WAVELET SOLUTIONS OF THE KLEIN-GORDON EQUATION

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Abstract. In this paper, The numerical solutions of the Klein-Gordon equations using Legendre wavelets are investigated. The interest is in solving the problem using the wavelet basis due to its simplicity and efficiency in numerical approximations. The approach of creating Legendre wavelets and their main properties were briefly mentioned. Also, the numerical results were presented for demonstrating the validity and applicability of the current technique.

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1. Introduction

Many positions in science and engineering lead to partial differential equations but determining the exact solutions may be difficult. Therefore, for these systems of PDEs, to the way of approximating unknown functions should be chosen. Several numerical techniques have been developed in order to compute approximate solutions of these equations, such as the finite difference, pseudo-spectral and adaptive grid methods. In this paper, a wavelet-based method is illustrated in order to solve partial differential equations, which are applied in mathematical physics. In the numerical analysis, wavelet-based methods become important tools because of the

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properties of localization. Wavelet transform or wavelet analysis has been recently developed as a powerful tool for numerical applications[5,7]; also, wavelets serve as a Galerkin basis for solving partial differential equations[13].

There have been a number of investigations in order to use wavelet expansions for the numerical computation of solutions for differential equations[5,10,14].

1) The possibility of compressing representations of functions on a wavelet basis.

2) In the wavelet coordinate, differential operators may be preconditioned by a diagonal matrix.

3) The properties of localization.

In wavelet methods, there are two ways of improving the approximation of the solutions: increasing the resolution level and increasing the order of the wavelet family[17].

The currently existing wavelet-based methods can be classified as Galerkin wavelet methods and wavelet collocation methods[8,9,15,16]. The present study's goal is to show how wavelets and multi-resolution analysis can be used for improving the method in terms of easy implementability and achieving the rapidity of its convergence; however, a disadvantage of wavelet is the action of differential and integral operator on the basis functions that can be difficult to determine, depending on the choice of wavelets. Thus, the Legendre wavelets are used here[11,12].

The accuracy and efficiency of the method are demonstrated for the solutions of two dimensional problems, such as Klein-Gordon equation. The Klein - Gordon equation is considered in the following form:

$$u_{tt} - u_{xx} + b_1 u + b_2 g(u) = f(x,t), \tag{1}$$

by subject to initial conditions:

$$u(x,0) = a_0(x), \qquad u_t(x,0) = a_1(x), \tag{2}$$

where b_1 and b_2 are real numbers, g is a given nonlinear function and f is a known function.

The Klein-Gordon equation is one of the most important mathematical models in quantum mechanics[3]. This equation has attracted much attention in studying the solutions and condensed matter physics, investigation of the interaction of solutions in a collisionless plasma, the Recurrence of initial states and examination of the nonlinear wave equations.

There are several methods for evaluating the approximate solutions, such as Variational Iteration Method (VIM), Homotopy Analysis Method (HAM) and Homotopy Perturbation Method(HPM), but also the numerical methods are very complicated and difficult. VIM is based on the general Lagrange's multiplier method, HAM contains an auxiliary parameter, $\overline{h}$, which provides a simple way for adjusting and controlling the convergence region, and the rate of convergence of the series solution; and HPM deforms a difficult problem into a set of problems which are easier to solve[1,2,4].

In this paper, the wavelet collocation method and Legendre wavelets are combined to obtain the approximate solutions of (1). Thus, continuous, orthonormal and compactly supported wavelets, called Legendre wavelets, which is specially constructed for the bounded interval is applied.
The present method consists of reducing (1) to a set of algebraic equations by expanding unknown functions with unknown coefficient and then, the solutions are compared with those resulting from the use of the other numerical methods and the efficiency of this method is demonstrated.

This paper is organized as follows; Wavelets, Legendre wavelets and their main properties are described in Section 2. Then, The Klein-Gordon equations and derivatives are stated in Section 3 and these problems are solved using the Legendre wavelets. In the final Section, numerical results are presented for comparison with the obtained solutions from the other methods.

2. Preliminaries and Conventions of Wavelet Analysis

In this section, an overview of wavelets is expressed and a brief introduction to wavelets, the Legendre wavelets and their properties is presented.

2.1. **Wavelets.** Wavelets are the family of functions which are derived from the family of scaling function $\{\phi_{j,k} : k \in Z\}$ where:

$$\phi(x) = \sum_k a_k \phi(2x - k). \tag{3}$$

For the continuous wavelets, the following equations can be represented:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}} \psi(\frac{x-b}{a}), \qquad a, b \in R, \quad a \neq 0, \tag{4}$$

where a and b are dilation and translation parameters, respectively, such that $\psi(x)$ is a single wavelet function[11].

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$a = a_0^{-j}, \qquad a_0 > 1, \qquad b_0 > 1, \tag{5}$$

$$b = kb_0 a_0^{-j},\ \ j, k \in Z.$$

Then, a family of discrete wavelets can be constructed as follows:

$$\psi_{j,k}(x) = |a_0|^{\frac{j}{2}} \psi(a_0^j x - kb_0), \tag{6}$$

where $\psi_{j,k}(x)$ are the wavelet basis for $L^2(R)$. Therefore, a wavelet basis is constructed in the following stage for $a_0 = 2$ and $b_0 = 1$.

Hence, the family of wavelets is in the following form:

$$\psi_{j,k} = |2|^{\frac{j}{2}} \psi(2^j x - k), \tag{7}$$

so, $\psi_{j,k}(x)$ constitutes an orthonormal basis in $L^2(R)$, where $\psi(x)$ is a single function.

2.2. **Legendre Wavelets and Their Properties.** The Legendre wavelets are in the following way,

$$\psi_{k,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{j}{2}} p_m(2^j x - k) : & \frac{k-1}{2^j} \le x < \frac{k}{2^j}, \\ 0 & otherwise, \end{cases}$$

where $m = 0, 1, 2, \ldots, M-1$ and $k = 1, 2, \ldots, 2^{j-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, then by (7), the wavelets $\psi_{k,m}(x)$ form an orthonormal basis for $L^2[0,1]$[5,11,12]. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$p_0 = 1,$$

$$p_1 = x,$$

$$p_{m+1}(x) = \frac{2m+1}{m+1} x p_m(x) - \frac{m}{m+1} p_{m-1}(x),$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval $[0,1]$. Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-j}$ and the translation parameter is $b = n2^j$[11,12].

2.3. **Function Approximation.** A given function $f(x)$ with the domain $[0, 1]$ can be approximated by:

$$f(x) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} c_{k,m}\psi_{k,m}(x) = C^T \cdot \Psi(x), \tag{8}$$

If the infinite series in Equation (8) is truncated, then this equation can be written as:

$$f(x) \simeq \sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1} c_{k,m}\psi_{k,m}(x) = C^T \cdot \Psi(x), \tag{9}$$

where C and Ψ are the matrices of size $(2^{j-1}M \times 1)$.

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,M-1}, \ldots, c_{2^{j-1},0}, c_{2^{j-1},1}, \ldots, c_{2^{j-1},M-1}]^T,$$

$$\Psi(x) = [\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{j-1},0}, \psi_{2^{j-1},1}, \ldots, \psi_{2^{j-1},M-1}]^T.$$

2.4. **Two Dimensional Case.** The generalization of the method for two dimensional problems is straightforward. For simplicity, a closed domain $\Omega = \Omega_x \times \Omega_t$ is considered where $\Omega_x = [x_1, x_2]$ and $\Omega_t = [t_1, t_2]$ and for each of the two dimensional domains Ω_x and Ω_t a one-dimensional wavelet basis can be defined as $\{\psi_{k,m_1}(x)\}$ and $\{\psi_{k,m_t}(t)\}$.

For the clarity of presentation, the subscript x is used to denote that the wavelet basis and all the parameters associated with it $(a_{x_0}, a_x, j_x, b_{x_0}, b_x, z_x)$ are defined for the domain Ω_x. Thus, relation (5) can be written as:

$$a_x = a_{x_0}^{-j_x}, \qquad a_{x_0} > 1 \qquad b_{x_0} > 1 \tag{10}$$

$$b_x = kb_{x_0}a_{x_0}^{-j_x}.$$

If there is a need to consider wavelet basis for another domain, a different subscript should be used. For example, the subscript t is used for the domain $\Omega_t = [t_1, t_2]$. Thus, the two dimensional basis $\psi_{k,m_1,m_2}(x, t)$ can be constructed as a combination of two one-dimensional translation and a dilation of a truly two-dimensional wavelet $\psi(x, t)$. Consequently, a functional element of the two-dimensional wavelet basis from the relation (6) can be written as:

$$\psi_{j,k_1,k_2}(x, t) = |a_{x_0}|^{\frac{j_x}{2}}\psi(a_{x_0}^{j_x}x - k_1b_{x_0})|a_{t_0}|^{\frac{j_t}{2}}\psi(a_{t_0}^{j_t}t - k_2b_{t_0}). \tag{11}$$

Consequently, the following relation can be concluded:

$$\{(b_{x_0}, b_{t_0})\} = \{b_{x_0}\} \times \{b_{t_0}\},$$

$$z = z_x \times z_y.$$

Consider a function $f(x,t)$ defind as a closed domain Ω and $j = \max\{j_x, j_t\}$, therefore, $f(x,t)$ can be approximated as:

$$f(x,t) = \sum_{k=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c_{k,m_1,m_2} \psi_{k,m_1,m_2}(x,t), \tag{12}$$

and the following relations between the sets of collocation points $\{(x_i^j, t_k^j) : (i,k) \in Z\}$ at different levels of resolution j $(0 \le j \le J-1)$ is satisfied :

$$\{(x_i^j, t_k^j\} \subset \{(x_i^{j+1}, t_k^{j+1}\}.$$

Thus, for a given function, $f(x,t)$ can be approximated by:

$$f(x,t) = \sum_{k=1}^{2^{j-1}} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} c_{k,m_1,m_2} \psi_{k,m_1}(x) \psi_{k,m_2}(t), \tag{13}$$

where $\{\psi_{k,m_1}(x)\}$ and $\{\psi_{k,m_2}(t)\}$ were described in Section (2.2).

3. Legendre Wavelet Method for Solving the Klein-Gordon Equation

Consider the general form of Klein-Gordon equation in (1). Thus by representing $u(x,t)$ in terms of linear combination from Legendre wavelets, the following relations can be presented:

$$u(x,t) = \sum_{k=1}^{2^{j-1}} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} c_{k,m_1,m_2} \psi_{k,m_1,m_2}(x,t)$$

$$= \sum_{k=1}^{2^{j-1}} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} c_{k,m_1,m_2} \psi_{k,m_1}(x) \psi_{k,m_2}(t), \tag{14}$$

now by inserting $u(x,t)$ in Equation (1), a system of the following equations is resulted:

$$\sum_{k=1}^{2^{j-1}} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} c_{k,m_1,m_2} \psi_{k,m_1}(x) \frac{d^2}{dt^2}(\psi_{k,m_2}(t))$$
$$-\sum_{k=1}^{2^{j-1}} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} c_{k,m_1,m_2} \frac{d^2}{dx^2}(\psi_{k,m_1}(x)) \psi_{k,m_2}(t)$$

$$+b_1(\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1}c_{k,m_1,m_2}\psi_{k,m_1}(x)\psi_{k,m_2}(t))+b_2g(u)=f(x,t), \tag{15}$$

under the following conditions:

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1}c_{k,m_1,m_2}\psi_{k,m_1}(x)\psi_{k,m_2}(0)=a_0(x), \tag{16}$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1}c_{k,m_1,m_2}\psi_{k,m_1}(x)\frac{d}{dt}(\psi_{k,m_2}(t))|_{t=0}=a_1(x), \tag{17}$$

where $g(u)$ and $f(x,t)$ may be approximated as:

$$g(u)=\sum_{k=1}^{2^{j-1}}\sum_{m_1'=0}^{M-1}\sum_{m_2'=0}^{M-1}d_{k,m_1',m_2'}\psi_{k,m_1',m_2'}(x,t), \tag{18}$$

$$f(x,t)=\sum_{k=1}^{2^{j-1}}\sum_{m_1''=0}^{M-1}\sum_{m_2''=0}^{M-1}e_{k,m_1'',m_2''}\psi_{k,m_1'',m_2''}(x,t). \tag{19}$$

Therefore, by choosing the arbitrary values of M and j, the coefficients of the solutions to the Klein-Gordon Equation are computed. At the final step, the approximate solutions of this equation are evaluated by inserting the values of c_{k,m_1,m_2} in (15).

3.1. **Solving the Linear Homogeneous Klein-Gordon Equation.** First, the Legendre wavelet method is described for solving the linear Klein-Gordon Equation,

$$u_{tt}-u_{xx}=u, \tag{20}$$

by being subject to the initial conditions:

$$u(x,0)=1+sin(x), \qquad u_t(x,0)=0. \tag{21}$$

By using this method, the collocation points for the domain $\Omega_x=[0,3]$ and $\Omega_t=[0,1]$ are $x=x[i]$ $(i=1,2,\dots,n)$ and $t=t[i']$ $(i'=1,2,\dots,n')$, such that:

$$u(x,t)=\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1}c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x,t), \tag{22}$$

thus, one has:

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial t^2}(\psi_{k,m_1,m_2}(x_i^j,t_{i'}^j))-$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial x^2}(\psi_{k,m_1,m_2}(x_i^j,t_{i'}^j))$$

$$=\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x_i^j,t_{i'}^j),$$

and the initial conditions are in the following form:

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x_i^j,0)=1+sin(x_i^j),$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial}{\partial t}(\psi_{k,m_1,m_2}(x_i^j,t))|_{t=0}=0.$$

By substituting the collocation points, $n=30$, $n^{'}=20$ and $M=3$, $j=1$ in the above relations, the numerical results are evaluated. These results are presented in Tables (1) and (2).

3.2. **Solving the Linear Non-Homogeneous Klein-Gordon Equation.** Consider linear non-homogeneous Klein-Gordon equation,

$$u_{tt}-u_{xx}-2u=-2sin(x)sin(t), \tag{23}$$

with the initial conditions

$$u(x,0)=0, \qquad u_t(x,0)=sin(x). \tag{24}$$

By inserting Eq.(14) in the Eq.(23), a system of nonlinear equations is resulted:

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial t^2}(\psi_{k,m_1,m_2}(x,t))-$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial x^2}(\psi_{k,m_1,m_2}(x,t))$$

$$-2(\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x,t))=-2sin(x)sin(t),$$

with the initial conditions,

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x,0)=0,$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial}{\partial t}(\psi_{k,m_1,m_2}(x,t))|_{t=0}=sin(x),$$

where $x=x[i]$ $(i=1,2,\dots,n)$ and $t=t[i']$ $(i'=1,2,\dots,n')$. Similar to the homogeneous case, numerical results are calculated. Then, the approximate solutions are shown in Tables (3) and (4).

3.3. **Solving the Non-Linear Non-Homogeneous Klein-Gordon Equation.** Finally, we consider the non-Linear non-homogeneous Klein-Gordon equation

$$u_{tt}-u_{xx}+u^2=-xcos(t)+x^2cos^2(t), \tag{25}$$

subject to the initial conditions

$$u(x,0)=x, \qquad u_t(x,0)=0. \tag{26}$$

Similar to the above two examples, we can rewrite(25) as follows;

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial t^2}(\psi_{k,m_1,m_2}(x,t))-$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial^2}{\partial x^2}(\psi_{k,m_1,m_2}(x,t))$$

$$+(\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x,t))^2=-xcos(t)+x^2cos^2(t),$$

and;

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\psi_{k,m_1,m_2}(x,0)=x,$$

$$\sum_{k=1}^{2^{j-1}}\sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1} c_{k,m_1,m_2}\frac{\partial}{\partial t}(\psi_{k,m_1,m_2}(x,t))|_{t=0}=0,$$

where $x=x[i]$ $(i=1,2,\dots,n)$ and $t=t[i']$ $(i'=1,2,\dots,n')$. The numerical results for this example are shown in Tables (5) and (6).

4. Numerical Results

In this section, the results of numerical experiments computed for the approximations of the solutions for the Klein-Gordon Equations are presented. In each problem, the different values for j and M are considered. Thus by setting these values, the best approximations are obtained.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	1.000038077	1.000038078
0.2	1.003528728	1.003528890
0.4	1.007019337	1.007018559
0.6	1.010509861	1.010512672
0.8	1.014000257	1.014001910
1	1.017490483	1.017491874
1.2	1.020980497	1.020981756
1.4	1.024470255	1.024471470
1.6	1.027959716	1.027960992
1.8	1.031448836	1.031450311
2	1.034937574	1.034939422
2.2	1.038425886	1.038428329
2.4	1.041913731	1.041917024
2.6	1.045401065	1.045405508
2.8	1.048887847	1.048893785
3	1.052374033	1.052381848

Table 1. The values of $u(x)$ for equation (20) with $t = 0.5$.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	1.000013708	1.000013708
0.2	1.003504359	1.003504707
0.4	1.006994968	1.006995322
0.6	1.010485492	1.010485894
0.8	1.013975888	1.013976217
1	1.017466114	1.017466329
1.2	1.020956128	1.020956221
1.4	1.024445886	1.024445904
1.6	1.027935347	1.027935371
1.8	1.031424467	1.031424628
2	1.034913205	1.034913672
2.2	1.038401517	1.038402499
2.4	1.041889362	1.041891117
2.6	1.045376696	1.045379519
2.8	1.048863478	1.048867711
3	1.052349664	1.052355690

Table 2. The values of $u(x)$ for equation (20) with $t = 0.3$.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	0	0
0.2	0.00003046129349	0.3046168223e-4
0.4	0.6092221583e-4	0.6092299520e-4
0.6	0.00009138239578	0.9138355165e-4
0.8	0.0001218414624	0.0001218430009
1	0.0001522990444	0.1523009752e-3
1.2	0.1827547706e-3	0.0001827570851
1.4	0.2132082700e-3	0.2132109814e-3
1.6	0.2436591716e-3	0.0002436622556
1.8	0.0002741071042	0.2741105826e-3
2	0.0003045516969	0.0003045555471
2.2	0.0003349925787	0.3349968378e-3
2.4	0.3654293788e-3	0.3654340152e-3
2.6	0.3958617262e-3	0.3958667460e-3
2.8	0.4262892505e-3	0.0004262946345
3	0.4567115802e-3	0.4567173835e-3

Table 3. The values of $u(x)$ for equation (23) with $t = 0.5$.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	0	0
0.2	0.0000182769256	0.00001827700809
0.4	0.00003655362643	0.00003655379464
0.6	0.000054829688287	0.00005483013328
0.8	0.00007310547129	0.00007310580701
1	0.00009138016892	0.00009138059107
1.2	0.0001096537531	0.0001096542571
1.4	0.0001279260012	0.0001279265866
1.6	0.0001461966905	0.0001461973541
1.8	0.0001644655985	0.0001644663547
2	0.000182732502	0.0001827333406
2.2	0.0002009971800	0.0002009981034
2.4	0.0002192594088	0.0002192604240
2.6	0.0002375189652	0.0002375200530
2.8	0.0002649028045	0.0002557768006
3	0.0002740291741	0.0002740304272

table 4. The values of $u(x)$ for equation (23) with $t = 0.3$.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	0	0.00006928203230
0.2	0.1999695390	0.2000415692
0.4	0.3999390781	0.4000138564
0.6	0.5999086171	0.5999861436
0.8	0.7998781562	0.7999584309
1	0.9998476952	0.9999307178
1.2	1.199817234	1.199903005
1.4	1.399786773	1.399875292
1.6	1.599756312	1.599847580
1.8	1.799725851	1.799819865
2	1.999695390	1.999792151
2.2	2.199664929	2.199764437
2.4	2.399634468	2.399736727
2.6	2.599604008	2.599709025
2.8	2.799573547	2.799681278
3	2.999543086	2.999653582

Table 5. The values of $u(x)$ for equation (25) with $t = 1$.

x	The values of u(x) obtained by the analytic method	The values of u(x) obtained by the wavelet method
0	0	0
0.2	0.1999972584	0.2000000000
0.4	0.3999945169	0.4000000000
0.6	0.5999917753	0.6000000000
0.8	0.7999890338	0.8000000000
1	0.9999862922	1.000000000
1.2	1.199983551	1.2000000000
1.4	1.399980809	1.4000000001
1.6	1.599978068	1.600000002
1.8	1.799975326	1.799999999
2	1.999972584	1.999999997
2.2	2.199969843	2.199999996
2.4	2.399967101	2.399999992
2.6	2.599964360	2.600000000
2.8	2.799961618	2.799999999
3	2.999958877	2.999999969

table 6. The values of $u(x)$ for equation (25) with $t = 0.3$.

5. Conclusions

Several numerical methods are used for solving the Klein-Gordon equation such as, ADM, VIM, HPM and HAM. In this paper, an approach which was composed of wavelet collocation method and Legendre wavelets was presented.

The Legendre wavelets were used for transforming the partial differential Equation (1) to a system of equations in terms of wavelets. In other words, The approximations were expressed in terms of bases functions.

In the Legendre wavelet method, the bases of Legendre wavelets were polynomials. Thus, the computations were preformed slowly. Also, these wavelets were continuous and orthonormal bases functions with compactly support[5,11]. Because of these properties, the Legendre wavelets were implemented and the best approximation of the solutions was computed. Numerical results indicated (expressed) the efficiency and accuracy of the proposed method in comparison with the other methods.

References

[1] S. Abbasbandy, Numerical solution of non-linear Klein-Gordon equations by variational iteration method, Int. J. Numer. Methods. Eng, 70 (2007), 876-881.

[2] A.K. Alomari, M. Salamimd Noorani and R. Mohd Nazar, Approximate Analytical Solutions of Klein-Gordon Equation by Means of Homotopy Analysis Method, Journal of Quality Measurement and analysis(JQMA), Jurnal Pengukuran Kualiti and Analisis, 4 (1) (2008), 45-57.

[3] B. Batiha, A Variational Iteration Method for Solving the Nonlinear Klein-Gordon equations, Journal of Basic and Applied Sciences, 3(4) (2009), 3876-3890.

[4] M.S.H. Chowdhury, I. Hashim, Application of homotopy-perturbation method to Klein-Gordon and sine-Gordon equations, www.elsevier.com/locate Chaos, Solitons and Fractals, 39 (2009), 1928-1935.

[5] E. Hesameddini, and S. Shekarpaz, Wavelet solutions of the second painlevé equation, Iran. J. Sci. Technol(to be Published) (2011).

[6] L. Gagnon, and J.M. Lina, Wavelets and numerical split-step method, A global adaptive scheme, Opt. Soc. Am. B, to appear.

[7] A.W. Galli, G.T. Heydt, and P F Ribeiro, Exploring the power of wavelet analysis, IEEE Computer Application in Power (1996), 37-41.

[8] M. Holmstr, Solving Hyperbolic PDEs Using Interpolating Wavelets, SIAM J. Sci. Comput., Vol. 21, No. 2 (2006), 405-420.

[9] Yoon Young Kim, and Gang-Won Jang, Hat interpolation wavelet-based multi-scale Galerkin method for thin-walled box beam analysis, International Journal for numerical methods in Engineering ;53 (2002), 1575-1592 (DOI:10.1002/NME.352).

[10] J. Liandrat, V. Perrier, and Ph. Tchamitchian, Numerical Resolution of Nonlinear Partial Differential Equations using the Wavelet Approach. Wavelets and Their Applications, edited by M B Ruskai, G Raphael (Jones Bartlett), Boston, 1992, Print 1996.

[11] M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, Math. Comp. Simulat. 53 (2000), 185-192.

[12] M. Razzaghi, S. Yousefi, Legendre wavelets method for constrained optimal control problems, Math. Methods Appl. Sci. 25 (2002), 529-539.

[13] M.N.O. Sadiku, C M Akujuobi, and R C Garcia, An introduction to wavelets in electromagnetics, Microwave Magazine, IEEE, vol. 6 (2) (2005), 63-72 (2005).

[14] R.L. Schult, and H.W. Wyld, Using wavelets to solve the Burgers' equation, A comparative study, Phys. Rev., A46, 12 (1992).

[15] Oleg V. Vasilyev and Christopher Bowman, Second-Generation Wavelet Collocation Method for the Solution of Partial Differential Equations, J. Comput. Phys, 165 (2000), 660-693.

[16] Oleg V. Vasilyev, A. Nicholas, K.R. Kevlahan, An adaptive multilevel wavelet collocation method for elliptic problems, J. Comput. Phys, 206 (2005), 412-431.

[17] A.S. Yakovlev, Wavelets as a Galerkin basis, Department of Computational Physics, St. Petersburg State University, 1994.