\left[\frac{\sin^{-1}\left[\cos[\theta]\right]}{n}\right] \right. \\ \left. \quad +\frac{n}{m}\cos\left[\frac{\sin^{-1}\left[\cos[\theta]\right]}{n}\right]\left(m\,\sin[\theta]\sin\left[\frac{\sin[\theta]}{m}\right] - \cos\left[\frac{\sin[\theta]}{m}\right]\right)\right] ds, \\ \Psi_{22}(\theta) = \int \left[-\cos[\theta]\sin\left[\frac{\sin[\theta]}{m}\right]\cos\left[\frac{\sin^{-1}\left[\cos[\theta]\right]}{n}\right] \right. \\ \left. \quad +\frac{n}{m}\sin\left[\frac{\sin^{-1}\left[\cos[\theta]\right]}{n}\right]\left(m\,\sin[\theta]\sin\left[\frac{\sin[\theta]}{m}\right] - \cos\left[\frac{\sin[\theta]}{m}\right]\right)\right] ds, \\ \Psi_{23}(\theta) = \frac{n}{m}\int \left(m\cos\left[\frac{\sin[\theta]}{m}\right] + \sin[\theta]\sin\left[\frac{\sin[\theta]}{m}\right]\right) ds, \end{cases}$$

where

$$ds = \frac{\sin[\theta]}{m}\left(c\,\cos\left[\frac{\sin[\theta]}{m}\right] + \sqrt{1-c^2}\,\sin\left[\frac{\sin[\theta]}{m}\right]\right) d\theta.$$

Remark 5.5. *When $c = 1$, the intrinsic equations of a slant-slant helix (41) takes the form*

(42) $$\kappa_2 = 1, \qquad \tau_2 = -\tan\left[\frac{\sin[\theta]}{m}\right].$$

We called this curve by a Salkowski slant-slant helix and the position vector takes the form (41) such that $ds = \left(\frac{\sin[\theta]}{m}\right)\cos\left[\frac{\sin[\theta]}{m}\right] d\theta$.

Remark 5.6. *When $c = 0$, the intrinsic equations of a slant-slant helix (41) takes the form*

$$\kappa_2 = \cot\left[\frac{\sin[\theta]}{m}\right], \qquad \tau_2 = -1. \tag{43}$$

We called this curve by an Anti-Salkowski slant-slant helix and the position vector takes the form (41) such that $ds = \left(\frac{\sin[\theta]}{m}\right)\sin\left[\frac{\sin[\theta]}{m}\right]d\theta$.

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