SOME RESULTS ON CONVERGENCE AND EXISTENCE OF BEST PROXIMITY POINTS

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Abstract. In this paper, we introduce generalized cyclic φ-contraction maps in metric spaces and give some results of best proximity points of such mappings in the setting of a uniformly convex Banach space. Moreover, we obtain convergence and existence results of proximity points of the mappings on reflexive Banach spaces.

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1. Introduction

As a generalization of Banach contraction principle, Kirk and et al [7] proved the following Theorem.

Theorem 1.1. *Let A and B be non-empty closed subsets of a compelet metric space $X := (X, d)$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic map (i.e. $T(A) \subseteq B$ and*

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$T(B) \subseteq A$) such that $d(Tx, Ty) \leq kd(x, y)$ for some $k \in (0, 1)$ and for all $x \in A$ and $y \in B$. Then T has a unique fixed point in $A \cap B$.

In 2011, Karapínar and Erhan [6], generalized Theorem1.1 as follows.

Theorem 1.2. *Let A and B be non-empty closed subsets of a compelet metric space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic map such that $d(Tx, Ty) \leq k\{d(x, y) + d(Tx, x) + d(Ty, y)\}$ for some $k \in (0, \frac{1}{3})$ and for all $x \in A$ and $y \in B$. Then T has a unique fixed point in $A \cap B$.*

Let A and B be non-empty subsets of a metric space X and $T : A \cup B \to A \cup B$ be a cyclic map. Then
(i) T is a cyclic contraction [4] if

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$$

for some $k \in (0, 1)$ and for all $x \in A$ and $y \in B$, where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

(ii) T is a generalized cyclic contraction [5] if

$$d(Tx, Ty) \leq (k/3)\{d(x, y) + d(Tx, x) + d(Ty, y)\} + (1 - k)d(A, B)$$

for some $k \in (0, 1)$ and for all $x \in A$ and $y \in B$,
(iii) T is a cyclic $\varphi-$contraction [1] if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing map and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$.

Definition 1.1. *[1] Let A and B be non-empty subsets of a normed space X, $T : A \cup B \to A \cup B$ be a cyclic map. We say that T satisfies the proximal property if for $\{x_n\}_{n\geq 0} \in A \cup B$,*

$$x_n \overset{w}{\to} x \in A \cup B,\ \|x_n - Tx_n\| \to d(A, B) \Longrightarrow \|x - Tx\| = d(A, B).$$

Definition 1.2. *A Banach space X is said to be*
(i) uniformly convex if there exists a stricly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x_1, x_2, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\|x_i - p\| \leq R,\ i = 1, 2 \text{ and } \|x_1 - x_2\| \geq r \Rightarrow \|(x_1 + x_2)/2 - p\| \leq (1 - \delta(r/R))R$$

(ii) strictly convex if the following impilication holds for all $x_1, x_2, p \in X$ *and* $R > 0$

$$\|x_i - p\| \leq R, \ i = 1, 2 \text{ and } x_1 \neq x_2 \Rightarrow \|(x_1 + x_2)/2 - p\| < R.$$

The following Theorems extend Theorem 1.1 and Theorem 1.2 to include the case $A \cap B = \emptyset$.

Theorem 1.3. *[4] Let* A *and* B *be non-empty closed and convex subsets of a uniformly convex Banach space* X*. Let* $T : A \cup B \to A \cup B$ *be a cyclic contraction map, for* $x_0 \in A$*, define* $x_{n+1} = Tx_n$ *for each* $n \geq 0$*. Then there exists a unique* $x \in A$ *such that* $x_{2n} \to x$ *and* $\|x - Tx\| = d(A, B)$*.*

Theorem 1.4. *[5] Let* A *and* B *be non-empty closed and convex subsets of a uniformly convex Banach space* X*. Let* $T : A \cup B \to A \cup B$ *be a generalizied cyclic contraction map. Then there exists a unique best proximity point* $x \in A$ *for* T*.*

Best proximity point theory of cyclic contraction maps has been studied by many authors see [2, 3, 8] and references therein. In 2009, Al-Thagafi and Shahzad [1], introduced cyclic $\varphi-$contraction maps and proved Convergence and existence results of best proximity points for such maps. In 2012, Karapínar [5], obtaiend best proximity points for cyclic maps. In 2010 Rezapour and et al [9], have elicited a best proximity point theorem for cyclic $\varphi-$contractions on reflexive Banach spaces. In this paper, we shall introduced the concept of generalized cyclic $\varphi-$contraction map, which contains the generalized cyclic contractions in [5]. Then, we give existence and convergence results of best proximity points for such maps in metric spaces, uniformly convex Banach spaces and reflexive Banach spaces.

2. Main results

We introduce the following generalized cyclic $\varphi-$contraction map in metric spaces.

Definition 2.1. *Let* A *and* B *be non-empty subsets of a metric space* X*. The cyclic mapping* $T : A \cup B \to A \cup B$ *is said to be a generalized cyclic* $\varphi-$*contraction, if* $\varphi : [0, +\infty) \to [0, +\infty)$ *is a strictly increasing map and*

$$\begin{aligned} d(Tx, Ty) &\leq (1/3)\{d(x, y) + d(Tx, x) + d(Ty, y)\} \\ &- \varphi(d(x, y) + d(Tx, x) + d(Ty, y)) + \varphi(3d(A, B)), \end{aligned}$$

for all $x \in A$ *and* $y \in B$*.*

Example 2.1. *A generalized cyclic contraction map is generalized cyclic $\varphi-$ contraction with $\varphi(t) = (1-k)t/3$ for $t \geq 0$ and $0 < k < 1$.*

Example 2.2. *Let $X = \mathbb{R}$ with the usual metric. For $A = B = [0,1]$, define $T : A \cup B \to A \cup B$ by $T(x) = \frac{x}{3(1+x)}$. Clearly T is a cyclic map. If $\varphi(t) = \frac{t^2}{3+3t}$ for $t \geq 0$, then T is a generalized cyclic $\varphi-$contraction map but is not generalized cyclic contraction.*

Example 2.3. *Let $X = \mathbb{R}$ with the usual metric. For $A = [0,1]$ and $B = [-1,0]$, define $T : A \cup B \to A \cup B$ by*

$$T(x) = \begin{cases} \frac{-x}{3(1+x)} & x \in A \\ \\ \frac{-x}{3(1-x)} & x \in B. \end{cases}$$

Clearly T is a cyclic map. If $\varphi(t) = \frac{t^2}{3+3t}$ for $t \geq 0$, then T is a generalized cyclic $\varphi-$contraction map but is not generalized cyclic contraction.

Lemma 2.1. *Let A and B be non-empty subsets of a metric space X and let $T : A \cup B \to A \cup B$ be generalized cyclic $\varphi-$contraction map. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then*
(a) $-\varphi(d(x,y) + d(Tx,x) + d(Ty,y)) + \varphi(3d(A,B)) \leq 0$ for all $x \in A$ and $y \in B$,
(b) $d(Tx,Ty) \leq (1/3)\{d(x,y) + d(Tx,x) + d(Ty,y)\}$, for all $x \in A$ and $y \in B$,
(c) $d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)$ for all $n \geq 0$.

We prove the following results which will be needed in what follows.

Proposition 2.1. *Let A and B be non-empty subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \geq 0$, then $d(x_n, x_{n+1}) \to d(A,B)$ as $n \to \infty$.*

Proof. Let $d_n = d(x_n, x_{n+1})$. It follows from Lemma 2.1(c), that $\{d_n\}$ is decreasing and bounded, so $\lim_{n\to\infty} d_n = t_0$. Since T is a generalized cyclic $\varphi-$ contraction map, we obtain

$$d_{n+1} \leq \frac{1}{3}\{2d_n + d_{n+1}\} - \varphi(2d_n + d_{n+1}) + \varphi(3d(A,B)).$$

Hence,

$$\varphi(3d(A,B)) \leq \varphi(2d_n + d_{n+1}) \leq \frac{2}{3}d_n - \frac{2}{3}d_{n+1} + \varphi(3d(A,B)).$$

Thus

$$\varphi(3d(A,B)) \leq \lim_{n\to\infty} \varphi(2d_n + d_{n+1}) \leq \varphi(3d(A,B)),$$

which shows that

$$\lim_{n\to\infty} \varphi(2d_n + d_{n+1}) = \varphi(3d(A,B)). \tag{1}$$

Since φ is strictly increasing and $d_n \geq d_{n+1} \geq t_0 \geq d(A,B)$, we have

$$\lim_{n\to\infty} \varphi(2d_n + d_{n+1}) \geq \varphi(3t_0) \geq \varphi(3d(A,B)).$$

From (1), we get

$$\lim_{n\to\infty} \varphi(2d_n + d_{n+1}) = \varphi(3t_0) = \varphi(3d(A,B)).$$

As φ is strictly increasing, we have $t_0 = d(A,B)$. □

Theorem 2.1. *Let A and B be non-empty subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ with $x_{2n_k} \to x \in A$. Since

$$d(A,B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}),$$

for each $k \geq 1$, it follows from Proposition 2.1 that $\lim_{k\to\infty} d(x_{2n_k-1}, x) \to d(A,B)$. Since

$$\begin{aligned} d(x_{2n_k}, Tx) &\leq (1/3)\{d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k-1}, x) + d(x, Tx)\} \\ &\leq (1/3)\{d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k-1}, x) + d(x, x_{2n_k}) + d(x_{2n_k}, Tx)\}. \end{aligned}$$

Letting $k \to \infty$,

$$(2/3) \lim_{k\to\infty} d(x_{2n_k}, Tx) \leq (2/3) d(A,B),$$

it follows that, $\lim_{k\to\infty} d(x_{2n_k}, Tx) = d(A,B)$. So $d(x,Tx) = d(A,B)$. □

Lemma 2.2. *Let A and B be non-empty subsets of a uniformly convex Banach space X such that A is convex and let $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then $\|x_{2n+2} - x_{2n}\| \to 0$ and $\|x_{2n+3} - x_{2n+1}\| \to 0$ as $k \to \infty$.*

Proof. To prove that $\|x_{2n+2} - x_{2n}\| \to 0$ as $k \to \infty$, assume the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ so that

$$\|x_{2n_k+2} - x_{2n_k}\| \geq \epsilon_0. \tag{2}$$

Choose $0 < \gamma < 1$ such that $d(A, B) < \frac{\epsilon_0}{\gamma}$ and choose ϵ such that

$$0 < \epsilon < \min\left\{\frac{\epsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)}\right\}.$$

By Proposition 2.1, there exists N_1 such that

$$\|x_{2n_k+2} - x_{2n_k+1}\| \leq d(A, B) + \epsilon, \tag{3}$$

for all $n_k \geq N_1$. Also, there exists N_2 such that

$$\|x_{2n_k} - x_{2n_k+1}\| \leq d(A, B) + \epsilon \tag{4}$$

for all $n_k \geq N_2$. Let $N = \max\{N_1, N_2\}$. From (2)-(4) and the uniform convexity of X, we get

$$\|(x_{2n_k+2} + x_{2n_k})/2 - x_{2n_k+1}\| \leq \left(1 - \delta\left(\frac{\epsilon_0}{d(A, B) + \epsilon}\right)\right)(d(A, B) + \epsilon),$$

for all $n_k \geq N$. As $(x_{2n_k+2} + x_{2n_k})/2 \in A$, the choice of ϵ implies that

$$\|(x_{2n_k+2} + x_{2n_k})/2 - x_{2n_k+1}\| < d(A, B),$$

for all $n_k \geq N$, a contradiction. A similar argument shows $\|x_{2n+3} - x_{2n+1}\| \to 0$ as $k \to \infty$. □

Theorem 2.2. *Let A and B be non-empty subsets of a uniformly convex Banach space X such that A is closed and convex and let $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$, $T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

Proof. First, we show for each $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$,

$$\|x_{2m} - x_{2n+1}\| < d(A, B) + \epsilon. \tag{5}$$

Suppose the contrary. So there exists $\epsilon_0 > 0$ such that for each $k \geq 1$. there is $m_k > n_k \geq k$ satisfying

$$\|x_{2m_k} - x_{2n_k+1}\| \geq d(A,B) + \epsilon_0 \tag{6}$$

and

$$\|x_{2(m_k-1)} - x_{2n_k+1}\| < d(A,B) + \epsilon_0. \tag{7}$$

Now from (6) and (7), we get

$$d(A,B) + \epsilon_0 \leq \|x_{2m_k} - x_{2n_k+1}\| \leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - x_{2n_k+1}\|$$

$$< \|x_{2m_k} - x_{2(m_k-1)}\| + d(A,B) + \epsilon_0.$$

Letting $k \to \infty$, Lemma 2.2 implies

$$\lim_{k\to\infty} \| x_{2m_k} - x_{2n_k+1}\| = d(A,B) + \epsilon_0, \tag{8}$$

it follows from Lemma 1.2(b), and the generalized cyclic $\varphi-$contraction property of T that

$$\begin{aligned}
\|x_{2m_k} - x_{2n_k+1}\| &\leq \|x_{2m_k} - x_{2m_k+2} + \|x_{2m_k+2} - x_{2n_k+3}\| + \|x_{2n_k+3} - x_{2n_k+1}\| \\
&\leq \|x_{2m_k} - x_{2m_k+2}\| + (1/3)\{\|x_{2m_k+1} - x_{2n_k+2}\| \\
&+ \|x_{2m_k+1} - x_{2m_k+2}\| + \|x_{2n_k+2} - x_{2n_k+3}\|\} + \|x_{2n_k+3} - x_{2n_k+1}\| \\
&\leq \|x_{2m_k} - x_{2m_k+2}\| + (1/3)\{(1/3)\{\|x_{2m_k} - x_{2n_k+1}\| \\
&+ \|x_{2m_k} - x_{2m_k+1}\| + \|x_{2n_k+2} - x_{2n_k+1}\|\} \\
&+ (1/3)\{\|x_{2m_k+1} - x_{2m_k+2}\| + \|x_{2n_k+2} - x_{2n_k+3}\|\} \\
&+ \|x_{2n_k+3} - x_{2n_k+1}\|.
\end{aligned}$$

Letting $k \to \infty$, by using (8), Lemma 2.2 and Proposition 2.1, we get

$$d(A,B) + \epsilon_0 \leq (1/9)(d(A,B) + \epsilon_0) + (2/9)d(A,B) + (2/3)d(A,B),$$

so

$$d(A,B) + \epsilon_0 \leq d(A,B) + (1/9)\epsilon_0,$$

this is a contradiction. Now, we show $\{x_{2n}\}$ Cauchy sequence in A. If $d(A,B) = 0$, then let $\epsilon_0 > 0$ be given. By Proposition 2.1, there exists N_1 such that

$$\|x_{2n} - x_{2n+1}\| < \epsilon$$

for every $n \geq N_1$. From (5), there exists N_2 such that

$$\|x_{2m} - x_{2n+1}\| < \epsilon$$

for every $m > n \geq N_2$. Let $N = \max\{N_1, N_2\}$. It follows that

$$\|x_{2m} - x_{2n}\| \leq \|x_{2m} - x_{2n+1}\| + \|x_{2n} - x_{2n+1}\| < 2\epsilon$$

for all $m > n \geq N$. So $\{x_{2n}\}$ is a Cauchy sequence in A. So we assume that $d(A, B) > 0$. To show that $\{x_{2n}\}$ is a Cauchy sequence in A, we assume the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there exists $m_k > n_k \geq k$ so that

$$\|x_{2m_k} - x_{2n_k}\| \geq \epsilon_0. \tag{9}$$

Choose $0 < \gamma < 1$ such that $d(A, B) < \frac{\epsilon_0}{\gamma}$ and choose ϵ such that

$$0 < \epsilon < \min\left\{\frac{\epsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)}\right\}.$$

By Proposition 2.1, there exists N_1 such that

$$\|x_{2n_k} - x_{2n_k+1}\| \leq d(A, B) + \epsilon, \tag{10}$$

for all $n_k \geq N_1$. From (5), there exists N_2 such that

$$\|x_{2m_k} - x_{2n_k+1}\| \leq d(A, B) + \epsilon, \tag{11}$$

for all $m_k > n_k \geq N_2$. Let $N = \max\{N_1, N_2\}$. From (9)-(11) and the uniform convexity of X, we get

$$\|(x_{2m_k} + x_{2n_k})/2 - x_{2n_k+1}\| \leq \left(1 - \delta\left(\frac{\epsilon_0}{d(A, B) + \epsilon}\right)\right)(d(A, B) + \epsilon),$$

for all $m_k > n_k \geq N$. As $(x_{2n_k+2} + x_{2n_k})/2 \in A$, the choice of ϵ implies that

$$\|(x_{2n_k+2} + x_{2n_k})/2 - x_{2n_k+1}\| < d(A, B),$$

for all $m_k > n_k \geq N$, a contradiction. Thus $\{x_{2n}\}$ Cauchy sequence in A. The completeness of X and the closedness of A imply that $x_{2n} \to x$ as $n \to \infty$. By Theorem 2.1, $\|x - Tx\| = d(A, B)$. Now from Lemma 2.1(b), we have

$$\|T^2x - Tx\| \leq (1/3)\{2\|Tx - x\| + \|T^2x - Tx\|\}.$$

Hence $\|T^2x - Tx\| = d(A, B)$, therefore $T^2x = x$. Next, suppose $y \in A$ and $x \neq y$ such that $\|y - Ty\| = d(A, B)$ with $T^2y = y$. By Lemma 2.1(b)

$$\begin{aligned} \|Tx - y\| &\leq (1/3)\{\|x - Ty\| + \|Tx - x\| + \|y - Ty\|\} \\ &\leq (1/3)\{(1/3)\{\|Tx - y\| + \|Tx - x\| + \|y - Ty\|\} \\ &+ \|Tx - x\| + \|y - Ty\|\}. \end{aligned}$$

So

$$(8/9)\|Tx - y\| \leq (4/9)\{\|Tx - x\| + \|y - Ty\|\},$$

which implies that $\|Tx - y\| = d(A, B)$. It follows from convexity of A and the strict convexity of X that

$$\|(x + y)/2 - Tx\| = \|(x - Tx)/2 + (y - Tx)/2\| < d(A, B),$$

a contradiction. Thus $x = y$. □

Now, we show existence of a best proximity point for generalized cyclic $\varphi-$contraction map in reflexive Banach space.

Proposition 2.2. *Let A and B be non-empty subsets of a metric space X, $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map, $x_0 \in A \cup B$, and $x_{n+1} = Tx_n$ for each $n \geq 0$. Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose that $x_0 \in A$ (the proof when $x_0 \in B$ is similar). From Proposition 2.1, either $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded or both sequences are unbounded. Fix $n_1 \in N$ and define

$$e_{n,k} = d(x_{2n}, x_{2(n_1+k)+1})$$

for all $n, k \geq 1$. Since $\{x_{2n+1}\}$ is unbounded, $\limsup_{k\to\infty} e_{n,k} = \infty$ for every $n \geq 1$. Therefore we choose a strictly increasing subsequence $\{e_{1,k_i^1}\}_{i\geq 1}$ of the sequence $\{e_{1,k}\}_{k\geq 1}$. Since

$$e_{1,k_i^1} \leq d(x_2, x_4) + e_{2,k_i^1},$$

we have $\limsup_{i\to\infty} e_{2,k_i^1} = \infty$. Again, we can choose strictly increasing subsequence $\{e_{2,k_i^2}\}_{i\geq 1}$ of the sequence $\{e_{2,k_i^1}\}_{i\geq 1}$ such that $\limsup_{i\to\infty} e_{2,k_i^2} = \infty$. By continuing this process, for every $n \in N$, we can choose strictly increasing subsequence $\{e_{n,k_i^n}\}_{i\geq 1}$ of the sequence $\{e_{n,k_i^{n-1}}\}_{i\geq 1}$ such that $\limsup_{i\to\infty} e_{n,k_i^n} = \infty$. By the construction, if we consider the sequence $\{k_i^i\}_{i\geq 1}$, then $\lim_{i\to\infty} k_i^i = \infty$, $\{e_{n,k_i^i}\}_{i\geq 1}$ is a strictly increasing subsequence of $\{e_{n,k_i^n}\}_{i\geq 1}$ and $\limsup_{i\to\infty}$

$e_{n,k_i^i} = \infty$. for every $n \geq 1$. We define $n_2 = n_1 + k_2^2 - k_1^1$, by induction define the sequence $\{n_m\}_{m\geq 1}$ with $n_m = n_1 + k_m^m - k_1^1$. the sequence $\{n_m\}_{m\geq 1}$ is strictly increasing and $\limsup_{m\to\infty} n_m = \infty$. By Lemma 2.1(c), $\{d(x_{2n_m}, x_{2(n_m+k_1^1)+1})\}_{m\geq 1}$ is a decreasing sequence. By the construction of the sequence $\{n_m\}_{m\geq 1}$, $\{d(x_{2n_m}, x_{2(n_1+k_m^m)+1})\}_{m\geq 1}$ is a decreasing sequence. Let $m \geq 1$, since $e_{n_m,k_1^1} \leq e_{n_m,k_m^m}$ and decreasing the sequence $\{d(x_{2n_m}, x_{2(n_1+k_m^m)+1})\}_{m\geq 1}$ we have

$$d(x_{2n_m}, x_{2(n_1+k_1^1)+1}) \leq d(x_{2n_1}, x_{2(n_1+k_1^1)+1}), \tag{12}$$

for all $m \geq 1$. By the construction of the sequence $\{n_m\}_{m\geq 1}$, inequality (12) and Lemma 2.1(c), we obtain

$$\begin{aligned} d(x_{2(n_1+k_m^m)+1}, x_{2(n_1+k_1^1)+1}) &\leq d(x_{2n_m}, x_{2(n_1+k_1^1)+1}) + d(x_{2n_m}, x_{2(n_1+k_m^m)+1}) \\ &\leq d(x_{2n_1}, x_{2(n_1+k_1^1)+1}) + d(x_{2n_m}, x_{2(n_m+k_1^1)+1}) \\ &\leq d(x_{2n_1}, x_{2(n_1+k_1^1)+1}) + d(x_{2n_m-1}, x_{2(n_m+k_1^1)}) \\ &\leq d(x_{2n_1}, x_{2(n_1+k_1^1)+1}) + d(x_0, x_{2k_1^1+1}), \end{aligned}$$

for all $m \geq 1$. Thus

$$\begin{aligned} d(x_{2(n_1+k_m^m)+1}, x_{2(n_1+k_1^1)}) &\leq d(x_{2(n_1+k_m^m)+1}, x_{2(n_1+k_1^1)+1}) + d(x_{2(n_1+k_1^1)+1}, x_{2(n_1+k_1^1)}) \\ &\leq d(x_{2n_1}, x_{2(n_1+k_1^1)+1}) + d(x_0, x_{2k_1^1+1}) \\ &+ d(x_{2(n_1+k_1^1)+1}, x_{2(n_1+k_1^1)}), \end{aligned}$$

for all $m \geq 1$. That is a contradiction, because $\limsup_{i\to\infty} e_{n,k_i^i} = \infty$ for all $n \geq 1$. This completes the proof. □

Next, we give best proximity pair for weakly closed subsets of a reflexive Banach space.

Theorem 2.3. *Let A and B be non-empty weakly closed subsets of a reflexive Banach space X, $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - y\| = d(A, B).$$

Proof. If $d(A, B) = 0$, by Theorem 1.2 the result follows. So we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. The sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded from Proposition 2.2. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \xrightarrow{w} x \in A$. Also

B is weakly closed, hence $x_{2n_k+1} \overset{w}{\to} y \in B$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \overset{w}{\to} x - y \neq 0$ as $k \to \infty$, there exists a bounded liner functional $f : X \to [0, \infty)$ such that $\|f\| = 1$ and $f(x - y) = \|x - y\|$. For all $k \geq 1$, we have

$$|f(x_{2n_k} - x_{2n_k+1})| \leq \|f\|\|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|.$$

Since $\lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| = \|x - y\|$, by appling Proposition 2.1, we give

$$\|x - y\| = \lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = d(A, B),$$

and this compeletes the proof. □

Theorem 2.4. *Let A and B be non-empty subsets of a reflexive Banach space X such that A is weakly closed and $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*
(i) T is weakly continuous on A.
(ii) T is satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.2. So we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. By Proposition 2.2 the sequence $\{x_{2n}\}$ is bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \overset{w}{\to} x \in A$.
From (i), $x_{2n_k+1} \overset{w}{\to} Tx \in B$ as $k \to \infty$. So $x_{2n_k} - x_{2n_k+1} \overset{w}{\to} x - Tx \neq 0$ as $k \to \infty$. Now the proof continues similar to that of Theorem 2.3.
From (ii), by Proposition 2.1, $\|x_{2n_k} - Tx_{2n_k}\| \to d(A, B)$ as $k \to \infty$. Thus $\|x - Tx\| = d(A, B)$. □

In following Theorems, we consider reflexive and strictly convex Banach space and obtain best proximity point for generalized cyclic $\varphi-$contraction map.

Theorem 2.5. *Let A and B be non-empty closed and convex subsets of a reflexive and strictly convex Banach space X such that $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. If $(A - A) \cap (B - B) = \{0\}$, then there exists a unique $x \in A$ such that $\|x - Tx\| = d(A, B)$.*

Proof. If $d(A, B) = 0$, by Theorem 1.2 the result follows. So we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.3, there exists $(x, y) \in A \times B$ with $\|x - y\| = d(A, B)$. Suppose that there exists

$(a, b) \in A \times B$ with $\|a - b\| = d(A, B)$. Since $(A - A) \cap (B - B) = \{0\}$, thus $x - y \neq a - b$. By the strict convexity of X, as convexity of A and B, we have

$$\|(x + a)/2 - (y + b)/2\| = \|(x - y)/2 + (a - b)/2\| < d(A, B),$$

which is a contraction. This show (x, y) is unique. □

Theorem 2.6. *Let A and B be non-empty subsets of a reflexive and strictly convex Banach space X such that A is closed and convex and $T : A \cup B \to A \cup B$ be a generalized cyclic $\varphi-$contraction map. Then there exists a unique $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*
(i) T is weakly continuous on A.
(ii) T is satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.2. So we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.4, there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$. □

References

[1] M. A. Al-Thagafi and N. Shahzad, Convergence and existence result for best proximity points, Nonliner Analysis, Theory, Methods and Applications 70, (10) 3665-3671 (2009).

[2] A. Amini-Harandi, N. Hussain and F. Akbar, Best proximity point results for generalized contractions in metric spaces. Fixed Point Theory Appl, 2013:164, 13 pp, (2013).

[3] S. S. Basha, Best proximity points: optimal solutions, Journal of optimization theory and applications 151(1) 210-216 (2011).

[4] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl 323, (2) 1001-1006 (2006).

[5] E. Karapínar, Best proximity points of cyclic mappings, Appl. Math. Letters 25, (11) 1761-1766 (2012).

[6] E. Karapínar and I. M. Erhan, Best proximity point on different type contractions, Appl. Math. Inf. Sci 3. (3) 342-353 (2011).

[7] W. A. Kirk, P. S. Serinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4, (1) 79-89 (2003).

[8] V. S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Analysis: Theory, Methods and Applications 74, no. 14, 4804-4808 (2011).

[9] Sh. Rezapour, M. Derafshpour and N. Shahzad, Best proximty points of cyclic $\varphi-$contractions on reflexive Banach spaces, Fixed Point Theory and Application, 7 pp, (2010).