APPROXIMATELY INNER $\sigma-$ DYNAMICS ON C^*- ALGEBRAS

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Abstract. Let D be a $*-$ subalgebra of C^*- algebra $\mathcal{A}$ and $\sigma : D \to \mathcal{A}$ be a linear operator. In this paper we introduce the notions of (approximately inner) $\sigma-$ derivations and (approximately inner) $\sigma-$ dynamics on C^*- algebras and present several results concerning on the approximately innerness of such dynamics. In particular we prove that if $\{\varphi_t\}_{t\in R}$ is a $\sigma-$ dynamics on the C^*- algebra $\mathcal{A}$ satisfying $\| \varphi_t \| \leq 2 \| \sigma \| +1$ and there exists a core D_0 for the generator d of $\{\varphi_t\}_{t\in R}$ such that d (as a $\sigma-$ derivation) is approximately inner on D_0, then $\{\varphi_t\}_{t\in R}$ is an approximately inner $\sigma-$ dynamics.

AMS Classification: 47D03, 46L57, 16W20.
Keywords: (inner) $\sigma-$ derivation; (inner) $\sigma-$ endomorphism; Approximately inner $C^*-(\sigma-)$ dynamics.

1. Introduction

Throughout the paper D is a $*-$ subalgebra of C^*- algebra $\mathcal{A}$ and $\sigma : D \to \mathcal{A}$ is a $*$- linear operator. Also $\mathcal{N}$ be considered as the set of all natural numbers.

JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER
VOL. 1, NUMBER 1 (2012) 55-63.

A one parameter group $\{\varphi_t\}_{t\in R}$ of bounded linear operators on $\mathcal{A}$ is a mapping $\varphi : R \to \mathbf{B}(\mathcal{A})$ from the additive group R of real numbers into the set $\mathbf{B}(\mathcal{A})$ of all bounded linear operators on $\mathcal{A}$ with the following properties:

(i) $\varphi_0 := I$ the identity operator on $\mathcal{A}$, and

(ii)$\varphi_{t+s} := \varphi_t\varphi_s$, for all $t, s \in R$.

A one parameter group $\{\varphi_t\}_{t\in R}$ is called uniformly (strongly) continuous if $\varphi : R \to \mathbf{B}(\mathcal{A})$ is continuous with respect to norm (strong) operator topology. A strongly continuous group of bounded linear operators on $\mathcal{A}$ is called a group of class C_0- or simply a C_0- group. We define the infinitesimal generator d of φ as a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - a}{t}$ where $D(d) = \{a \in \mathcal{A} : \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} exists\}$. Also we define the resolvent set $\rho(d)$ to be the set of all complex number λ for which $\lambda I - d$ is invertible, [9].

A $*-$ automorphism on $\mathcal{A}$ is an invertible linear operator $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$, for all $a, b \in \mathcal{A}$. An automorphism φ on $\mathcal{A}$ is called inner if there exists a unitary element $u \in \mathcal{A}$ such that $\varphi(a) = uau^*$. We denote by $aut(\mathcal{A})$ the set of all $*-$ automorphisms on $\mathcal{A}$.

Let G be a locally compact group and $t \to \varphi_t$ $(t \in G)$ be a norm continuous group homomorphism of G into $aut(\mathcal{A})$. Then the triple $\{\mathcal{A}, G, \varphi\}$ is called uniformly continuous C^*- dynamical system. In the case of $G = R$, as in quantum field theory, we call a one parameter group $\{\varphi_t\}_{t\in R}$ of $*-$ automorphisms on $\mathcal{A}$ a C^*- dynamics. It is easy to check that if $\{\varphi_t\}_{t\in R}$ is a group of $*-$ automorphisms with the infinitesimal generator d, then d is a $*-$ derivation and conversely if d is a bounded $*-$ derivation on the C^*- algebra $\mathcal{A}$, then d induces a uniformly continuous group of $*-$ automorphisms $\{e^{td}\}_{t\in R}$. In particular, if h is a self adjoint element in the C^*- algebra $\mathcal{A}$, then by Stone's theorem ih is the infinitesimal generator of a uniformly continuous group $\{u_t\}_{t\in R}$ of unitaries in $\mathcal{A}$, such that $u_t = e^{ith}$ and $d(a) = i[h, a]$, where $[h, a] = ha - ah$, is an inner derivation which is infinitesimal generator of the uniformly continuous group of inner $*-$ automorphisms $\{u_t a u_t^*\}_{t\in R}$. Conversely each uniformly continuous group $\{\varphi_t\}_{t\in R}$ of the form $\varphi_t(a) = e^{ith}ae^{-ith}$ of inner $*-$ automorphisms has an inner derivation as its infinitesimal generator.

A one parameter group of $*-$ automorphisms $\{\varphi_t\}_{t\in R}$ on the C^*- algebra $\mathcal{A}$ is said to be approximately inner if there exists a sequence $\{h_n\}$ of self adjoint elements in $\mathcal{A}$ such that for each $t \in R$, $\varphi_t = s - \lim_{n\to\infty} \varphi_{n,t}$, where $\varphi_{n,t}(a) = e^{ih_n t}ae^{-ih_n t}$

which means for each $t \in R$ and $a \in A$, $\| \varphi_{n,t}(a) - \varphi_t(a) \| \to 0$ uniformly on every compact subset of R, [10].

Now let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear operator. By a $\sigma-$ endomorphism we mean a linear mapping $\varphi : \mathcal{A} \to \mathcal{A}$ such that $(\varphi+\sigma-I)(ab)-(\varphi+\sigma-I)(a)(\varphi+\sigma-I)(b) = \sigma(ab)-\sigma(a)\sigma(b)$, for all $a, b \in \mathcal{A}$. In order to construct a $\sigma-$ endomorphism suppose that u is a unitary element of $\mathcal{A}$ satisfying $u\,(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))\,u$. Then the mapping $\varphi : \mathcal{A} \to \mathcal{A}$ defined by $\varphi(a) = u\sigma(a)u^* - \sigma(a) + a$ is a $\sigma-$ endomorphism which is called inner. Note that if σ is an endomorphism, then u can be any arbitrary unitary element of $\mathcal{A}$.

Let $\{\varphi_t\}_{t\in R}$ be a uniformly continuous one parameter group of bounded linear operators on $\mathcal{A}$. Following [7], we call $\{\varphi_t\}_{t\in R}$ a $\sigma-$ dynamics on the C^*- algebra $\mathcal{A}$ (or briefly $C^* - \sigma-$ dynamics) if additionally φ_t's are $\sigma-$ endomorphisms.

A linear mapping $d : D \to \mathcal{A}$ is called a $\sigma-$ derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in D$. (see [4], [5], [6], [7] and references therein.) The infinitesimal generator of $* - \sigma-$ dynamics is an everywhere defined $* - \sigma-$ derivation, [7].

In this paper we introduce the notions of (approximately inner) $\sigma-$ derivations and (approximately inner) $\sigma-$ dynamics on C^*- algebras and present several results concerning on the approximately innerness of such dynamics. In particular we prove that if $\{\varphi_t\}_{t\in R}$ is a $\sigma-$ dynamics on the C^*- algebra $\mathcal{A}$ satisfying $\| \varphi_t \| \leq 2 \| \sigma \| + 1$ and there exists a core D_0 for the generator d of $\{\varphi_t\}_{t\in R}$ such that d (as a $\sigma-$ derivation) is approximately inner on D_0, then $\{\varphi_t\}_{t\in R}$ is an approximately inner $\sigma-$ dynamics.

The reader is referred to [1],[3] and [8] for more details on Banach(C^*-) algebras and to [2] and [10] for more information on dynamical systems.

2. Preliminaries

Definition 2.1 A linear mapping $d : D \to \mathcal{A}$ is called a $\sigma-$ *derivation* if

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$$

(for all $a, b \in D$)

Example 2.2 Let σ be an arbitrary linear mapping on D and suppose that h is a self adjoint element of $\mathcal{A}$ satisfying $h\left(\sigma(ab) - \sigma(a)\sigma(b)\right) = \left(\sigma(ab) - \sigma(a)\sigma(b)\right)h$, for all $a, b \in D$. Then the mapping d_h^σ defined by $d_h^\sigma(a) = i[h, \sigma(a)]$ is a $* - \sigma-$ derivation which is called *inner* . Note that if σ is an endomorphism, then h can be any arbitrary self adjoint element of $\mathcal{A}$.

Definition 2.3 Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A linear mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called $\sigma-$ *endomorphism* if for all $a, b \in \mathcal{A}$

$$(\varphi + \sigma - I)(ab) - (\varphi + \sigma - I)(a)(\varphi + \sigma - I)(b) = \sigma(ab) - \sigma(a)\sigma(b).$$

Example 2.4 Let A and B be C^*- algebras. Then $A \times B$ is also a C^*- algebra by regarding the following structure:

(i) $(a, b) + (c, d) = (a + c, b + d)$

(ii) $\lambda(a, b) = (\lambda a, \lambda b)$

(iii) $(a, b).(c, d) = (ac, bd)$, $(a, b)^* = (a^*, b^*)$

(iv) $\| (a, b) \| = \max\{\| a \|, \| b \|\}$

Now define the maps φ and σ as follows:

$\varphi(a, b) = (2a, b)$

$\sigma(a, b) = (0, b)$

then φ is a $* - \sigma-$ endomorphism.

Definition 2.5 A linear mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called an *inner* $\sigma-$ *endomorphism* if there exist a unitary element $u \in \mathcal{A}$ such that for each $a, b \in \mathcal{A}$

(i)$(\varphi + \sigma - I)(a) = u\sigma(a)u^*$

(ii)$u\left(\sigma(ab) - \sigma(a)\sigma(b)\right) = \left(\sigma(ab) - \sigma(a)\sigma(b)\right)u$

Example 2.6 Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an arbitrary $*-$ linear endomorphism and h be a self adjoint element of $\mathcal{A}$. Then the mapping $\varphi : \mathcal{A} \to \mathcal{A}$ given by $\varphi(a) = e^{ih}\sigma(a)e^{-ih} - \sigma(a) + a$ is an inner $* - \sigma-$ endomorphism.

Definition 2.7 Let $\{\varphi_t\}_{t \in R}$ be a one parameter group of bounded linear operators on $\mathcal{A}$ such that for each $t \in R$, φ_t is a $\sigma-$ endomorphism. If moreover, $\{\varphi_t\}_{t \in R}$ is uniformly continuous, then it is called a $\sigma-$ *dynamics* on the C^*- algebra $\mathcal{A}$ (or briefly $C^* - \sigma-$ dynamics). We define the *infinitesimal generator* d of φ as a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - a}{t}$ where $D(d) = \{a \in \mathcal{A} : \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} exists\}$.

Remark 2.8 (i) Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a $*-$ endomorphism, $\{u_t\}_{t\in R}$ be a uniformly continuous one parameter group of unitary elements of $\mathcal{A}$ and let $\varphi_t : \mathcal{A} \to \mathcal{A}$ be the uniformly continuous one parameter group $\varphi_t(a) = u_t\sigma(a)u_t^* - \sigma(a) + a$ of inner $*-\sigma-$ endomorphisms. Applying Stone's theorem [9], there is a self adjoint element $h \in \mathcal{A}$ such that $u_t = e^{ith}$. Therefore the inner $* - \sigma-$ derivation $d_h^\sigma(a) = i[h, \sigma(a)]$ is the generator of $\{\varphi_t\}_{t\in R}$, [7].

(ii) Let h be a self adjoint element in the C^*- algebra $\mathcal{A}$, $\sigma : \mathcal{A} \to \mathcal{A}$ be an idempotent bounded $*-$ linear operator and $d_h^\sigma : \mathcal{A} \to \mathcal{A}$ be the inner $* - \sigma-$ derivation $d_h^\sigma(a) = i[h, \sigma(a)]$. It is known [7] that if for all $a \in \mathcal{A}$, $\sigma(ah) := \sigma(a)h$ and $\sigma(ha) := h\sigma(a)$, then d_h^σ induces the uniformly continuous one parameter group $\varphi_t(a) = e^{ith}\sigma(a)e^{-ith} - \sigma(a) + a$ of inner $* - \sigma-$ endomorphisms.

We end this section with the following well-known theorem entitled " Trotter-Kato Approximation Theorem" which can be found in [9]:

Theorem 2.9 Let d_n, $n \in \mathcal{N}$ be the generator of a C_0- semigroup $\{\varphi_n(t)\}_{t\in R}$ satisfying $\| \varphi_n(t) \| \leq Me^{\omega t}$ on the Banach algebra A. If for some complex number λ_0 with $Re(\lambda_0) > \omega$ we have:

(i) $(\lambda_o - d_n)^{-1}$ converges strongly to some operator $R(\lambda_0)$ on A, and

(ii) the range of $R(\lambda_0)$ is dense in A

then there exists a unique operator d which is the generator of a C_0- semigroup $\{\varphi_t\}_{t\in R}$ on A of the same type as $\{\varphi_n(t)\}_{t\in R}$ such that $(\lambda_o - d)^{-1} = R(\lambda_0)$ and $\varphi_n(t)$ converges strongly to $\varphi(t)$.

3. Approximately Inner $\sigma-$ Dynamics on C^*- Algebras

Throughout this section σ is an idempotent bounded $*-$ linear operator on the C^*- algebra $\mathcal{A}$.

Definition 3.1 A $* - \sigma-$ dynamics $\{\varphi_t\}_{t\in R}$ on $\mathcal{A}$ is called *approximately inner $\sigma-$ dynamics* if there exists a sequence $\{h_n\}$ of self adjoint elements of $\mathcal{A}$ such that for each $a, b \in \mathcal{A}$

(i) $h_n(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))h_n$

(ii) $\sigma(h_n a) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$

(iii) for each $t \in R$, $\varphi_t = s-\lim\limits_{n\to\infty} \varphi_{n,t}$, where $\varphi_{n,t}(a) = e^{ih_nt}\sigma(a)e^{-ih_nt} - \sigma(a) + a$

which means for each $t \in R$

$$\lim_{n\to\infty} \varphi_{n,t}(a) = \varphi_t(a), \text{ for all } a \in \mathcal{A}$$

Theorem 3.2 Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an endomorphism and d be the generator of a $\sigma-$ dynamics $\{\varphi_t\}_{t\in R}$ satisfying $\parallel \varphi_t \parallel \leq 2 \parallel \sigma \parallel +1$. If $\{h_n\}$ is a sequence of self adjoint elements of $\mathcal{A}$ such that for each $a \in \mathcal{A}$

(i) $\sigma(h_n a) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$

(ii) $(1-d)^{-1} = s - \lim\limits_{n\to\infty}(1 - d^\sigma_{h_n})^{-1}$, where $d^\sigma_{h_n}(a) = i[h_n, \sigma(a)]$.

Then $\{\varphi_t\}_{t\in R}$ is approximately inner.

Proof. By Remark 2.8 $d^\sigma_{h_n}$ induces the uniformly continuous one parameter group $\varphi_n(t)(a) = e^{ih_nt}\sigma(a)e^{-ih_nt} - \sigma(a) + a$ and the condition (ii) implies that the range of $(1-d)^{-1}$ is dense in $\mathcal{A}$. Therefore by Trotter-Kato approximation Theorem for each $t \in R$, $\varphi_t = \lim\limits_{n\to\infty} \varphi_{n,t}$.□

Corollary 3.3 Let $\{\varphi_t\}_{t\in R}$ be a C^*- dynamics on $\mathcal{A}$ with the generator d. If $\{h_n\}$ is a sequence of self adjoint elements of $\mathcal{A}$ such that $(1-d)^{-1} = s-\lim\limits_{n\to\infty}(1-d_{h_n})^{-1}$, where $d_{h_n}(a) = i[h_n, a]$, then $\{\varphi_t\}_{t\in R}$ is approximately inner.

Remark 3.4 In the sense of Theorem 3.2 and Corollary 3.3

(*) A C^*- dynamics $\{\varphi_t\}_{t\in R}$ on the C^*- algebra $\mathcal{A}$ with the generator d is approximately inner if there exists a sequence $\{h_n\}$ of self adjoint elements in $\mathcal{A}$ such that $(1-d)^{-1} = s - \lim\limits_{n\to\infty}(1 - d_{h_n})^{-1}$, where $d_{h_n}(a) = i[h_n, a]$, for all $a \in \mathcal{A}$.

(**) Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an endomorphism. A $\sigma-$ dynamics $\{\varphi_t\}_{t\in R}$ on $\mathcal{A}$ satisfying $\parallel \varphi_t \parallel \leq 2 \parallel \sigma \parallel +1$ with the generator d is approximately inner if there exists a sequences $\{h_n\}$ of self adjoint elements in the C^*- algebra $\mathcal{A}$ such that for each $a \in \mathcal{A}$

(i) $\sigma(h_n a) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$

(ii) $(1-d)^{-1} = s - \lim\limits_{n\to\infty}(1 - d^\sigma_{h_n})^{-1}$, where $d^\sigma_{h_n}(a) = i[h_n, \sigma(a)]$

The following definition is a natural generalization of definition of approximately inner derivations:

Definition 3.5 A $* - \sigma-$ derivation $d : D \to \mathcal{A}$ is said to be *approximately inner*

$\sigma-$ *derivation on* D if there exists a sequence $\{h_n\}$ of self adjoint elements of $\mathcal{A}$ such that for each $a, b \in \mathcal{A}$

(i) $h_n(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))h_n$

(ii) $\sigma(h_n a) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$

(iii) for each $a \in D$, $d(a) = \lim_{n\to\infty} d^{\sigma}_{h_n}(a)$, where $d^{\sigma}_{h_n}(a) = i[h_n, \sigma(a)]$.

Theorem 3.6 Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an endomorphism and d be the generator of a $\sigma-$ dynamics $\{\varphi_t\}_{t\in R}$ satisfying $\| \varphi_t \| \leq 2 \| \sigma \| +1$. If d is approximately inner and $(1 - d)(D(d))$ is dense in $\mathcal{A}$, then $\{\varphi_t\}_{t\in R}$ is approximately inner.

Proof. Since d is an approximately inner $\sigma-$ derivation, so there exists a sequence $\{h_n\}$ of self adjoint elements of $\mathcal{A}$ such that for each $a \in \mathcal{A}$

(i) $\sigma(h_n a) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$

(ii) for each $a \in D(d)$, $d(a) = \lim_{n\to\infty} d^{\sigma}_{h_n}(a)$.

Also $d^{\sigma}_{h_n}$ induces the uniformly continuous one parameter group
$\varphi_n(t)(a) = e^{ih_n t}\sigma(a)e^{-ih_n t} - \sigma(a) + a$. By Remark 3.4 it is enough to show that $(1 - d)^{-1} = s - \lim_{n\to\infty}(1 - d^{\sigma}_{h_n})^{-1}$. For this aim we have

$$\begin{aligned} &||(1 - d^{\sigma}_{h_n})^{-1}(1 - d)(a) - (1 - d)^{-1}(1 - d)(a)|| \\ = &||(1 - d^{\sigma}_{h_n})^{-1}(1 - d)(a) - (1 - d^{\sigma}_{h_n})^{-1}(1 - d^{\sigma}_{h_n})(a)|| \\ \leq &||(1 - d^{\sigma}_{h_n})^{-1}|| \ \ \| (1 - d)(a) - (1 - d^{\sigma}_{h_n})(a)|| \\ \leq &(2||\sigma|| + 1)||(1 - d)(a) - (1 - d^{\sigma}_{h_n})(a)|| \to 0. \end{aligned}$$

Since $\| (1 - d^{\sigma}_{h_n})^{-1} \| \leq 2 \| \sigma \| +1$ (By Hille-Yosida Theorem [9]). Now the density of $(1 - d)(D(d))$ in $\mathcal{A}$ implies that $(1 - d^{\sigma}_{h_n})^{-1} \to (1 - d)^{-1}$ (strongly).□

Before we state the next theorem, we need the following well-known definition:

Definition 3.7 A subset D_0 of domain D of a closed linear operator d on the Banach space A is called a *core* for d, if d is the closure of its restriction on D_0. In the other words D_0 is a core for d if for each $a \in D$, there exists a sequence $\{a_n\}$ in D_0 such that $a_n \to a$ and $d(a_n) \to d(a)$.

Theorem 3.8 Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an endomorphism and $\{\varphi_t\}_{t\in R}$ be a $\sigma-$ dynamics on $\mathcal{A}$ satisfying $\| \varphi_t \| \leq 2 \| \sigma \| +1$. If there exists a core D_0 for the generator d of $\{\varphi_t\}_{t\in R}$ such that d is approximately inner on D_0, then $\{\varphi_t\}_{t\in R}$ is an approximately inner $\sigma-$ dynamics.

Proof. First note that since d is approximately inner on D_0, so there exists a sequence $\{h_n\}$ of self adjoint elements of $\mathcal{A}$ such that $d^{\sigma}_{h_n}$ induces the uniformly continuous one parameter group $\varphi_n(t)(a) = e^{ih_n t}\sigma(a)e^{-ih_n t} - \sigma(a) + a$. Also by the extended Hille-Yosida Theorem $\lambda = 1 \in \rho(d) \cap \rho(d^{\sigma}_{h_n})$ and the range $R(1-d)$ of $1-d$ is $\mathcal{A}$. Further $\| (1-d)^{-1} \| \leq 2 \| \sigma \| +1$ and $\| (1-d^{\sigma}_{h_n})^{-1} \| \leq 2 \| \sigma \| +1$. Applying Remark 3.4 it suffices to show that

$$(1-d^{\sigma}_{h_n})^{-1} \to (1-d)^{-1} \qquad (strongly \ \ on \ \ \mathcal{A})$$

For this aim, let $\mathcal{B} = \{(1-d)(b) : \ b \in D_0\} = R(1-d\mid_{D_0})$. First we show that $\mathcal{B}$ is dense in $\mathcal{A}$. Let $a \in \mathcal{A}$, since $R(1-d) = \mathcal{A}$, so there exists $c \in D(d)$ such that $a = c - d(c)$. But D_0 is a core for d. Thus there exists a sequence $\{b_n\}$ in D_0 such that $b_n \to c$ and $b_n - d(b_n) \to c - d(c) = a$. Hence $\mathcal{B}$ is dense in $\mathcal{A}$. Now we show that $(1-d^{\sigma}_{h_n})^{-1}$ converges strongly on $\mathcal{B}$ to $(1-d)^{-1}$. For, let $b \in \mathcal{B}$. There exists $b_0 \in D_0$ such that $b = b_0 - d(b_0)$ and by assumption $d^{\sigma}_{h_n}(b_0) \to d(b_0)$. Therefore

$$\begin{aligned} \| (1-d^{\sigma}_{h_n})^{-1}(b) - (1-d)^{-1}(b) \| &= \| (1-d^{\sigma}_{h_n})^{-1}(d^{\sigma}_{h_n} - d)(1-d)^{-1}(b) \| \\ &\leq (2 \| \sigma \| +1) \| (d^{\sigma}_{h_n} - d)(1-d)^{-1}(b) \| \\ &= (2 \| \sigma \| +1) \| (d^{\sigma}_{h_n} - d)(b_0) \| \to 0 \end{aligned}$$

which implies that for each $b \in \mathcal{B}$

$$(1-d^{\sigma}_{h_n})^{-1}(b) \to (1-d)^{-1}(b)$$

Finally given $a \in \mathcal{A}$ and $\epsilon > 0$. Since $\mathcal{B}$ is dense in $\mathcal{A}$, so there exist $b \in \mathcal{B}$ and $N_\epsilon \in \mathcal{N}$ such that

$$\| b - a \| \leq \frac{\epsilon}{3(2 \| \sigma \| +1)}$$

and for each $n \geq N_\epsilon$

$$\| (1-d^{\sigma}_{h_n})^{-1}(b) - (1-d)^{-1}(b) \| \leq \frac{\epsilon}{3}.$$

Therefore

$$\begin{aligned} \| (1-d^{\sigma}_{h_n})^{-1}(a) - (1-d)^{-1}(a) \| &\leq \| (1-d^{\sigma}_{h_n})^{-1}(a) - (1-d^{\sigma}_{h_n})^{-1}(b) \| \\ &+ \| (1-d^{\sigma}_{h_n})^{-1}(b) - (1-d)^{-1}(b) \| \\ &+ \| (1-d)^{-1}(b) - (1-d)^{-1}(a) \| \\ &< 2(2 \| \sigma \| +1) \| b - a \| + \tfrac{\epsilon}{3} < \epsilon. \square \end{aligned}$$

4. Acknowledgment

The authors would like to thank the referee for his/her useful comments and suggestions.

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