A NEW MODIFICATION OF LEGENDRE-GAUSS COLLOCATION METHOD FOR SOLVING A CLASS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

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(Received: 7 February 2017, Accepted: 21 January 2018)

Abstract. In this paper, the optimal conditions for fractional optimal control problems (FOCPs) were derived in which the fractional differential operators defined in terms of Caputo sense and reduces this problem to a system of fractional differential equations (FDEs) that is called two-point boundary value (TPBV) problem. An approximate solution of this problem is constructed by using the Legendre-Gauss collocation method such that the exact boundary conditions are satisfied. Several example are given and the optimal errors are obtained for the sake of comparison. The obtained results are shown that the technique introduced here is accurate and easily applied to solve the FOCPs.

AMS Classification: 65L03, 49K30, 26A33.

Keywords: Fractional optimal control problem, Fractional differential equation, Legendre-Gauss collocation method.

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JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER
VOL. 6, NUMBERS 1-2 (2017) 81-94.
DOI: 10.22103/JMMRC.2018.9861.1033

1. Introduction

Fractional calculus is a generalization of ordinary calculus which introduces derivatives and integrals of fractional order. The most important types of fractional derivatives are Riemann-Liouville (RLFD) and Caputo fractional derivatives (CFD) that we adopt here the Caputo definition. Major reviews on the concepts and history of fractional calculus can be found in the books of [25, 12]. Interesting and promising applications of fractional calculus have been proposed to model the physical and engineering processes, science and economics, gravity , medicine, [10, 13, 34], that have found to be best described by FDEs. In general, most of FDEs do not have exact analytic solutions, so, numerical methods have been used widely to find the approximate solutions of these equations such as Homotopy Perturbation Method (HPM) [9], the Adomian Decomposition Method (ADM) [26], the Variational Iteration Method (VIM) [11], the Generalized Differential Transformation Method (GDTM) [27], the Measurable functions approach [35] and the Fractional Difference Method (FDM) [28].

FOCPs are a subclass of optimal control problems whose dynamics are described by FDEs. This type of problems can be defined with different definitions of fractional derivatives. With the emerging number of the applications of FOCPs [15, 8], the solution of these kind of problems has become an important topic for researchers. Since, it is difficult to obtain the exact solutions of most FOCPs, approximate and numerical methods are used extensively. Work on FOCPs has started by [19] and was extended by [20, 21], where the necessary conditions of optimization are achieved with derivatives of fractional order. By using necessary optimality conditions, the FOCP is reduced to a system of FDEs and by finding its solutions, one approximates the solution to the original fractional problem. The interested reader can see [22, 37, 29, 7, 30, 1, 31, 23, 16, 17, 24, 36] for some recent study in FOCPs.

In this paper, we would like to investigate the possibility of presence numerical approximated solutions for a class of FOCPs. To proceed, we achieved the necessary conditions of optimization for this class of FOCPs with a system of FDEs. To solve this system, first using a modified approach Caputo fractional derivatives (CFD) that our problem relies on. By using this approach and a joint application of Legendre polynomials, we transform the original system of FDEs into a discrete system of ordinary differential equations, in way by obtaining the optimal solutions of this system, we obtain the approximate solution of the FOCP.

The motivation of this research is transforming the FOCP to a system of FDEs. Then by approximating FDs, the Legendre-Gauss collocation method have been used for this system. However, as the problem have the singular point, discrete methods cannot be used in this case. So, we have used and showed the efficiency of Legendre-Gauss collocation method for our problem. In the next section, we described the method in detail.

This study is organized as follows: In Section 2, some important definitions and necessary preliminaries of fractional derivatives and Legendre-Gauss collocation method are described. We summarize the necessary optimality conditions of FOCPs and the reconstraction approach of it's solutions in Section 3. Finally, some examples are solved to demonstrate the performance of the method in Section 4 and a brief summary is given in the final section.

2. Preliminary Considerations

In this section, we present some notations, definitions and preliminary facts of the fractional calculus theory which will be used further in this work. For the sake of simplicity, we consider $0 < \alpha < 1$. These considerations don't affect the generalization of the derivation procedure. For the definitions of fractional derivatives and some of their applications, see [2, 3, 14, 32, 6]

2.1. **Fractional Calculus.**

Definition 2.1. *Let $f \in L_1[t_0, t_f]$. The left Riemann-Liouville Fractional Integral (RLFI) of order α of function $f \in L_1[t_0, t_f]$ is defined as*

$$I_{t_0}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{x} (x-t)^{\alpha-1} f(t) dt,$$

and the right RLFI as

$$I_{t_f}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{t_f} (t-x)^{\alpha-1} f(t) dt,$$

for all $x \in [t_0, t_f]$ and $\Gamma(.)$ is known as the Euler-Gamma function.

We denote $I_t^0 f(x) = f(x)$. Now, we define the Left and Right RLFDs.

Definition 2.2. *For a continuous function $f : [t_0, +\infty) \to \mathbb{R}$, the left RLFD of fractional order $\alpha > 0$, is defined by*

$$ {}_{t_0}D_t^{\alpha} f(x) := \frac{d^n}{dx^n} I_{t_0}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{t_0}^{x} (x-t)^{n-\alpha-1} f(t)dt, \tag{2.1} $$

that provided the right-hand side of (2.1) is pointwise defined on (t_0, ∞), where $x \in [t_0, t_f]$, $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α. Similary, the right RLFD of order α of function f, defined by

$$ {}_{t}D_{t_f}^{\alpha} f(x) := (-1)^n \frac{d^n}{dx^n} I_{t_f}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{t_f} (x-t)^{n-\alpha-1} f(t)dt. \tag{2.2} $$

Definition 2.3. *The left and right CFDs of order $\alpha \in \mathbb{R}_+$ are defined respectively, by ${}_{t_0}^{C}D_t^{\alpha} f(x) := I_{t_0}^{n-\alpha} \, {}_{t_0}D_t^{n} f(x)$ and ${}_{t}^{C}D_{t_f}^{\alpha} f(x) := (-1)^n I_{t_f}^{n-\alpha} \, {}_{t}D_{t_f}^{n} f(x)$ with $n = [\alpha] + 1$; that is*

$$ {}_{t_0}^{C}D_t^{\alpha} f(x) := \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{x} \frac{f^{(n)}(t)}{(x-t)^{n-\alpha-1}} dt \tag{2.3} $$

and

$$ {}_{t}^{C}D_{t_f}^{\alpha} f(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{t_f} \frac{f^{(n)}(t)}{(t-x)^{n-\alpha-1}} dt \tag{2.4} $$

where $t_0 \le x \le t_f$ and $f^{(n)}(t) = \frac{d^n f(t)}{dt^n} \in L_1[t_0, t_f]$ is the ordinary derivative of integer order n.

Some useful properties of fractional integrals and derivatives, include all fractional operators are linear, that is, if D is an arbitrary fractional operator, then

$$ D(tf + sg) = tD(f) + sD(g). \tag{2.5} $$

Also, for all function $f \in L_1[t_0, t_f]$; if $\alpha, \beta > 0$, then

- $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x) = I^{\alpha+\beta} f(x)$,
- $D^{\alpha} D^{\beta} f(x) = D^{\alpha+\beta} f(x)$,
- $D^{\alpha} I^{\alpha} f(x) = f(x)$,
- $D^{\alpha} I^{\beta} f(x) = I^{\beta-\alpha} f(x)$.

For approximation in numerical computations of fractional derivatives, a modified Grunwald-Letnikov approach was proposed in [5]. Authors in [18] focused on the Jacobi polynomials to solve fractional variational problems and FOCPs. Later, a new expanision formula was obtained in [33] as the following form:

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}x(t) &\simeq A(\alpha,N)(t-t_0)^{-\alpha}x(t)+B(\alpha,N)(t-t_0)^{1-\alpha}\dot{x}(t)\\
&-\sum_{p=2}^{N}C(\alpha,N)(t-t_0)^{1-p-\alpha}V_p(t)-\frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad (2.6)
\end{aligned}
$$

where $V_p(t)$ is defined as the solution of the system

$$
\begin{cases}
\dot{V}_p(t)=(1-p)(t-t_0)^{p-2}x(t),\\
V_p(t_0)=0, \quad p=2,3,\cdots,N,
\end{cases}
$$

and

$$
\begin{aligned}
{}_{t}^{C}D_{t_f}^{\alpha}x(t) &\simeq A(\alpha,N)(t_f-t)^{-\alpha}x(t)-B(\alpha,N)(t_f-t)^{1-\alpha}\dot{x}(t)\\
&+\sum_{p=2}^{N}C(\alpha,N)(t_f-t)^{1-p-\alpha}W_p(t)-\frac{x(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad (2.7)
\end{aligned}
$$

where $W_p(t)$ is the solution of the differential equation

$$
\begin{cases}
\dot{W}_p(t)=-(1-p)(t_f-t)^{p-2}x(t),\\
W_p(t_f)=0, \quad p=2,3,\cdots,N,
\end{cases}
$$

and $A(\alpha,N)$, $B(\alpha,N)$, $C(\alpha,p)$ are defined by:

$$
A(\alpha,N)=\frac{1}{\Gamma(1-\alpha)}\Big[1+\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!}\Big],
$$

$$
B(\alpha,N)=\frac{1}{\Gamma(2-\alpha)}\Big[1+\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\Big],
$$

$$
C(\alpha,p)=\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\frac{\Gamma(p-1+\alpha)}{(p-1)!}, \quad p=2,3,\cdots,N.
$$

Theorem 2.1. *If we approximate the left fractional derivative by the finite sum* (2.6)*, then the error $E_{tr(.)}$ is bounded by:*

$$|E_{tr}(t)| \leq \max_{s\in[a,t]} |x''(s)| \frac{exp((1-\alpha)^2+1-\alpha)}{\Gamma(2-\alpha)(1-\alpha)S^{1-\alpha}}(t-a)^{2-\alpha}. \tag{2.8}$$

Proof. See [33] for proofs and other details. □

2.2. **Legendre-Gauss Collocation Method.** Let $P_n(t)$ be the n th-degree Legendre polynomial defined on $[-1,1]$. Any piecewise continuous function in this interval can be expresses as follows:

$$f(t) \sim \sum_{n=0}^{\infty} c_n P_n(t); \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)dt, \;\; t\in[-1,1]. \tag{2.9}$$

Let us define the modified Legendre polynomials (MLPs) of degree at most n as follows:

$$\hat{P}_n(t) = P_n\Big(\frac{2t}{t_f-t_0} - \frac{t_f+t_0}{t_f-t_0}\Big), \qquad t\in[t_0,t_f]. \tag{2.10}$$

According to the properties can be achieved of these functions, any function $f(t) \in L_2[t_0,t_f]$ can be approximated as follows [4]:

$$f(t) \sim \sum_{n=0}^{\infty} \hat{c}_n \hat{P}_n(t); \quad \hat{c}_n = \frac{2n+1}{t_f-t_0}\int_{t_0}^{t_f} f(t)\hat{P}_n(t)dt, \;\; t\in[t_0,t_f]. \tag{2.11}$$

Therefore, if we have:

$$\begin{cases} \dfrac{d}{dt}u(t) = f(u(t),t), \;\; t_0 \leq t \leq t_f \\ u(t_0) = u_0, \end{cases} \tag{2.12}$$

the Legendre-Gauss collocation method for solving this problem is equivalent to solve the following problem:

$$\begin{cases} \dfrac{d}{dt}u^M(\hat{t}_j^M) = f(u^M(\hat{t}_j^M),\hat{t}_j^M), \\ u^M(t_0) = u_0, \;\; 1\leq j\leq M, \end{cases} \tag{2.13}$$

where:

$$u^M(t) = \sum_{n=0}^{M} \hat{c}_n \hat{P}_n(t), \tag{2.14}$$

and $\hat{P}_M(t_0, t_f)$ is the set of MLPs of degree at most M and $\hat{t}_j^M$, $1 \leq j \leq M$, are the nodes of the MLPs interpolation on $[t_0, t_f]$. To get the answer of equation (2.12), it's enough to obtain coefficients $\hat{c}_n$ from equations (2.13)-(2.14).

3. Numerical Scheme for Solving FOCPs

In this section we want to present a formulation and a numerical approximations for solving fractional order optimal control problem of Caputo type, which we define as follow:

$$\min J(u) = \frac{1}{2}\int_{t_0}^{t_f} \Big\{ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \Big\} dt \tag{3.1}$$

$$s.t. \ {}_{t_0}^{C}D_t^{\alpha} x(t) = A(t)x(t) + B(t)u(t),$$

$$x(t_0) = x_0, \ 0 \leq \alpha \leq 1$$

where $x(t)$ is the state variable, $u(t)$ is the control variable, $Q(t)$ and $R(t)$ are chosen to be positive semidefinite and positive definite matrices respectively. The aim is to find a control vector $u^*(t)$ such that the cost functional (3.1) is minimized while the dynamic equality constraint is satisfied. To obtain the necessary conditions, we define the following Hamiltonian function:

$$H(x(t), u(t), \lambda(t), t) = \frac{1}{2}\Big\{ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \Big\} + \lambda^T \Big\{ A(t)x(t) + B(t)u(t) \Big\}, \tag{3.2}$$

where $\lambda \in \mathbb{R}^n$ is the vector of the Lagrange multiplier. By application of the maximum principle for problem (3.2), we can obtain the following nonlinear TPBVP (see [5]):

$${}_{t}^{C}D_{t_f}^{\alpha} \lambda(t) = \frac{\partial H}{\partial x} = Q(t)x(t) + A^T(t)\lambda(t), \ \lambda(t_f) = 0 \tag{3.3}$$

$$\frac{\partial H}{\partial u} = R(t)u(t) + B^T(t)\lambda(t) = 0$$

$${t_0}^{C}D_t^{\alpha} x(t) = \frac{\partial H}{\partial \lambda} = A(t)x(t) + B(t)u(t), \ x(t_0) = x_0.$$

From this system of equations we obtain $u(t) = -R^{-1}B^T\lambda$. So, it can be demonstrated that the necessary conditons for problem (3.1) are as follows:

$$\tag{3.4} \begin{aligned} &{}_{t}^{C}D_{t_f}^{\alpha}\lambda(t) = Q(t)x(t) + A^T(t)\lambda(t), \;\; \lambda(t_f) = 0 \\ &{}_{t_0}^{C}D_{t}^{\alpha}x(t) = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t), \;\; x(t_0) = x_0 \end{aligned}$$

Then, by solving the above system of FDEs, we will find the optimal value of $x(t)$ and $u(t)$ in a way which the function $J(u)$ in problem (3.1) becomes minimum. To reach this goal, we use approximation (2.6), which can rewrite equations (3.3) as the following form:

$$\tag{3.5} \begin{cases} A(\alpha,N)(t_f-t)^{-\alpha}\lambda(t) - B(\alpha,N)(t_f-t)^{1-\alpha}\dot{\lambda}(t) + \sum_{p=2}^{N} C(\alpha,p)(t_f-t)^{1-p-\alpha}W_p(t) \\ \qquad -\dfrac{\lambda(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)} = Q(t)x(t) + A^T(t)\lambda(t), \\ A(\alpha,N)(t-t_0)^{-\alpha}x(t) + B(\alpha,N)(t-t_0)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha,p)(t-t_0)^{1-p-\alpha}V_p(t) \\ \qquad -\dfrac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t), \\ \dot{V}_p(t) = (1-p)(t-t_0)^{p-2}x(t), \quad V_p(t_0) = 0, \qquad p = 2,3,\cdots,N, \\ \dot{W}_p(t) = -(1-p)(t_f-t)^{p-2}\lambda(t), \;\; W_p(t_f) = 0, \qquad p = 2,3,\cdots,N, \\ \qquad x(t_0) = x_0, \quad \lambda(t_f) = 0. \end{cases}$$

Now, instead of solving the system of fractional differential equations (3.4), we can solve the system of ordinary differential equations (3.5). For applying the Legendre Gauss collocation method for solving this system of equations, it is necessary to assume that:

$$\tag{3.6} \begin{aligned} &\lambda^M(t) = \sum_{n=0}^{M} \hat{a}_n \hat{P}_n(t), \qquad x^M(t) = \sum_{n=0}^{M} \hat{b}_n \hat{P}_n(t), \\ &V_p^M(t) = \sum_{n=0}^{M} \hat{c}_n \hat{P}_n(t), \qquad W_p^M(t) = \sum_{n=0}^{M} \hat{d}_n \hat{P}_n(t), \qquad p = 2,3,\cdots,N, \\ &\lambda^M(t), x^M(t), \; V_p^M(t), W_p^M(t) \in \hat{P}_M(t_0,t_f). \end{aligned}$$

Now, from equations (3.5)-(3.6), we have:

$$
\begin{cases}
A(\alpha,N)(t_f-t)^{-\alpha}\lambda^M(t) - B(\alpha,N)(t_f-t)^{1-\alpha}\dfrac{d\lambda^M(t)}{dt} + \sum_{p=2}^{N} C(\alpha,p)(t_f-t)^{1-p-\alpha}W_p^M(t) \\
\qquad -\dfrac{\lambda^M(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)} = Q(t)x^M(t) + A(t)\lambda^M(t), \\
A(\alpha,N)(t-t_0)^{-\alpha}x^M(t) + B(\alpha,N)(t-t_0)^{1-\alpha}\dfrac{dx^M(t)}{dt} - \sum_{p=2}^{N} C(\alpha,p)(t-t_0)^{1-p-\alpha}V_p^M(t) \\
\qquad -\dfrac{x^M(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} = A(t)x^M(t) - B(t)R^{-1}(t)B^T(t)\lambda^M(t), \\
\dfrac{dV_p^M(t)}{dt} = (1-p)(t-t_0)^{p-2}x^M(t), \qquad V_p^M(t_0)=0, \qquad p=2,3,\cdots,N, \\
\dfrac{dW_p^M(t)}{dt} = -(1-p)(t_f-t)^{p-2}\lambda^M(t), \quad W_p^M(t_f)=0, \qquad p=2,3,\cdots,N, \\
\qquad x^M(t_0)=x_0, \quad \lambda^M(t_f)=0.
\end{cases} \tag{3.7}
$$

Finally, by obtaining the solution of this system of equations, we recognize the approximated solution of FOCP (3.1).

4. Numerical Examples

Here, we apply the approach presented in the last section to some FOCPs. The test problems demonstrate the validity and efficiency of this approximation.

Example 4.1. *Consider the following FOCP:*

$$\min J(u(.)) = \frac{1}{2}\int_0^1 \Big(x^2(t)+u^2(t)\Big)dt \tag{4.1}$$

subject to

$${}_0^C D_t^\alpha x(t) = -x(t)+u(t), \;\; x(0)=1. \tag{4.2}$$

The necessary conditions for this problem are as follows [19]*:*

$$
\begin{aligned}
&{}_t^C D_1^\alpha \lambda(t) = x(t)+u(t) \\
&{}_0^C D_t^\alpha x(t) = -x(t)-\lambda(t) \\
&u(t)+\lambda(t)=0 \\
&\lambda(1)=0, \;\; x(0)=1.
\end{aligned} \tag{4.3}
$$

For this example, we have $Q(t) = R(t) = -A(t) = B(t) = x_0 = 1$. *Assuming* $N = 2$ *and using approximations* (2.6)-(2.7) *for the system of FDEs* (4.3). *The absolute errors of the cost functional values at different values of* M, *are listed in Table 1. Also, simulation results of the optimal control* $u(t)$ *and the corresponding state* $x(t)$ *are presented in Table 2 and Figure 1. It can be seen that when* M *increased, the better approximations to both the state and the control functions and than the better approximation of the optimal cost will be obtained.*

Table 1. Absolute errors of cost functional values at $\alpha = 0.9$ and different values of M.

M	E_J
5	0.1041
10	0.0232
15	0.0033
20	0.0019

Table 2. Absolute errors of the optimal control and optimal state at $M = 20$ and different values of α.

α	$u(t)$	$x(t)$
0.5	4.21340×10^{-2}	3.40323×10^{-3}
0.8	2.51714×10^{-4}	2.21262×10^{-4}
0.9	1.32072×10^{-7}	2.64992×10^{-6}
1	3.97889×10^{-9}	1.43634×10^{-9}

Example 4.2. *Consider the following FOCP:*

$$\min J(u) = \frac{1}{2}\int_0^1 \Big\{(x(t) - t^2)^2 + (u(t) - t + 1)^2\Big\}dt \tag{4.4}$$

$$s.t.\ {}_0^C D_t^\alpha x(t) = \frac{\Gamma(3)}{\Gamma(2)}(x(t) - tu(t)), \qquad x(0) = 0.$$

The exact solution of this equation is given by $x(t) = t^2$, $u(t) = t - 1$ *when* $\alpha = 1$. *In Figure 2, we present the approximation state and approximation control by Legendre-Gauss collocation method with* $M = 20$ *and different values of* α. *It shows that as*

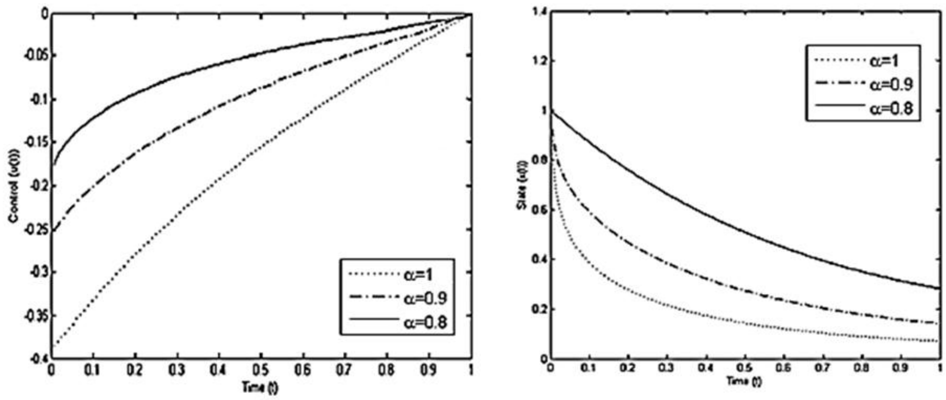

FIGURE 1. Simulation curve at different values of α for Example 4.1.

the values of α approaches to 1, the numerical solutions of system (3.7)*, approach into the analytical solutions of FOCP* (4.4)*. Also, Table 3 shows the maximum absolute errors of this approximation for $x(t)$ and $u(t)$.*

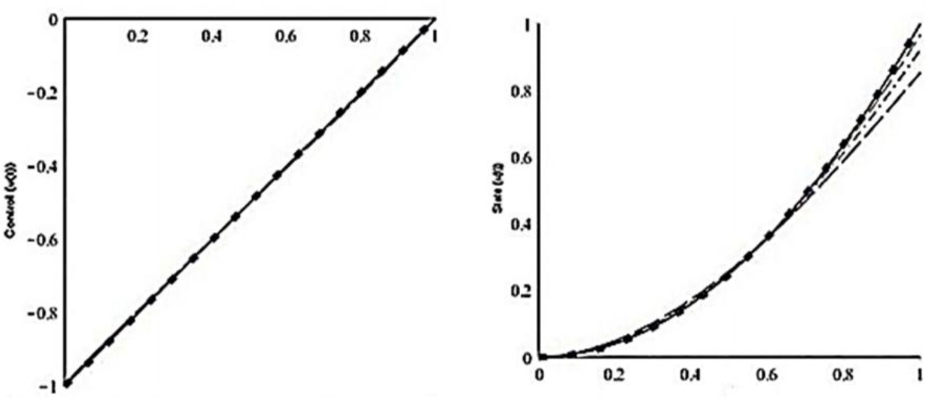

FIGURE 2. Approximate solutions of $x(t)$ and $u(t)$ for different values of α ($\mathbf{.}:\alpha=1$, $--:\alpha=0.98$, $-.-:\alpha=0.95$, $---:\alpha=0.9$).

TABLE 3. Absolute errors of $x(t)$ and $u(t)$ at $\alpha = 1$.

t	$x(t)$	$u(t)$
0.0	$0.50E-10$	0
0.2	$0.52E-10$	$0.3E-12$
0.4	$0.61E-10$	$0.1E-12$
0.6	$0.12E-11$	$0.1E-12$
0.8	$0.20E-11$	$0.1E-12$
1.0	$0.35E-11$	$0.1E-12$

5. Conclusions

In the present work, we developed a new approximation for the fractional differential problem has been derived in which, the fractional derivatives are described in the Caputo sense. The properties of the CFDs are used to reduce the given FOCP into a system of FDEs. Numerical approach for solving this system of FDEs is based on the Legendre-Gauss collocation method to approximate the solutions of the original FOCPs. Numerical results show that this approximation is computationally attractive and also reduces keeping the accuracy of the solution.

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