EXPONENTIAL MAP OF A CLASS OF TOP SPACES

NEDA EBRAHIMI
DEPARTMENT OF MATHEMATICS
SHAHID BAHONAR UNIVERSITY OF KERMAN
KERMAN, IRAN
E-MAIL: N_EBRAHIMI@MAIL.UK.AC.IR

Abstract. In this paper we wish to investigate right top spaces [1]. It is proved that if T is a right Rees matrix with the Lie algebra τ then (a) given a Lie subalgebra h of τ there exists a sub top space of T with the Lie algebra h, (b) given a morphism of Lie algebras $\psi : g \to \tau$ and $t \in T$, where g is the Lie algebra of a simply connected Lie group G, there exists a unique homomorphism $\varphi : G \to T$ such that $\varphi(e) = e(t)$ and $(\varphi)_* = \psi$. Finally exponential map for right Rees matrixes is defined.

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1. Introduction

The notion of top space as a generalization of Lie group is considered in [2]. Let us recall its definition.

Definition 1.1. A top space T is a smooth manifold admitting an operation called multiplication, subject to the set of rules given below:

- $(xy)z = x(yz)$ for all $x, y, z \in T$;

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- For each $x \in T$ there exists a unique $z \in T$ such that $xz = zx = x$ (we denote z by $e(x)$);
- For each $x \in T$ there exists $y \in T$ such that $xy = yx = e(x)$ (we denote y by x^{-1});
- The mapping $m_1 : T \to T$ is defined by $m_1(x) = x^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(x, y) = xy$ are smooth maps;
- $e(xy) = e(x)e(y)$ for all $x, y \in T$.

The reader can see [3], [4], [5] for recent works on top spaces.

We also recall that the map $l_g : T \to T$ ($r_g : T \to T$) defined by $l_g(x) = gx$ ($r_g(x) = xg$) is called left (right) translation by g. Left and right translations in Lie groups are diffeomprphism but top spaces don't have this property in general. The following theorem implies that if there is $g \in T$ that $Tg = T$ then r_g is diffeomorphism, for all $g \in T$. In particular $r_{e(g)} = Id$, for all $g \in T$.

Theorem 1.2.[2] If $Tg \cap Th \neq \emptyset$, then $Tg = Th$, where $g, h \in T$.

Definition 1.3.[1] A top space T is called a right top space if there is $g \in T$ that $Tg = T$.

A vector field X on a top space T is a left invariant vector field if $(l_g)_*(X) = X$, for all $g \in T$. In addition a form ω on T is left invariant if $(l_g)^*\omega = \omega$.

There are right top spaces with infinite number of identities.

Example 1.4.[2] The n- dimensional torus $T^n = R^n/Z^n$ with the product

$$((a_1, a_2, ..., a_n) + Z^n, (b_1, b_2, ..., b_n) + Z^n) = (a_1 + b_1, a_2 + b_2, ..., a_{n-1} + b_{n-1}, a_n) + Z^n$$

is a top space. $e((a_1, a_2, ..., a_n) + Z^n) = (0, 0, ..., a_n) + Z^n$. Hence T^n has infinite number of identities. In addition $T^n a = T^n$, for all $a \in T^n$.

Example 1.5. Suppose that G is a lie group and two smooth manifolds Λ and I are given. If $p : I \times \Lambda \to G$ is a smooth mapping, then $M(G, \Lambda, I, p) = \Lambda \times G \times I$ with the product $(\lambda, g, i)(\lambda_1, g_1, i_1) = (\lambda, gp(i, \lambda_1)g_1, i_1)$ is a top space, which is called Rees Matrix. Suppose that I is a one point set then $\Lambda \times G \times I \simeq \Lambda \times G$ is a right top space which we call right Rees matrix. In addition $e((\lambda, g)) = (\lambda, p(\lambda)^{-1})$ and consequently $\mathrm{card}(e(\Lambda \times G)) = \mathrm{card}(\Lambda)$.

Definition 1.6. (H, φ) is a sub top space of the top space T if

- H is a top space;
- (H, φ) is a submanifold of T;

- $\varphi : H \to T$ is a homomorphism.

Example 1.7. $(e(T^n), i)$ in example 1.4 is a sub top space. $(\{\lambda\} \times G, i)$ in the right Rees matrix $M(G, \Lambda, p)$ is a sub top space too.

By Ado's theorem any finite dimensional Lie algebra is isomorphic to the Lie algebra of a Lie group. Left invariant vector fields of a right top space form a Lie algebra [1]. The following theorem shows that in a class of right top spaces this Lie algebra is isomorphic to the Lie algebra of a sub top space which is a Lie group too.

Theorem 1.8.[1] Let T be a right top space. If for $t \in e(T)$, tT has a manifold structure which makes (tT, i) an imbedding then:

- gT is a Lie group and is diffeomorphic to tT, for every $g \in T$;
- There is a one to one correspondence between left invariant vector fields of T and tT.

2. Preliminaries

The following notions and theorems are applied in section 3.

Let M be a smooth manifold. We denote the set of all differential forms by $E^*(M)$. An ideal $I \subset E^*(M)$, is called a differential ideal if it is closed under exterior differentiation.

Theorem 2.1.[6] Let N and M be differentiable manifolds, and let π_1 and π_2 be the canonical projections of $N \times M$ on to N and M respectively. Suppose that there exists a basis $\{\omega_i, i = 1, \ldots.d\}$ for the 1- forms on M. If $\{\alpha_i : i = 1, \ldots, d\}$ are 1-forms on N and if the ideal of forms on $N \times M$ generated by $\{(\pi_1)^*(\alpha_i) - (\pi_2)^*(\omega_i) : i = 1, \ldots, d\}$ is a differential ideal, then given $n_0 \in N$ and $m_0 \in M$ there exists a neighborhood U of n_0 and a C^∞ map $f : U \to M$ such that $f(n_0) = m_0$ and $f^*(\omega_i) = \alpha_i|U$ for $i = 1, \ldots, d$. Moreover, if U is any connected open set containing n_0 for which there exists a C^∞ map $f : U \to M$ satisfying both $f(n_0) = m_0$ and $f^*(\omega_i) = \alpha_i|U$, then there exists a unique such map on U.

Sketch of proof. Since $\{(\pi_1)^*(\alpha_i) - (\pi_2)^*(\omega_i) : i = 1, \ldots, d\}$ is a differential ideal it has an integral manifold, I, through (n_0, m_0). $d\pi_1|I_p$, for $p \in I$ is nonsingular and consequently it is a local diffeomorphism. Hence there are open neighborhoods $V \subseteq I$ of (n_0, m_0) and $U \subseteq N$ of n_0 that $\pi_1 : V \to U$ is a diffeomorphism. The function $f = \pi_2 \circ (\pi_1|V)^{-1}$ is the desired map.

Remark 2.2. Using homomorphisms from a Lie group to a right top space one can construct a differential ideal. Let $\varphi : G \to T$ be a homomorphism of the Lie group G to the right top space T that $\varphi(e) = t$ for some $t \in e(T)$ and $\{\omega_i, i = 1, ...d\}$ be a basis for the space of left invariant 1- forms on T. The pull back of a left invariant form on T is a left invariant form on G. Let π_1 and π_2 be the canonical projection of $G \times T$ on to G and T respectively. The ideal I of left invariant one forms on $G \times T$ generated by the collection of independent one forms $\{\pi_1^* \varphi^*(\omega_i) - \pi_2^*(\omega_i), i = 1, \ldots d\}$ is a differential ideal. The proof is similar to the case that T is a Lie group [6].
In addition if $\psi : g \to \tau$ is a homomorphism of Lie algebras then it has a transpose ψ^*. Let $\{\omega_i, i = 1, \ldots, d\}$ be a basis of the space of left invariant 1- forms on T. The ideal generated by the collection of independent left invariant one forms $\{\pi_1^* \psi^*(\omega_i) - \pi_2^*(\omega_i), i = 1, \ldots d\}$ is a differential ideal [6].

3. Homomorphisms from a Lie group to a right top space

In this section we generalize main theorems in Lie group theory. One of these theorems is as follows.

Theorem 3.1.[6] Let G be a Lie group with Lie algebra g, and let $\widetilde{h} \subseteq g$ be a subalgebra. Then there is a unique connected Lie subgroup (H, φ) of G such that $d\varphi(h) = \widetilde{h}$.

Sketch of proof. We define a distribution D on G by setting $D(g) = \{X(g) : X \in \widetilde{h}\}$, for any $g \in G$. This distribution is smooth and involutive and its maximal integral manifold through e is the desired Lie subgroup.

Lemma 3.2. Let T be a right top space that tT, for some $t \in e(T)$, and $e(T)$ are embedded in T. Then T is diffeomorphic with a right Rees matrix.

Proof. tT is a Lie group by using theorem 1.8. Let $p : e(T) \to tT$ be the constant map, $p(s) = t$ for every $s \in e(T)$. We prove that the right Rees matrix $M(tT, e(T), p)$ is diffeomorphic with T. Let $\alpha : T \to e(T) \times tT$ be the map $\alpha(g) = (e(g), tg)$ for every $g \in T$. Since $\beta : e(T) \times tT \to T$, $\beta(s, tg) = stg$ is the inverse of α it is injective and surjective. α and β are smooth since the functions e and l_t are smooth and $e(T)$ and tT are imbedded in T. In addition $\alpha(gg') = \alpha(g)\alpha(g')$.

Theorem 3.3. Let $M(G, \Lambda, p)$ be a right Rees matrix with the Lie algebra τ and D be a distribution defined by setting $D(r) = \{X(r) : X \in \kappa\}$, for every $r \in M(G, \Lambda, p)$. If $t \in e(M(G, \Lambda, p))$ then for every subalgebra $\kappa \subseteq \tau$:

- There is a connected sub top space (H, φ) of $M(G, \Lambda, p)$ which is also a Lie group with identity t, such that $d\varphi(h) = \kappa$;
- (H, φ) is the maximal connected integral manifold of the distribution D through t.

Proof. Since $t \in e((G, \Lambda, p))$, $t = (i, p(i)^{-1})$, for some $i \in \Lambda$. $(i, p(i)^{-1})(\Lambda \times G) = \{i\} \times G$ is a Lie group and theorem 1.8 implies that its Lie algebra is τ. By theorem 3.1 for every subalgebra κ of τ there is a Lie subgroup, (ψ, K), of $\{i\} \times G$ that its Lie algebra is κ. This Lie subgroup is also a sub top space of $M(G, \Lambda, p)$ and by the proof of theorem 3.1 is the maximal integral manifold of D through $(i, p(i)^{-1})$ in $\{i\} \times G$.

Now let (H, φ) be the maximal integral manifold of D through $(i, p(i)^{-1})$ in $M(G, \Lambda, p)$. By pervious part $\psi(K) \subseteq \varphi(H)$. If there is $(j, g) \in \varphi(H) - \psi(K)$ then (j, g) is not in $\{i\} \times G$, for if $(j, g) \in \{i\} \times G$ then $l_{(j,g)^{-1}} \circ \psi(K)$ is an integral manifold of D through $(i, p(i)^{-1})$ in $\{i\} \times G$. Hence by maximality of (ψ, K), $l_{(j,g)^{-1}} \circ \psi(K) \subseteq \psi(K)$ and consequently $(j, g)^{-1} \in \psi(K)$. Since $\psi(K)$ is a Lie group $(j, g) \in \psi(K)$ which is a contradiction with the fact that $(j, g) \in \varphi(H) - \psi(K)$. Hence $e((j, g)) \neq (i, p(i)^{-1})$ and consequently $i \neq j$. There is a piecewise smooth curve $\gamma = (\gamma_1, \gamma_2)$ from $(i, p(i)^{-1})$ to (j, g). Since γ_1 is not a single point there is $s \in R$ that the tangent vector to $\gamma(s)$ has tangent element in Λ. Hence $dim(\varphi(H)) > dim(D)$ which is a contradiction.

Corollary 3.4.[6] Suppose that the ideal I generated by a collection $\{\omega_1, \omega_2, ..., \omega_{c-d}\}$ of independent left invariant 1- forms on the right Rees matrix $M(G, \Lambda, p)$ is a differential ideal. Then the maximal connected integral manifold of I through $t \in e(M(G, \Lambda, p))$ is a sub top space of $M(G, \Lambda, p)$ which is also a Lie group.

Theorem 3.5. Let G be a connected Lie group, T a right top space, and φ and ψ homomorphisms of G into T. If $\psi(e) = \varphi(e) = e(t)$ and $(\varphi)_* = (\psi)_*$ then $\varphi = \psi$.

Proof. Since $(\varphi)_* = (\psi)_*$ the transpose of ψ and φ are identical. In addition $\psi(e) = \varphi(e) = e(t)$. Let $\{\omega_i, i = 1, ...d\}$ be a basis for the space of left invariant 1- forms on T. By using remark 2.2, the ideal of forms on $G \times T$ generated by the one forms $\{\pi_1^* \varphi^*(\omega_i) - \pi_2^*(\omega_i), i = 1, \ldots d\}$ is a differential ideal. It follows from theorem 2.1 that $\psi = \varphi$.

Theorem 3.6.[6] Let G and H be connected Lie groups, and let $\varphi : G \to H$ be a homomorphism. Then φ is a covering map if and only if $d\varphi : G_e \to H_e$ be an isomorphism.

Theorem 3.7. Let H be a simply connected Lie group with Lie algebra h and $M(G, \Lambda, p)$ be a right Rees matrix with Lie algebra τ. If $\psi : g \to \tau$ be a homomorphism of Lie algebras then for every $t \in M(G, \Lambda, p)$, there exists a unique homomorphism $\varphi : H \to M(G, \Lambda, p)$ such that $\varphi(e) = e(t)$ and $(\varphi)_* = \psi$.

Proof. Uniqueness follows from theorem 3.5. Let $\{\omega_i, i = 1, \ldots d\}$ be a basis of left invariant one forms on $M(G, \Lambda, p)$, and ψ^* be the transpose of ψ. Then the ideal J generated by the collection of independent left invariant one forms $\{\pi_1^* \psi^*(\omega_i) - \pi_2^*(\omega_i), i = 1, \ldots d\}$ is a differential ideal by remark 2.2. According to the corollary 3.4 the maximal connected integral manifold I of J through $(e, e(t))$ is a sub top space of $H \times M(G, \Lambda, p)$ which is also a Lie group. By the proof of theorem 2.1, $\pi_1 \mid I : I \to H$ is nonsingular and by theorem 3.6, $\pi_1 \mid I$ is a covering homomorphism. Sine H is simply connected and $\pi_1|I$ is a covering it is a homeomorphism. By inverse function theorem $\pi_1 \mid I$ is an isomorphism. We define $\varphi = \pi_2 \circ (\pi_1 \mid I)^{-1}$. φ is a homomorphism and according to the theorem 2.1, $\varphi^*(\omega_i) = \psi^*(\omega_i)$.

Theorem 3.8. Let T be a connected right top space, and let U be an open neighborhood of $e(T)$. Then $T = \bigcup_{n=1}^{\infty} U^n$.

Proof. Let V be an open subset of U containing $e(T)$ such that $V = V^{-1}$. $H = \bigcup V^n$ is a sub generalized group of T (note that if $a, b \in V$ then $(ab)^{-1} = e(a)b^{-1}a^{-1}e(b) \in V^4$). If $a \in H$ then Va is an open neighborhood of a containing in H, since r_a is an open map. Hence H is open. In addition H is the complement of the disjoint union of all cosets mod H different from H itself, and consequently H is closed. Since T is connected, $T = H$.

4. Exponential map of right Rees matrixes

In this section we define exponential map for right Rees matrixes.

Definition 4.1. Let $M(G, \Lambda, p)$ be a right Rees matrix with the Lie algebra τ. Then $exp_t : \tau \to M(G, \Lambda, p)$, for $t \in M(G, \Lambda, p)$, is defined by $exp_t(X) = exp_t^X(1)$, where exp_t^X is the one parameter group of X which contains t.

Remark 4.2. Note that by theorem 1.8 one parameter groups of $M(G,\Lambda,p)$ that contain t are subset of $tM(G,\Lambda,p)$, $exp_t(\tau) \subseteq tM(G,\Lambda,p)$. $tM(G,\Lambda,p)$ is a Lie group with identity t and is embedded in $M(G,\Lambda,p)$. This implies the following theorem on right Rees matrixes.

Theorem 4.2. Let $M(G,\Lambda,p)$ be a right Rees matrix with Lie algebra τ. If X belongs to τ then:

- $exp_t(sX) = exp_t^X(s)$;
- $exp_t((t_1+t_2)X) = (exp_t(t_1X))(exp_t(t_2X))$;
- $exp_t(-tX) = (exp_t(tX))^{-1}$;
- $l_g \circ exp_t^X$ is the unique integral curve of X which takes the value g at 0;
- exp is a smooth map.

Example 4.3. Let $M(Gl(n,R),\Lambda,p)$ be a right Rees matrix that $Gl(n,R)$ is the set of $n\times n$ non singular matrixes and $p:\Lambda\to Gl(n,R)$ is the constant map $p(\lambda)=I$, for every $\lambda\in\Lambda$ where I is the identity element of $Gl(n,R)$. Using theorem 1.8, the Lie algebra of $M(Gl(n,R),\Lambda,p)$ and $(\lambda,p(\lambda)^{-1})M(Gl(n,R),\Lambda,p)=\{\lambda\}\times Gl(n,R)$ are the same. In addition $Gl(n,R)$ and $\{\lambda\}\times Gl(n,R)$ are Lie group isomorphic. Hence the Lie algebra of $\{\lambda\}\times Gl(n,R)$ is $gl(n,R)$, the set of $n\times n$ matrixes, and one parameter groups of $M(G,\Lambda,p)$ which contains $(\lambda,p(\lambda)^{-1})$ are $t\mapsto(\lambda,e^{tA})$, that $e^A = I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots$. Consequently $exp_{(\lambda,p(\lambda)^{-1})}: gl(n,R)\to\Lambda\times Gl(n,R)$ is $exp(A)=(\lambda,e^A)$, for every $A\in gl(n,R)$.

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References

[1] N. Ebrahimi, Left invariant vector fields of a class of top spaces, preprint.

[2] M.R. Molaei, Top spaces, J. Interdiscip. Math., Volume 7, Number 2 (2004), 173-181.

[3] M.R. Molaei, G.S. Khadekar and M.R. Farhangdoost, On Top spaces, Balkan J. Geom. Appl., Volume 11, Number 1 (2006), 101-106.

[4] M.R. Molaei and M.R. Farhangdoost, Lie algebras of a class of top spaces, Balkan J. Geom. Appl., Volume 14, Number 1 (2009), 46-51.

[5] M.R. Molaei and M.R. Farhangdoost, Upper top spaces, Appl. Sci., Volume 8 (2006), 128-131.

[6] F. Warner, Foundations of Differentiable manifolds and Lie groups, Springer, 1971.