

APPROXIMATELY INNER σ - DYNAMICS ON C^* - ALGEBRAS

¹M. MOSADEQ, ¹M.HASSANI, AND ²A. NIKNAM

¹DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, MASHHAD BRANCH
ISLAMIC AZAD UNIVERSITY, MASHHAD, IRAN

MOSADEQ@MSHDIAU.AC.IR

HASSANI@MSHDIAU.AC.IR

²DEPARTMENT OF MATHEMATICS AND CENTER OF EXCELLENCE IN ANALYSIS ON
ALGEBRAIC STRUCTURES(CEAAS), FERDOWSI UNIVERSITY, MASHHAD, IRAN

ABSTRACT. Let D be a $*$ - subalgebra of C^* - algebra \mathcal{A} and $\sigma : D \rightarrow \mathcal{A}$ be a linear operator. In this paper we introduce the notions of (approximately inner) σ - derivations and (approximately inner) σ - dynamics on C^* - algebras and present several results concerning on the approximately innerness of such dynamics. In particular we prove that if $\{\varphi_t\}_{t \in \mathbb{R}}$ is a σ - dynamics on the C^* - algebra \mathcal{A} satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$ and there exists a core D_0 for the generator d of $\{\varphi_t\}_{t \in \mathbb{R}}$ such that d (as a σ - derivation) is approximately inner on D_0 , then $\{\varphi_t\}_{t \in \mathbb{R}}$ is an approximately inner σ - dynamics.

AMS Classification: 47D03, 46L57, 16W20.

Keywords: (inner) σ - derivation; (inner) σ - endomorphism; Approximately inner C^* -(σ -) dynamics.

1. INTRODUCTION

Throughout the paper D is a $*$ - subalgebra of C^* - algebra \mathcal{A} and $\sigma : D \rightarrow \mathcal{A}$ is a $*$ - linear operator. Also \mathbb{N} be considered as the set of all natural numbers.

JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER

VOL. 1, NUMBER 1 (2012) 55-63.

©MAHANI MATHEMATICAL RESEARCH CENTER

A one parameter group $\{\varphi_t\}_{t \in R}$ of bounded linear operators on \mathcal{A} is a mapping $\varphi : R \rightarrow \mathbf{B}(\mathcal{A})$ from the additive group R of real numbers into the set $\mathbf{B}(\mathcal{A})$ of all bounded linear operators on \mathcal{A} with the following properties:

- (i) $\varphi_0 := I$ the identity operator on \mathcal{A} , and
- (ii) $\varphi_{t+s} := \varphi_t \varphi_s$, for all $t, s \in R$.

A one parameter group $\{\varphi_t\}_{t \in R}$ is called uniformly (strongly) continuous if $\varphi : R \rightarrow \mathbf{B}(\mathcal{A})$ is continuous with respect to norm (strong) operator topology. A strongly continuous group of bounded linear operators on \mathcal{A} is called a group of class C_0 – or simply a C_0 – group. We define the infinitesimal generator d of φ as a mapping $d : D(d) \subseteq \mathcal{A} \rightarrow \mathcal{A}$ such that $d(a) = \lim_{t \rightarrow 0} \frac{\varphi_t(a) - a}{t}$ where $D(d) = \{a \in \mathcal{A} : \lim_{t \rightarrow 0} \frac{\varphi_t(a) - a}{t} \text{ exists}\}$. Also we define the resolvent set $\rho(d)$ to be the set of all complex number λ for which $\lambda I - d$ is invertible, [9].

A $*$ – automorphism on \mathcal{A} is an invertible linear operator $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$, for all $a, b \in \mathcal{A}$. An automorphism φ on \mathcal{A} is called inner if there exists a unitary element $u \in \mathcal{A}$ such that $\varphi(a) = uau^*$. We denote by $aut(\mathcal{A})$ the set of all $*$ – automorphisms on \mathcal{A} .

Let G be a locally compact group and $t \rightarrow \varphi_t$ ($t \in G$) be a norm continuous group homomorphism of G into $aut(\mathcal{A})$. Then the triple $\{\mathcal{A}, G, \varphi\}$ is called uniformly continuous C^* – dynamical system. In the case of $G = R$, as in quantum field theory, we call a one parameter group $\{\varphi_t\}_{t \in R}$ of $*$ – automorphisms on \mathcal{A} a C^* – dynamics. It is easy to check that if $\{\varphi_t\}_{t \in R}$ is a group of $*$ – automorphisms with the infinitesimal generator d , then d is a $*$ – derivation and conversely if d is a bounded $*$ – derivation on the C^* – algebra \mathcal{A} , then d induces a uniformly continuous group of $*$ – automorphisms $\{e^{td}\}_{t \in R}$. In particular, if h is a self adjoint element in the C^* – algebra \mathcal{A} , then by Stone's theorem ih is the infinitesimal generator of a uniformly continuous group $\{u_t\}_{t \in R}$ of unitaries in \mathcal{A} , such that $u_t = e^{ith}$ and $d(a) = i[h, a]$, where $[h, a] = ha - ah$, is an inner derivation which is infinitesimal generator of the uniformly continuous group of inner $*$ – automorphisms $\{u_t a u_t^*\}_{t \in R}$. Conversely each uniformly continuous group $\{\varphi_t\}_{t \in R}$ of the form $\varphi_t(a) = e^{ith} a e^{-ith}$ of inner $*$ – automorphisms has an inner derivation as its infinitesimal generator.

A one parameter group of $*$ – automorphisms $\{\varphi_t\}_{t \in R}$ on the C^* – algebra \mathcal{A} is said to be approximately inner if there exists a sequence $\{h_n\}$ of self adjoint elements in \mathcal{A} such that for each $t \in R$, $\varphi_t = s - \lim_{n \rightarrow \infty} \varphi_{n,t}$, where $\varphi_{n,t}(a) = e^{ih_n t} a e^{-ih_n t}$

which means for each $t \in R$ and $a \in A$, $\|\varphi_{n,t}(a) - \varphi_t(a)\| \rightarrow 0$ uniformly on every compact subset of R , [10].

Now let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be a linear operator. By a σ -endomorphism we mean a linear mapping $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ such that $(\varphi + \sigma - I)(ab) - (\varphi + \sigma - I)(a)(\varphi + \sigma - I)(b) = \sigma(ab) - \sigma(a)\sigma(b)$, for all $a, b \in \mathcal{A}$. In order to construct a σ -endomorphism suppose that u is a unitary element of \mathcal{A} satisfying $u(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))u$. Then the mapping $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\varphi(a) = u\sigma(a)u^* - \sigma(a) + a$ is a σ -endomorphism which is called inner. Note that if σ is an endomorphism, then u can be any arbitrary unitary element of \mathcal{A} .

Let $\{\varphi_t\}_{t \in R}$ be a uniformly continuous one parameter group of bounded linear operators on \mathcal{A} . Following [7], we call $\{\varphi_t\}_{t \in R}$ a σ -dynamics on the C^* -algebra \mathcal{A} (or briefly C^* - σ -dynamics) if additionally φ_t 's are σ -endomorphisms.

A linear mapping $d : D \rightarrow \mathcal{A}$ is called a σ -derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in D$. (see [4], [5], [6], [7] and references therein.) The infinitesimal generator of $*$ - σ -dynamics is an everywhere defined $*$ - σ -derivation, [7].

In this paper we introduce the notions of (approximately inner) σ -derivations and (approximately inner) σ -dynamics on C^* -algebras and present several results concerning on the approximately innerness of such dynamics. In particular we prove that if $\{\varphi_t\}_{t \in R}$ is a σ -dynamics on the C^* -algebra \mathcal{A} satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$ and there exists a core D_0 for the generator d of $\{\varphi_t\}_{t \in R}$ such that d (as a σ -derivation) is approximately inner on D_0 , then $\{\varphi_t\}_{t \in R}$ is an approximately inner σ -dynamics.

The reader is referred to [1],[3] and [8] for more details on Banach(C^* -) algebras and to [2] and [10] for more information on dynamical systems.

2. PRELIMINARIES

Definition 2.1 A linear mapping $d : D \rightarrow \mathcal{A}$ is called a σ -derivation if

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$$

(for all $a, b \in D$)

Example 2.2 Let σ be an arbitrary linear mapping on D and suppose that h is a self adjoint element of \mathcal{A} satisfying $h(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))h$, for all $a, b \in D$. Then the mapping d_h^σ defined by $d_h^\sigma(a) = i[h, \sigma(a)]$ is a $*$ - σ -derivation which is called *inner*. Note that if σ is an endomorphism, then h can be any arbitrary self adjoint element of \mathcal{A} .

Definition 2.3 Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be a linear mapping. A linear mapping $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ is called σ -*endomorphism* if for all $a, b \in \mathcal{A}$

$$(\varphi + \sigma - I)(ab) - (\varphi + \sigma - I)(a)(\varphi + \sigma - I)(b) = \sigma(ab) - \sigma(a)\sigma(b).$$

Example 2.4 Let A and B be C^* -algebras. Then $A \times B$ is also a C^* -algebra by regarding the following structure:

- (i) $(a, b) + (c, d) = (a + c, b + d)$
- (ii) $\lambda(a, b) = (\lambda a, \lambda b)$
- (iii) $(a, b) \cdot (c, d) = (ac, bd)$, $(a, b)^* = (a^*, b^*)$
- (iv) $\| (a, b) \| = \max\{\| a \|, \| b \|\}$

Now define the maps φ and σ as follows:

$$\begin{aligned}\varphi(a, b) &= (2a, b) \\ \sigma(a, b) &= (0, b)\end{aligned}$$

then φ is a $*$ - σ -endomorphism.

Definition 2.5 A linear mapping $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ is called an *inner σ -endomorphism* if there exist a unitary element $u \in \mathcal{A}$ such that for each $a, b \in \mathcal{A}$

- (i) $(\varphi + \sigma - I)(a) = u\sigma(a)u^*$
- (ii) $u(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))u$

Example 2.6 Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an arbitrary $*$ -linear endomorphism and h be a self adjoint element of \mathcal{A} . Then the mapping $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ given by $\varphi(a) = e^{ih}\sigma(a)e^{-ih} - \sigma(a) + a$ is an inner $*$ - σ -endomorphism.

Definition 2.7 Let $\{\varphi_t\}_{t \in R}$ be a one parameter group of bounded linear operators on \mathcal{A} such that for each $t \in R$, φ_t is a σ -endomorphism. If moreover, $\{\varphi_t\}_{t \in R}$ is uniformly continuous, then it is called a σ -*dynamics* on the C^* -algebra \mathcal{A} (or briefly C^* - σ -dynamics). We define the *infinitesimal generator* d of φ as a

mapping $d : D(d) \subseteq \mathcal{A} \rightarrow \mathcal{A}$ such that $d(a) = \lim_{t \rightarrow 0} \frac{\varphi_t(a) - a}{t}$ where $D(d) = \{a \in \mathcal{A} : \lim_{t \rightarrow 0} \frac{\varphi_t(a) - a}{t} \text{ exists}\}$.

Remark 2.8 (i) Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be a $*$ - endomorphism, $\{u_t\}_{t \in \mathbb{R}}$ be a uniformly continuous one parameter group of unitary elements of \mathcal{A} and let $\varphi_t : \mathcal{A} \rightarrow \mathcal{A}$ be the uniformly continuous one parameter group $\varphi_t(a) = u_t \sigma(a) u_t^* - \sigma(a) + a$ of inner $*$ - σ - endomorphisms. Applying Stone's theorem [9], there is a self adjoint element $h \in \mathcal{A}$ such that $u_t = e^{ith}$. Therefore the inner $*$ - σ - derivation $d_h^\sigma(a) = i[h, \sigma(a)]$ is the generator of $\{\varphi_t\}_{t \in \mathbb{R}}$, [7].

(ii) Let h be a self adjoint element in the C^* - algebra \mathcal{A} , $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an idempotent bounded $*$ - linear operator and $d_h^\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be the inner $*$ - σ - derivation $d_h^\sigma(a) = i[h, \sigma(a)]$. It is known [7] that if for all $a \in \mathcal{A}$, $\sigma(ah) := \sigma(a)h$ and $\sigma(ha) := h\sigma(a)$, then d_h^σ induces the uniformly continuous one parameter group $\varphi_t(a) = e^{ith} \sigma(a) e^{-ith} - \sigma(a) + a$ of inner $*$ - σ - endomorphisms.

We end this section with the following well-known theorem entitled " Trotter-Kato Approximation Theorem" which can be found in [9]:

Theorem 2.9 Let $d_n, n \in \mathbb{N}$ be the generator of a C_0 - semigroup $\{\varphi_n(t)\}_{t \in \mathbb{R}}$ satisfying $\|\varphi_n(t)\| \leq M e^{\omega t}$ on the Banach algebra A . If for some complex number λ_0 with $Re(\lambda_0) > \omega$ we have:

- (i) $(\lambda_0 - d_n)^{-1}$ converges strongly to some operator $R(\lambda_0)$ on A , and
- (ii) the range of $R(\lambda_0)$ is dense in A

then there exists a unique operator d which is the generator of a C_0 - semigroup $\{\varphi_t\}_{t \in \mathbb{R}}$ on A of the same type as $\{\varphi_n(t)\}_{t \in \mathbb{R}}$ such that $(\lambda_0 - d)^{-1} = R(\lambda_0)$ and $\varphi_n(t)$ converges strongly to $\varphi(t)$.

3. APPROXIMATELY INNER σ - DYNAMICS ON C^* - ALGEBRAS

Throughout this section σ is an idempotent bounded $*$ - linear operator on the C^* - algebra \mathcal{A} .

Definition 3.1 A $*$ - σ - dynamics $\{\varphi_t\}_{t \in \mathbb{R}}$ on \mathcal{A} is called *approximately inner σ - dynamics* if there exists a sequence $\{h_n\}$ of self adjoint elements of \mathcal{A} such that for each $a, b \in \mathcal{A}$

- (i) $h_n(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))h_n$

- (ii) $\sigma(h_n a) = h_n \sigma(a)$ and $\sigma(a h_n) = \sigma(a) h_n$
- (iii) for each $t \in R$, $\varphi_t = s - \lim_{n \rightarrow \infty} \varphi_{n,t}$, where $\varphi_{n,t}(a) = e^{i h_n t} \sigma(a) e^{-i h_n t} - \sigma(a) + a$

which means for each $t \in R$

$$\lim_{n \rightarrow \infty} \varphi_{n,t}(a) = \varphi_t(a), \text{ for all } a \in \mathcal{A}$$

Theorem 3.2 Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism and d be the generator of a σ -dynamics $\{\varphi_t\}_{t \in R}$ satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$. If $\{h_n\}$ is a sequence of self adjoint elements of \mathcal{A} such that for each $a \in \mathcal{A}$

- (i) $\sigma(h_n a) = h_n \sigma(a)$ and $\sigma(a h_n) = \sigma(a) h_n$
- (ii) $(1 - d)^{-1} = s - \lim_{n \rightarrow \infty} (1 - d_{h_n}^\sigma)^{-1}$, where $d_{h_n}^\sigma(a) = i[h_n, \sigma(a)]$.

Then $\{\varphi_t\}_{t \in R}$ is approximately inner.

Proof. By Remark 2.8 $d_{h_n}^\sigma$ induces the uniformly continuous one parameter group $\varphi_n(t)(a) = e^{i h_n t} \sigma(a) e^{-i h_n t} - \sigma(a) + a$ and the condition (ii) implies that the range of $(1 - d)^{-1}$ is dense in \mathcal{A} . Therefore by Trotter-Kato approximation Theorem for each $t \in R$, $\varphi_t = \lim_{n \rightarrow \infty} \varphi_{n,t}$. \square

Corollary 3.3 Let $\{\varphi_t\}_{t \in R}$ be a C^* -dynamics on \mathcal{A} with the generator d . If $\{h_n\}$ is a sequence of self adjoint elements of \mathcal{A} such that $(1 - d)^{-1} = s - \lim_{n \rightarrow \infty} (1 - d_{h_n})^{-1}$, where $d_{h_n}(a) = i[h_n, a]$, then $\{\varphi_t\}_{t \in R}$ is approximately inner.

Remark 3.4 In the sense of Theorem 3.2 and Corollary 3.3

(*) A C^* -dynamics $\{\varphi_t\}_{t \in R}$ on the C^* -algebra \mathcal{A} with the generator d is approximately inner if there exists a sequence $\{h_n\}$ of self adjoint elements in \mathcal{A} such that $(1 - d)^{-1} = s - \lim_{n \rightarrow \infty} (1 - d_{h_n})^{-1}$, where $d_{h_n}(a) = i[h_n, a]$, for all $a \in \mathcal{A}$.

(**) Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism. A σ -dynamics $\{\varphi_t\}_{t \in R}$ on \mathcal{A} satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$ with the generator d is approximately inner if there exists a sequences $\{h_n\}$ of self adjoint elements in the C^* -algebra \mathcal{A} such that for each $a \in \mathcal{A}$

- (i) $\sigma(h_n a) = h_n \sigma(a)$ and $\sigma(a h_n) = \sigma(a) h_n$
- (ii) $(1 - d)^{-1} = s - \lim_{n \rightarrow \infty} (1 - d_{h_n}^\sigma)^{-1}$, where $d_{h_n}^\sigma(a) = i[h_n, \sigma(a)]$

The following definition is a natural generalization of definition of approximately inner derivations:

Definition 3.5 A $*$ - σ -derivation $d : D \rightarrow \mathcal{A}$ is said to be *approximately inner*

σ - derivation on D if there exists a sequence $\{h_n\}$ of self adjoint elements of \mathcal{A} such that for each $a, b \in \mathcal{A}$

- (i) $h_n(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))h_n$
- (ii) $\sigma(h_na) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$
- (iii) for each $a \in D$, $d(a) = \lim_{n \rightarrow \infty} d_{h_n}^\sigma(a)$, where $d_{h_n}^\sigma(a) = i[h_n, \sigma(a)]$.

Theorem 3.6 Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism and d be the generator of a σ -dynamics $\{\varphi_t\}_{t \in \mathbb{R}}$ satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$. If d is approximately inner and $(1 - d)(D(d))$ is dense in \mathcal{A} , then $\{\varphi_t\}_{t \in \mathbb{R}}$ is approximately inner.

Proof. Since d is an approximately inner σ - derivation, so there exists a sequence $\{h_n\}$ of self adjoint elements of \mathcal{A} such that for each $a \in \mathcal{A}$

- (i) $\sigma(h_na) = h_n\sigma(a)$ and $\sigma(ah_n) = \sigma(a)h_n$
- (ii) for each $a \in D(d)$, $d(a) = \lim_{n \rightarrow \infty} d_{h_n}^\sigma(a)$.

Also $d_{h_n}^\sigma$ induces the uniformly continuous one parameter group

$\varphi_n(t)(a) = e^{ih_nt}\sigma(a)e^{-ih_nt} - \sigma(a) + a$. By Remark 3.4 it is enough to show that $(1 - d)^{-1} = s - \lim_{n \rightarrow \infty} (1 - d_{h_n}^\sigma)^{-1}$. For this aim we have

$$\begin{aligned} & \| (1 - d_{h_n}^\sigma)^{-1}(1 - d)(a) - (1 - d)^{-1}(1 - d)(a) \| \\ &= \| (1 - d_{h_n}^\sigma)^{-1}(1 - d)(a) - (1 - d_{h_n}^\sigma)^{-1}(1 - d_{h_n}^\sigma)(a) \| \\ &\leq \| (1 - d_{h_n}^\sigma)^{-1} \| \| (1 - d)(a) - (1 - d_{h_n}^\sigma)(a) \| \\ &\leq (2\|\sigma\| + 1) \| (1 - d)(a) - (1 - d_{h_n}^\sigma)(a) \| \rightarrow 0. \end{aligned}$$

Since $\| (1 - d_{h_n}^\sigma)^{-1} \| \leq 2\|\sigma\| + 1$ (By Hille-Yosida Theorem [9]). Now the density of $(1 - d)(D(d))$ in \mathcal{A} implies that $(1 - d_{h_n}^\sigma)^{-1} \rightarrow (1 - d)^{-1}$ (strongly). \square

Before we state the next theorem, we need the following well-known definition:

Definition 3.7 A subset D_0 of domain D of a closed linear operator d on the Banach space A is called a *core* for d , if d is the closure of its restriction on D_0 . In the other words D_0 is a core for d if for each $a \in D$, there exists a sequence $\{a_n\}$ in D_0 such that $a_n \rightarrow a$ and $d(a_n) \rightarrow d(a)$.

Theorem 3.8 Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism and $\{\varphi_t\}_{t \in \mathbb{R}}$ be a σ -dynamics on \mathcal{A} satisfying $\|\varphi_t\| \leq 2\|\sigma\| + 1$. If there exists a core D_0 for the generator d of $\{\varphi_t\}_{t \in \mathbb{R}}$ such that d is approximately inner on D_0 , then $\{\varphi_t\}_{t \in \mathbb{R}}$ is an approximately inner σ -dynamics.

Proof. First note that since d is approximately inner on D_0 , so there exists a sequence $\{h_n\}$ of self adjoint elements of \mathcal{A} such that $d_{h_n}^\sigma$ induces the uniformly

continuous one parameter group $\varphi_n(t)(a) = e^{ih_n t} \sigma(a) e^{-ih_n t} - \sigma(a) + a$. Also by the extended Hille-Yosida Theorem $\lambda = 1 \in \rho(d) \cap \rho(d_{h_n}^\sigma)$ and the range $R(1-d)$ of $1-d$ is \mathcal{A} . Further $\| (1-d)^{-1} \| \leq 2 \| \sigma \| + 1$ and $\| (1-d_{h_n}^\sigma)^{-1} \| \leq 2 \| \sigma \| + 1$. Applying Remark 3.4 it suffices to show that

$$(1-d_{h_n}^\sigma)^{-1} \rightarrow (1-d)^{-1} \quad (\text{strongly on } \mathcal{A})$$

For this aim, let $\mathcal{B} = \{(1-d)(b) : b \in D_0\} = R(1-d|_{D_0})$. First we show that \mathcal{B} is dense in \mathcal{A} . Let $a \in \mathcal{A}$, since $R(1-d) = \mathcal{A}$, so there exists $c \in D(d)$ such that $a = c - d(c)$. But D_0 is a core for d . Thus there exists a sequence $\{b_n\}$ in D_0 such that $b_n \rightarrow c$ and $b_n - d(b_n) \rightarrow c - d(c) = a$. Hence \mathcal{B} is dense in \mathcal{A} . Now we show that $(1-d_{h_n}^\sigma)^{-1}$ converges strongly on \mathcal{B} to $(1-d)^{-1}$. For, let $b \in \mathcal{B}$. There exists $b_0 \in D_0$ such that $b = b_0 - d(b_0)$ and by assumption $d_{h_n}^\sigma(b_0) \rightarrow d(b_0)$. Therefore

$$\begin{aligned} \| (1-d_{h_n}^\sigma)^{-1}(b) - (1-d)^{-1}(b) \| &= \| (1-d_{h_n}^\sigma)^{-1}(d_{h_n}^\sigma - d)(1-d)^{-1}(b) \| \\ &\leq (2 \| \sigma \| + 1) \| (d_{h_n}^\sigma - d)(1-d)^{-1}(b) \| \\ &= (2 \| \sigma \| + 1) \| (d_{h_n}^\sigma - d)(b_0) \| \rightarrow 0 \end{aligned}$$

which implies that for each $b \in \mathcal{B}$

$$(1-d_{h_n}^\sigma)^{-1}(b) \rightarrow (1-d)^{-1}(b)$$

Finally given $a \in \mathcal{A}$ and $\epsilon > 0$. Since \mathcal{B} is dense in \mathcal{A} , so there exist $b \in \mathcal{B}$ and $N_\epsilon \in \mathcal{N}$ such that

$$\| b - a \| \leq \frac{\epsilon}{3(2 \| \sigma \| + 1)}$$

and for each $n \geq N_\epsilon$

$$\| (1-d_{h_n}^\sigma)^{-1}(b) - (1-d)^{-1}(b) \| \leq \frac{\epsilon}{3}.$$

Therefore

$$\begin{aligned} \| (1-d_{h_n}^\sigma)^{-1}(a) - (1-d)^{-1}(a) \| &\leq \| (1-d_{h_n}^\sigma)^{-1}(a) - (1-d_{h_n}^\sigma)^{-1}(b) \| \\ &+ \| (1-d_{h_n}^\sigma)^{-1}(b) - (1-d)^{-1}(b) \| \\ &+ \| (1-d)^{-1}(b) - (1-d)^{-1}(a) \| \\ &< 2(2 \| \sigma \| + 1) \| b - a \| + \frac{\epsilon}{3} < \epsilon. \square \end{aligned}$$

4. ACKNOWLEDGMENT

The authors would like to thank the referee for his/her useful comments and suggestions.

REFERENCES

- [1] F.F. Bonsall and J. Duncan, Complete normed algebras, Springer-Verlag, New York, 1973.
- [2] O. Bratteli and D.R. Robinson, Operator algebras and quantum statistical mechanics, Vol 1, Springer-Verlag, New York, 1987.
- [3] H.G. Dales, Banach algebras and automatic continuity, Oxford. Univ. Press, 2001.
- [4] M. Mirzavaziri and M.S. Moslehian, A Kadison-Sakai Type Theorem, arXiv: math.FA/0510267v2.
- [5] M. Mirzavaziri and M.S. Moslehian, Automatic continuity of σ -derivations in C^* -algebras, Proc. Amer. Math. Soc., Vol.11, number 5 (2006), 805-813.
- [6] M. Mirzavaziri and M.S. Moslehian, Innermess of ρ -derivations on Hyperfinite Von Neumann algebras, Elec. J. Theor. Phys., Vol 6, Number 2 (2009), 321-326.
- [7] M. Mirzavaziri and M.S. Moslehian, σ -derivations in Banach algebras, Bull. Iranian Math. Soc., Vol 32, Number 1 (2006), 65-78.
- [8] G.J. Murphy, C^* -algebras and operator theory, Academic Press, 1990.
- [9] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.
- [10] S. Sakai, Operator algebras in dynamical systems, Cambridge Univ. Press, 1991.