PROXIMINALITY AND REMOTALITY IN NORMED LINEAR SPACES

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Abstract. In this paper, we consider **"Nearest points"** and **"Farthest points"** in normed linear spaces, We obtain necessary and sufficient conditions for subsets of normed linear spaces to be proximinal, Chebyshev, Remotal and uniquely remotal.

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1. Introduction

Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. Starting in 1853, a Russian mathematician P.L. Chebyshev made significant contributions in

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the theory of best approximation. The Weierstrass approximation theorem of 1885 by K. Weierstrass is well known. The study was followed in the first half of the 20th Century by L.N.H. Bunt (1934), T.S. Motzkin (1935) and B. Jessen (1940). B. Jessen was the first to make significant contributions in the theory of farthest points. This theory is less developed as compared to the theory of best approximation.

Let $(X, \|.\|)$ be a normed linear space, W a subset of X. A point $y_0 \in W$ is said to be a best approximation point (nearest point) for $x \in X$, if

$$\|x - y_0\| \leq \|x - y\|,$$

for each $y \in W$. If $d_x = d(x, W) = \inf_{y \in W} \|x - y\|$, then y_0 is a best approximation point of x, if $\|x - y_0\| = d_x$. For each $x \in X$, put

$$P_W(x) = \{y_0 \in W : \ \|x - y_0\| = d_x\}.$$

For each $x \in X$, if $P_W(x)$ is nonempty (a singleton), we say that W is proximinal (Chebyshev). For each $x \in X \backslash W$, if $P_W(x) = \emptyset$, we say that W is anti-proximinal (see [2 , 5 , 8 , 10]).

Let X be a normed linear space and W a bounded subset of X. A point $y_0 \in W$ is said to be a farthest point for $x \in X$, if

$$\|x - y_0\| \geq \|x - y\|,$$

for each $y \in W$. If $\delta_x = \delta(x, W) = \sup_{y \in W} \|x - y\|$, then y_0 is a farthest point of x, if $\|x - y_0\| = \delta_x$. For each $x \in X$ put

$$F_W(x) = \{y_0 \in W : \ \|x - y_0\| = \delta_x\}.$$

For each $x \in X$, if $F_W(x)$ is nonempty (a singleton), we say that W is remotal (uniquely remotal). For each $x \in X$, if $F_W(x) = \emptyset$, we say that W is anti-remotal. (see [3 , 4 , 6 , 7 , 9 , 10 , 12 , 13]).

Let X be a normed linear space , $x \in X, \ r \in \mathbb{R}^+$. We put

$$B[x, r] = \{z \in X : \ \|x - z\| \leq r\},$$

$$B^c[x, r] = \{z \in X : \ \|x - z\| \geq r\}.$$

2. Proximinal, Chebyshev, Remotal, Uniquely remotal sets

In this section we consider proximinal, Chebyshev, remotal and uniquely remotal sets,

Theorem 2.1. *Let $(X, \|.\|)$ be a normed linear space and A a closed subset of X.*

a) The set A in X is proximinal if and only if for every $x \in X$, $x \in H_{d_x}$, where $H_{d_x} = A + B[0, d_x]$.

b) The set A in X is anti-proximinal if and only if for every $x \in X$, $x \notin H_{d_x}$.

c) The proximinal set A is Chebyshev if and only if for every $x \in X$, $x \in H_{d_x}^{\bigoplus}$, (where $\bigoplus$ means that the sum decomposition of each element $x \in X$ is unique.)

Proof. a) Suppose that A is proximinal and $x \in X \backslash A$ we take $d_x = dist(x, A)$. It is clear that $d_x > 0$ and there exists $a \in A$ such that $\|x - a\| = d_x$. It follows that $x = a + (x - a) \in H_{d_x}$. Also if $x \in A$, then $d_x = 0$ and $B[0, d_x] = \{0\}$. Therefore $x = x + 0 \in H_{d_x}$.
Suppose $x \in X$ and $x \in H_{d_x}$. It follows that for some $a \in A$, $\|x-a\| \le d_x \le \|x-a\|$. Therefore $a \in P_A(x)$, and A is proximinal.

b) Suppose that A is anti-proximinal and $x \in X \backslash A$, then $P_A(x) = \emptyset$. If $x \in H_{d_x}$, there exists an $a \in A$ such that $x - a \in B[0, d_x]$. Therefore $a \in P_A(x)$, a contradiction. It follows that $x \notin H_{d_x}$. For each $x \in X$, if $x \notin H_{d_x}$ and there exists an $a \in P_A(x)$, then $x = a + (x - a) \in H_{d_x}$ so a contradiction. It follows that $P_A(x) = \emptyset$ and A is anti-proximinal.

c) Suppose that the set A is Chebyshev. Since A is proximinal, for each $x \in X \backslash A$, $x \in H_{d_x}$. Suppose there exist two representations for x. That $x = a_1 + u_1 = a_2 + u_2$, where $a_1, a_2 \in A$ and $u_1, u_2 \in B[0, d_x]$. Since $\|x - a_i\| \le d_x$ for $i = 1, 2$. Therefore $a_1, a_2 \in P_A(x)$ and A is Chebyshev, it follows that $a_1 = a_2$ and $u_1 = u_2$. Therefore $x \in H_{d_x}^{\bigoplus}$.

Conversely, if for $x \in X$, there exist $a_1, a_2 \in P_A(x)$, then $u_i = x - a_i \in B[0, d_x]$ and $x = a_1 + u_1 = a_2 + u_2$. Since $x \in H_{d_x}^{\bigoplus}$, we have $a_1 = a_2$. Therefore A is Chebyshev.

Theorem 2.2. *Let $(X, \|.\|)$ be a normed linear space and A a non-empty bounded subset of X.*

a) The set A in X is remotal if and only if for every $x \in X$, $x \in K_{\delta_x}$, where $K_{\delta_x} = A + B^c[0, \delta_x]$.

b) The set A in X is anti-remotal if and only if for every $x \in X$, $x \notin K_{\delta_x}$.

c) The remotal set A is uniquely remotal if and only if for every $x \in X$, $x \in K_{\delta_x}^{\bigoplus}$, then $x \in K_{\delta_x}^{\bigoplus}$, (where $\bigoplus$ means that the sum decomposition of each element $x \in X$ is unique.)

Proof. a) Suppose that the set A is remotal, for an arbitrary $x \in X$, there exists $a \in A$ such that $\|x-a\| = \delta_x$, it follows that $x = a+(x-a) \in K_{\delta_x}$. Suppose $x \in X$ and $x \in K_{\delta_x}$. It follows that for some $a \in A$, $\|x - a\| \geq \delta_x \geq \|x - a\|$. Therefore $a \in F_A(x)$, and A is remotal.

b) Suppose that A is anti-remotal and $x \in X$, then $F_A(x) = \emptyset$. If $x \in K_{\delta_x}$, there exists an $a \in A$ such that $x - a \in B^c[0, \delta_x]$. Therefore $a \in F_A(x)$, a contradiction. It follows that $x \notin K_{\delta_x}$.
For each $x \in X$, if $x \notin K_{\delta_x}$ and there exists an $a \in F_A(x)$. Then $x = a + (x - a) \in K_{\delta_x}$ and a contradiction. It follows that $F_A(x) = \emptyset$, A is anti-remotal.

c) Since A is remotal and $x \in X$ then $x \in K_{\delta_x}$. If there exist two representing for x. That is, $x = a_1 + u_1 = a_2 + u_2$ where $a_1, a_2 \in A$ and $u_1, u_2 \in B^c[0, \delta_x]$. Since $a_1, a_2 \in F_A(x)$ and A is uniquely remotal. It follows that $a_1 = a_2$ and $u_1 = u_2$. Therefore $x \in K_{\delta_x}^{\bigoplus}$.
Suppose A is remotal and for each $x \in X$, there exist $a_1, a_2 \in F_A(x)$. Then $u_i = x - a_i \in B[0, \delta_x]$ and $x = a_1 + u_1 = a_2 + u_2$. Since $x \in K_{\delta_x}^{\bigoplus}$, we have $a_1 = a_2$. Therefore A is uniquely remotal.

Example 2.1. *From Theorems 2.1, 2.2, we also obtain the following known result:*

Every non-empty compact set in a normed linear space X is proximinal and remotal.

Proof. Suppose A is a non-empty compact set in X and $x \in X$. For $n \in \mathbb{N}$, consider the set

$$H_n(x) = \{a \in A : \ \|x - a\| \leq d_x + \frac{1}{n}\}.$$

From the definition of d_x , $H_n(x) \neq \emptyset$, $H_{n+1}(x) \subseteq H_n(x)$ and $H_n(x) \subseteq A$ is compact. From Cantor's intersection Theorem, there exists an $a \in A$ such that for all $n \geq 1$, $a \in H_n(x)$. That is for all $n \geq 1$, $\|x - a\| \leq d_x + \frac{1}{n}$. Therefore $\|x - a\| \leq d_x$ and $x \in H_{d_x}$, it follows that A is proximinal.

For $x \in X$ and $n \in \mathbb{N}$, consider the set

$$K_n(x) = \{a \in A : \ \|x - a\| \geq \delta_x - \frac{1}{n}\}.$$

From the definition of δ_x, $K_n(x) \neq \emptyset$, $K_{n+1}(x) \subseteq K_n(x)$ and $K_n(x) \subseteq A$ is compact. Therefore there exists an $a \in A$ such that for all $n \geq 1$, $a \in K_n(x)$. That is for all $n \geq 1$, $\|x - a\| \geq \delta_x - \frac{1}{n}$. Therefore $\|x - a\| \geq \delta_x$, and $x \in K_{\delta_x}$, it follows that A is remotal.

In particular , suppose $A = \{a_1, a_2, a_3, ..., a_n\}$ is a finite set of a normed space $(X, \|.\|)$. Then A is remotal and proximinal. Because if $x \in X$, then $x = a_k + x - a_k \in K_{\delta_x}$, where $\|x - a_k\| = \max_{a_i \in A} \|x - a_i\|$.
Also $x = a_l + x - a_l \in H_{d_x}$, where $\|x - a_l\| = \min_{a_i \in A} \|x - a_i\|$.

Theorem 2.3. *Let $(X, \|.\|)$ be a normed linear space and A be a remotal (proximinal) subset of X, $0 \in A$ and $x \in X$. If the set $K_{\delta_x}(H_{d_x})$ is compact then the set $F_A(x)(P_A(x))$ is compact.*

Proof. Suppose that the set $K_{\delta_x} \neq \emptyset$ $(H_{d_x} \neq \emptyset)$. Suppose the set K_{δ_x} (H_{d_x}) is compact. If $\{g_n\} \subseteq F_A(x)(\{g_n\} \subseteq P_A(x))$, then $\|x - g_n - 0\| = \delta_x(\|x - g_n - 0\| = d_x)$. Since $0 \in A$, we have $\{x - g_n\} \subseteq K_{\delta_x}(\{x - g_n\} \subseteq H_{d_x})$ has a convergent subsequence $\{x - g_{n_k}\}$. Therefore there exists a $z_0 \in K_{\delta_x}(z_0 \in H_{d_x})$ such that $x - g_{n_k} \to z_0$, $g_{n_k} \to x - z_0$ as $k \to \infty$. Since $g_{n_k} \subseteq A$ and A is closed, we have $g_0 = x - z_0 \in A$ and $\|x - g_{n_k}\| = \delta_x(\|x - g_{n_k}\| = d_x)$. It follows that $\|x - g_0\| = \delta_x(\|x - g_0\| = d_x)$ and $g_0 \in F_A(x)(g_0 \in P_A(x))$. Therefore the set $F_A(x)(P_A(x))$ is compact.

Definition 2.1. [10] *Let $(X, \|.\|)$ be a normed linear space and W a bounded subset of X, $x \in X$, $\epsilon > 0$ and $w_0 \in W$. We say that w_0 is a ϵ-best approximation to x if $\|x - g_0\| \leq d(x, W) + \epsilon$.*

Definition 2.2. [10] *Let $(X, \|.\|)$ be a normed linear space and W a bounded subset of X, $x \in X$, $\epsilon > 0$ and $w_0 \in W$. We say that w_0 is a ϵ-farthest to x if $\|x - w_0\| \geq \delta(x, W) - \epsilon$.*

Theorem 2.4. *Let $(X, \|.\|)$ be a normed linear space and W a subset of X, $x \in X$ and $\epsilon > 0$. Then $X = \bigcup_{x \in X} H_{d_x^\epsilon}$. Where $H_{d_x^\epsilon} = W + B[0, d_x^\epsilon]$ and $d_x^\epsilon = d(x, W) + \epsilon$.*

Proof. Suppose $x \in X$, there exists a $w_0 \in W$, such that $\|x - w_0\| \leq d(x, W) + \epsilon$. Therefore $x = w_0 + (x - w_0) \in H_{d_x^\epsilon}$.

Theorem 2.5. *Let $(X, \|.\|)$ be a normed linear space and W a bounded subset of X, $x \in X$ and $\epsilon > 0$. Then $X = \bigcup_{x \in X} K_{\delta_x^\epsilon}$. Where $K_{\delta_x^\epsilon} = W + B^c[0, \delta_x^\epsilon]$ and $\delta_x^\epsilon = \delta(x, W) - \epsilon$.*

3. Co-proximinality and co-remotality

A kind of approximation, called best co-approximation was introduced by Franchettei and Furi in 1972 [1]. Some results in best co-approximation theory in linear normed spaces have been obtained by P. L. Papini and I. Singer (see [11]). In this section we consider co-proximinality and co-remotality in normed linear spaces.

Definition 3.1. *Let $(X, \|.\|)$ be a normed linear space, G a non-empty subset of X and $x \in X$. We say that $g_0 \in G$ is a best co-approximation of x whenever $\|g - g_0\| \leq \|x - g\|$ for all $g \in G$. We denote the set of all best co-approximations of x in G by $R_G(x)$.*
We say that G is a co-proximinal subset of X if $R_G(x)$ is a non-empty subset of G for all $x \in X$. Also, we say that G is a co-Chebyshev subset of X if $R_G(x)$ is a singleton subset of G for all $x \in X$.

Definition 3.2. *Let $(X, \|.\|)$ be a normed linear space, A a subset of X, $x \in X$ and $m_0 \in A$. We say that m_0 is co-farthest to x if $\|m_0 - a\| \geq \|x - a\|$ for every $a \in A$. The set of co-farthest points to x in A is denoted by*

$$C_A(x) = \{a_0 \in A : \ \|a_0 - a\| \geq \|x - a\| \ for \ every \ a \in A\}.$$

The set A is said to be co-remotal if $C_A(x)$ has at least one element for every $x \in X$. If for each $x \in X$, $C_A(x)$ has exactly one element in A, then the set A is called co-uniquely remotal. We define for $a_0 \in A$,

$${C_A}^{-1}(a_0) = \{x \in X : \ a_0 \in C_A(x)\} = \{x \in X : \ \|a_0 - a\| \geq \|x - a\| \ for \ every \ a \in A\}.$$

${C_A}^{-1}(a_0)$ is a closed set and $a_0 \in {C_A}^{-1}(a_0)$. Note that if $x \in A$, then $x \in C_A(x)$.

Example 3.1. *Suppose $X = \mathbb{R}$ and $A = [1, 2] \bigcup \{3\} \backslash \{1\}$. We set $x = 1$ and $a_0 = 3$. Then $a_0 \in C_A(x)$.*

Theorem 3.1. *Let $(X, \|.\|)$ be a normed linear space and A a subset of X.*

a) If for every $x \in X$ and for every $a \in A$, $a \in H_{d_x}$, then A is co-proximinal.

b) If for every $x \in X$ and for every $a \in A$, there exists a unique $b \in H^{\bigoplus}_{\|x-a\|}$, then A is co-Chebyshev.

Proof. a) Suppose $x \in X$, for every $a \in A$ there exists $a_0 \in A$ such that $a - a_0 \in B[0, d_x]$. Therefore for every $a \in A$

$$\begin{aligned} \|a - a_0\| &\leq d_x \\ &\leq \|x - a\|. \end{aligned}$$

That is $a_0 \in R_A(x)$ so A is co-proximinal.
b) Suppose $x \in X$, $a \in A$ and there exists an unique $b \in H^{\bigoplus}_{\|x-a\|}$, by part (a), $R_A(x)$ is non-empty. The set A is co-proximinal.

For each $x \in X$ if there exist $a_1, a_2 \in R_A(x)$, then for $a \in A$ we have $\|a_i - a\| \leq \|x - a\|$ for $i = 1, 2$. Therefore for $a \in A$, $a_i - a \in B[0, \|x - a\|]$, and for $a \in A$, we have $a_i \in H^{\bigoplus}_{\|x-a\|}$. This is a contraction It follows that A is co-Chebyshev.

Theorem 3.2. *Let $(X, \|.\|)$ be a normed linear space and A a subset of X.*

a) If for every $x \in X$ and for every $a \in A$, $a \in K_{\delta_x}$, then A is co-remotal.

b) If for every $x \in X$ and for every $a \in A$, there exists a unique $b \in K^{\bigoplus}_{\|x-a\|}$, then A is co-uniquely remotal.

Proof. a) Suppose $x \in X$ and $a \in A$. Suppose there exists an $a_0 \in A$ such that $a - a_0 \in B^c[0, \delta_x]$. Therefore for every $a \in A$

$$\begin{aligned} \|a - a_0\| &\geq \delta_x \\ &\geq \|x - a\|. \end{aligned}$$

That is $a_0 \in C_A(x)$ so A is co-remotal.
b) If $x \in X$ and $a \in A$ if there exists an unique $b \in K^{\bigoplus}_{\|x-a\|}$, then $C_A(x)$ is non-empty. The set A is co-remotal.

For $x \in X$ if there exist $a_1, a_2 \in C_A(x)$, then for $a \in A$ we have $\|a_i - a\| \leq \|x - a\|$ for $i = 1, 2$. Therefore for $a \in A$, $a_i - a \in B^c[0, \|x - a\|]$, and for $a \in A$, we have $a_i \in K^{\bigoplus}_{\|x-a\|}$. This is a contraction. It follows that A is co-uniquely remotal.

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