POSITIVE IMPLICATIVE HYPER MV-IDEALS OF TYPES 1,2,3, AND 4

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Abstract. In this paper first we define the notions of positive implicative hyper MV-ideals of types 1,2,3 and 4 in hyper MV-algebras and we investigate the relationship between of them . Then by some examples we show that these notions are not equivalent. Finally we give some relations between these notions and the notions of (weak) hyper MV-ideals and (weak) hyper MV-deductive systems of hyper MV-algebras.

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1. Introduction

The concept of an MV-algebra was introduced by C.C. Chang in 1958 [2] to prove the completeness theorem of infinite valued Łukasiewicz propositional calculus. The hyper structure theory was introduced by F. Marty at 8th congress of Scandinavian Mathematicians in 1934 [11]. Since then many researches have worked in these areas. Recently in [5], Sh. Ghorbani, A. Hasankhni and E. Eslami applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebras which are a generalization of MV-algebras and investigated some related results.

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Y.B. Jun et al. introduced (weak)hyper MV-deductive systems in hyper MV-algebras and gave several properties of them[9]. In paper [12] author investigated some results on hyper MV-algebras and defined (weak)hyper MV-ideals of hyper MV-algebras. Now in this paper we define 4 types of positive implicative hyper MV-ideals on hyper MV-algebras and investigate some results as mentioned in abstract.

In the next section some preliminary theorems and definitions are stated from [5,12]. In section 3, the notions of positive implicative hyper MV-ideals of types 1,2,3, and 4 are defined and some relations between of them are shown. Also by some examples we show that these notions are not equivalent. In section 4, we investigate the relation between (weak)hyper MV-ideals, (weak)hyper MV-deductive systems and positive implicative hyper MV-ideals of types 1,2,3 and 4 of hyper MV-algebras.

2. Preliminaries

Definition 2.1. [5] A hyper MV-algebra is a nonempty set M endowed with a hyper operation " $\oplus$ ", a unary operation "$*$" and a constant "0" satisfying the following axioms:

(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(hMV2) $x \oplus y = y \oplus x$,

(hMV3) $(x^*)^* = x$,

(hMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

(hMV5) $0^* \in x \oplus 0^*$,

(hMV6) $0^* \in x \oplus x^*$,

(hMV7) if $x \ll y$ and $y \ll x$, then $x = y$.

for all $x, y, z \in M$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$ and $A \oplus B = \bigcup\limits_{\substack{a\in A \\ b \in B}} a \oplus b$. Also, we define $0^* = 1$ and $A^* = \{a^* | a \in A\}$.

Proposition 2.2. [5] *Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra. Then for all $x, y, z \in M$ and for all nonempty subsets A, B and C of M the following statements hold:*

(a_1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,

(a_2) $0 \ll x$,

(a_3) *$x \ll x$,*

(a_4) *if $x \ll y$, then $y^* \ll x^*$ and $A \ll B$ implies $B^* \ll A^*$,*

(a_5) *$x \ll 1$,*

(a_6) *$A \ll A$,*

(a_7) *$A \subseteq B$ implies $A \ll B$,*

(a_8) *$x \ll x \oplus y$ and $A \ll A \oplus B$,*

(a_9) *$(A^*)^* = A$,*

(a_{10}) *$0 \oplus 0 = \{0\}$,*

(a_{11}) *$x \in x \oplus 0$,*

(a_{12}) *if $y \in x \oplus 0$, then $y \ll x$,*

(a_{13}) *if $y \oplus 0 = x \oplus 0$, then $x = y$.*

A hyper MV-algebra $(M, \oplus, ^*, 0)$ is called nontrivial if $M \neq \{0\}$. It is clear that a hyper MV-algebra is nontrivial if and only if $0 \neq 1$. In this paper, we consider nontrivial hyper MV-algebras.
On a hyper MV-algebra $(M, \oplus, ^*, 0)$ we define the hyper operation "$\otimes$" by: $x \otimes y = (x^* \oplus y^*)^*$.

Theorem 2.3. [12] *Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra and $x, y, z \in M$. Then the following hold:*

(a_{14}) *$x \otimes (y \otimes z) = (x \otimes y) \otimes z$,*

(a_{15}) *$x \otimes y = y \otimes x$,*

(a_{16}) *$0 \in x \otimes x^*$,*

(a_{17}) *$0 \in x \otimes 0$,*

(a_{18}) *$x \in x \otimes 1$,*

(a_{19}) *$x \otimes y \ll x, y$,*

(a_{20}) *if $x \in x \oplus x$, then $x \ll x \otimes x$,*

(a_{21}) *if $x = x^*$, then $x \in x \oplus x \Leftrightarrow x \in x \otimes x$,*

(a_{22}) *if $x \in x \otimes x$, then $x \ll x \oplus x$,*

(a_{23}) *$x \ll y$ implies $x \oplus z \ll y \oplus z$ and $x \otimes z \ll y \otimes z$,*

(a_{24}) *$z \otimes x \ll y \Leftrightarrow z \ll x^* \oplus y$,*

(a_{25}) *$(A^* \oplus y)^* \oplus y \subseteq (y^* \oplus A)^* \oplus A$, for all $A \subseteq M$.*

Definition 2.4. [12] A nonempty subset I of a hyper MV-algebra $(M, \oplus, ^*, 0)$ is called S-reflexive, if $(x \oplus y) \cap I \neq \emptyset$, then $x \oplus y \subseteq I$, for all $x, y \in M$.

In the rest of this paper, by M we denote a hyper MV-algebra.

Definition 2.5. [9, 12] Let I be a nonempty subset of a hyper MV-algebra M. Then I is called

(i) a hyper MV-ideal of M, if I satisfies the following conditions:

(I_0) if $x \in I$, $y \in M$ and $y \ll x$, then $y \in I$,

(I_1) $x \oplus y \subseteq I$, for all $x, y \in I$.

(ii) a weak hyper MV-ideal of M, if I satisfies (I_0) and

(I_2) $x \oplus y \ll I$, for all $x, y \in I$.

(iii) a weak hyper MV-deductive system of M, if $0 \in I$ and

(I_3) for all $x, y \in M$, $y \otimes x^* = (y^* \oplus x)^* \subseteq I$ and $x \in I$ imply that $y \in I$,

(iv) a hyper MV-deductive system of M, if $0 \in I$ and

(I_4) for all $x, y \in M$, $y \otimes x^* = (y^* \oplus x)^* \ll I$ and $x \in I$ imply that $y \in I$.

Theorem 2.6. [9, 12] *Let M a hyper MV-algebra M. Then the following statements hold:*

(i) *Every hyper MV-ideal of M is a weak hyper MV-ideal of M. The converse is not true in general,*

(ii) *Every hyper MV-deductive system of M is a weak hyper MV-deductive system of M. The converse is not true in general.*

Definition 2.7. [4] Let $(M_1, \oplus_1, ^{*_1}, 0_1)$ and $(M_2, \oplus_2, ^{*_2}, 0_2)$ be two hyper MV-algebras . A map $f : M_1 \to M_2$ is said to be a homomorphism, if for all $x, y \in M$:

(i) $f(0) = 0$,

(ii) $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$,

(iii) $f(x^{*_1}) = (f(x))^{*_2}$.

3. Some types of positive implicative hyper MV-ideals

Definition 3.1. Let I be a nonempty subset of M and $0 \in I$. Then I is called a positive implicative hyper MV-ideal of

(i) type 1, if for all $x, y, z \in M$,

$(x \otimes y^*) \otimes z^* \subseteq I$ and $y \otimes z^* \subseteq I$ imply that $x \otimes z^* \subseteq I$,

(ii) type 2, if for all $x, y, z \in M$,

$(x \otimes y^*) \otimes z^* \ll I$ and $y \otimes z^* \subseteq I$ imply that $x \otimes z^* \subseteq I$,

(iii) type 3, if for all $x, y, z \in M$,

$(x \otimes y^*) \otimes z^* \subseteq I$ and $y \otimes z^* \ll I$ imply that $x \otimes z^* \subseteq I$,

(iv) type 4, if for all $x, y, z \in M$,

$(x \otimes y^*) \otimes z^* \ll I$ and $y \otimes z^* \ll I$ imply that $x \otimes z^* \subseteq I$.

Theorem 3.2. *Let I be a nonempty subset of M. Then the following hold:*
(i) if I is a positive implicative hyper MV-ideal of type 4, then I is a positive implicative hyper MV-ideal of types 1, 2 and 3,
(ii) if I is a positive implicative hyper MV-ideal of type 2 or type 3, then I is a positive implicative hyper MV-ideal of type 1.

Proof. The proof is easy by (a_7). □

By the following we give some examples of positive implicative hyper MV-ideals of types 1,2,3 and 4 and show that the above positive implicative hyper MV-ideals are not equivalent in general.

Example 3.3. (i) The following tables show a hyper MV-algebra structure on $M = \{0, a, b, 1\}$(see [5]):

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{b\}$	$\{b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{b, 1\}$	$\{b, 1\}$
b	$\{b\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then

(a_1) $I = \{0, a\}$ is a positive implicative hyper MV-ideal of type 4.

(a_2) $I = \{0, a, b\}$ is a positive implicative hyper MV-ideal of type 1, while it is not a positive implicative hyper MV-ideal of type 2, because $(b \otimes a^*) \otimes 0^* = \{b, 1\} \ll I$ and $a \otimes 0^* = \{a\} \subseteq I$, while $b \otimes 0^* = \{b, 1\} \nsubseteq I$. Also I is not a positive implicative hyper MV-ideal of type 3, since $(1 \otimes 1^*) \otimes a^* = \{0, a\} \subseteq I$ and $1 \otimes a^* = \{b, 1\} \ll I$, while $1 \otimes a^* = \{b, 1\} \nsubseteq I$. By Theorem 3.2 we can obtain I is not a positive implicative hyper MV-ideal of type 4.

(a_3) $I = \{0\}$ is a positive implicative hyper MV-ideal of types 1,2 and 3, while it is not a positive implicative hyper MV-ideal of type 4.

(ii) Consider the following tables on $M = \{0, a, b, c, 1\}$.

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{0,a\}$	$\{0,b\}$	$\{0,c\}$	$\{0,a,b,c,1\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$
b	$\{0,b\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c\}$	$\{0,a,b,c,1\}$
c	$\{0,c\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$
1	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$

$*$	0	a	b	c	1
	1	b	a	c	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra [5]. Also $I = \{0, a, b\}$ is a positive implicative hyper MV-ideal of types $1, 2$ and 3 , while It is not a positive implicative hyper MV-ideal of type 4, since $(0 \otimes a^*) \otimes 0^* = \{0, a, b, c, 1\} \ll I$ and $a \otimes 0^* = \{a, 1\} \ll I$, while $0 \otimes 0^* = \{0, a, b, c, 1\} \nsubseteq I$.

(iii) The following tables show a hyper MV-algebra structure on $M = \{0, b, 1\}$ (see [4]):

$\oplus$	0	b	1
0	$\{0\}$	$\{b\}$	$\{b,1\}$
b	$\{b\}$	$\{b,1\}$	$\{b,1\}$
1	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$

$*$	0	b	1
	1	b	0

Then

(a_1) $I = \{0, b\}$ is not a positive implicative hyper MV-ideal of type 1.

(a_2) $I = \{0, 1\}$ is a positive implicative hyper MV-ideal of type 3, while it is not a positive implicative hyper MV-ideal of type 2, since $(0 \otimes 1^*) \otimes 0^* = \{0, b\} \ll I$ and $1 \otimes 0^* = \{1\} \subseteq I$, but $0 \otimes 0^* = \{0, b\} \nsubseteq I$.

Theorem 3.4. *Let $\{I_j\}_{j\in J}$ be a family of positive implicative hyper MV-ideals of type $1(2, 3, 4)$. Then $\bigcap\limits_{j\in J} I_j$ is a positive implicative hyper MV-ideal of type $1(2, 3, 4)$.*

Proof. Let $(x \otimes y^*) \otimes z^* \subseteq \bigcap\limits_{j\in J} I_j$ and $y \otimes z^* \subseteq \bigcap\limits_{j\in J} I_j$. Then $(x \otimes y^*) \otimes z^* \subseteq I_j$ and $y \otimes z^* \subseteq I_j$ for all $j \in J$. So by hypothesis we get that $x \otimes z^* \subseteq I_j$,for all $j \in J$.

Therefore $x \otimes z^* \subseteq \bigcap_{j\in J} I_j$, that is $\bigcap_{j\in J} I_j$ is a positive implicative hyper MV-ideal of type 1. Similarly we can prove theorem for the other types. □

Theorem 3.5. *Let I be a nonempty subset of M and $1 \in I$. Then*
(i) *I is a positive implicative hyper MV-ideal of type 2 iff for all $x, y, z \in M$, $y \otimes z^* \subseteq I$ implies that $x \otimes z^* \subseteq I$,*
(ii) *I is a positive implicative hyper MV-ideal of type 3 iff for all $x, y, z \in M$, $(x \otimes y^*) \otimes z^* \subseteq I$ implies that $x \otimes z^* \subseteq I$,*
(iii) *I is a positive implicative hyper MV-ideal of type 4 iff for all $x, z \in M$, $x \otimes z^* \subseteq I$.*

Proof. The proof follows from (a_5) and Definition 3.1. □

Theorem 3.6. *Let I be a nonempty subset of M and $1 \in I$. Then I is a positive implicative hyper MV-ideal of type 2 (4) if and only if I=M.*

Proof. Let I be a positive implicative hyper MV-ideal of type 2. Then by Theorem 3.5 part (i), $0^* \otimes 0^* = (0 \oplus 0)^* = \{1\} \subseteq I$ implies that $x \in x \otimes 0^* \subseteq I$, for all $x \in M$. Therefore I=M. The proof of the converse is trivial. □

By Example 3.3 part (iii), we can see that the above theorem may not be true for positive implicative hyper MV-ideals of types 3 and 1.

Theorem 3.7. *Let M be a hyper MV-algebra and $0 \in x \oplus y$, for all $x, y \in M$. If $0 \in I \subseteq M - \{1\}$, then I is a positive implicative hyper MV-ideal of types 1,2 and 3.*

Proof. Since $0 \in x \oplus y$, for all $x, y \in M$, then we get that $1 \in ((x \otimes y^*) \otimes z^*) \bigcap (y \otimes z^*)$, for all $x, y, z \in M$. So $(x \otimes y^*) \otimes z^* \nsubseteq I$ and $y \otimes z^* \nsubseteq I$,for all $x, y, z \in M$. Therefore by Definition 3.1, I is a positive implicative hyper MV-ideal of types 1,2 and 3. □

In Example 3.3 part (ii), we can see that $0 \in x \oplus y$, for all $x, y \in M$ and $I = \{0, a, b\}$ is not a positive implicative hyper MV-ideal of type 4, so the above theorem may not be true for positive implicative hyper MV-ideals of type 4.

Theorem 3.8. *Let $|x \oplus y| \geq 2$, for all $x, y \in M$ except x=y=0. Then $I = \{0\}$ is a positive implicative hyper MV-ideal of types 1,2 and 3.*

Proof. Since $|x \oplus y| \geq 2$, for all $x, y \in M$, then it is easy to check that for all $x, y, z \in M$, $x \otimes y^* \otimes z^* \not\subseteq I$ and $y \otimes z^* \not\subseteq I$. Therefore by Definition 3.1, I is a positive implicative hyper MV-ideal of types 1 ,2 and 3. □

Consider hyper MV-algebra in Example 3.3 part (ii). We can see that $|x \oplus y| \geq 2$ for all $x, y \in M$, except x=y=0, and also I=$\{0\}$ is not a positive implicative hyper MV-ideal of type 4, since $0 \otimes 0^* \otimes 1^* = \{0, a, b, c, 1\} \ll I$ and $0 \otimes 1^* = \{0, a, b, c, 1\} \ll I$, while $0 \otimes 1^* = \{0, a, b, c, 1\} \not\subseteq I$.

Theorem 3.9. *Let M be a hyper MV-algebra, $0 \in 1 \oplus x$, for some $x \in M$ and $1 \oplus 1 = \{1\}$. If $0 \in I \subseteq M - \{1\}$, then I is not a positive implicative hyper MV-ideal of type* 2.

Proof. By hypothesis we have $\exists x \in M$ such that $0 \in 1 \oplus x$. So $x^* \otimes 0^* \otimes 1^* \ll I$ and $0 \otimes 1^* = \{0\} \subseteq I$, while $x^* \otimes 1^* = (x \oplus 1)^* \not\subseteq I$. Therefore I is not a positive implicative hyper MV-ideal of type 2. □

Theorem 3.10. *Let M be a hyper MV-algebra and for some $x \in M$, $x^* = x$ and $x \oplus 0 = \{x\}$. If $\{0, x\} \subseteq I \subseteq M - \{1\}$, then I is not a positive implicative hyper MV-ideal of type* 1 *(*2,3,4*).*

Proof. By hypothesis we can obtain that $1 \otimes x^* \otimes 0^* = (0 \oplus x \oplus 0)^* = \{x\} \subseteq I$ and $x \otimes 0^* = (x \oplus 0)^* = \{x\} \subseteq I$, while $1 \otimes 0^* = (0 \oplus 0)^* = \{1\} \not\subseteq I$. Therefore I is not a positive implicative hyper MV-ideal of type 1. □

4. The relationship between (weak)hyper MV-ideals, (weak)hyper MV-deductive systems and positive implicative hyper MV-ideals of types 1,2,3 and 4

Theorem 4.1. *Let I be a nonempty subset of a hyper MV-algebra M. Then I is a hyper MV-deductive system of M if and only if I is a hyper MV-ideal of M.*

Proof. Let I be a hyper MV-deductive system of M, $x \ll y$ and $y \in I$. Then $0 \in x \otimes y^*$ and so $x \otimes y^* \ll I$. Thus by hypothesis we get that $x \in I$. Now let $x, y \in I$ and $t \in x \oplus y$. Since

$$0 \in t \otimes t^* \subseteq t \otimes (x \oplus y)^* = t \otimes (x^* \otimes y^*) = (t \otimes x^*) \otimes y^*,$$

then $(t \otimes x^*) \otimes y^* \ll I$. Thus $y \in I$ implies that $t \otimes x^* \ll I$, so by $x \in I$ we get that $t \in I$. Therefore $x \oplus y \subseteq I$.

Conversely, let I be a hyper MV-ideal of M. It is clear that $0 \in I$. Assume that $y \otimes x^* \ll I$ and $x \in I$. Then there are $a \in y \otimes x^*$ and $i \in I$ such that $a \ll i$ and so $a \in I$. Hence by (a_1) we have

$$1 \in a \oplus a^* \subseteq a \oplus (y \otimes x^*)^* = a \oplus (x \oplus y^*) = (a \oplus x) \oplus y^*.$$

Then $y \ll a \oplus x \subseteq I$ and so by hypothesis we get that $y \in I$. Therefore I is a hyper MV-deductive system. □

Consider hyper MV-algebra in Example 3.3 part (i). We can check that $I = \{0, a, b\}$ is a weak hyper MV-ideal of M, while it is not a weak hyper MV-deductive system of M, because $1 \otimes b^* = \{a\} \subseteq I$ and $b \in I$, while $1 \notin I$.

Example 4.2. Consider the following tables on $M = \{0, b, 1\}$:

$\oplus$	0	b	1
0	$\{0\}$	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{1\}$	$\{0, b, 1\}$
1	$\{0, b, 1\}$	$\{0, b, 1\}$	$\{0, b, 1\}$

$*$	0	b	1
	1	b	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra [12]. Also $I = \{0, b\}$ is a weak hyper MV-deductive system of M, while it is not a weak hyper MV-ideal of M, because $b \oplus b \not\ll I$.

Theorem 4.3. *Let I be a positive implicative hyper MV-ideal of type 4. Then I is a hyper MV-deductive system of M.*

Proof. Let $y \otimes x^* \ll I$ and $x \in I$. Then $y \otimes x^* \subseteq (y \otimes x^*) \otimes 1$ implies that $(y \otimes x^*) \otimes 0^* \ll I$. Also by $x \in I$ we can get that $x \otimes 0^* \ll I$. Thus by hypothesis we have $y \otimes 0^* \subseteq I$. Since $y \in y \otimes 0^*$, then we get that $y \in I$. Therefore I is a hyper MV-deductive system. □

The converse of the above theorem is not true in general. Consider the hyper MV-algebra in Example 3.3 part (ii). We can see that $I = \{0\}$ is a hyper MV-deductive system of M while it is not a positive implicative hyper MV-ideal of type 4.

Corollary 4.4. *Every positive implicative hyper MV-ideal of type 4 is a (weak)hyper MV-ideal and a weak hyper MV-deductive system of M.*

Example 4.5. Consider the following tables on $M = \{0, a, b, 1\}$:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a,b\}$	$\{0,b\}$	$\{0,a,b,1\}$
a	$\{0,a,b\}$	$\{0,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
b	$\{0,b\}$	$\{0,a,b,1\}$	$\{b\}$	$\{0,a,b,1\}$
1	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$

$*$	0	a	b	1
	1	b	a	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra [12]. It is easy to see that $I = \{0, a, 1\}$ is a positive implicative hyper MV-ideal of types 3 and 1 while I is not a weak hyper MV-deductive system , since $b \otimes a^* = \{0, 1\} \subseteq I$ and $a \in I$ but $b \notin I$. Also since $1 \in I$, then I is not a weak hyper MV-ideal.

Theorem 4.6. *Let I be a positive implicative hyper MV-ideal of type 1(2,3) and I^* be S-reflexive. Then I is a hyper MV-deductive system of M.*

Proof. Let I be a positive implicative hyper MV-ideal of type 1. We show that I is a weak hyper MV-deductive system. Let $x \otimes y^* \subseteq I$ and $y \in I$. Then for all $h \in x \otimes y^*$, $(h \otimes 0^*) \cap I \neq \emptyset$, by (a_{18}), and so $(h^* \oplus 0) \cap I^* \neq \emptyset$. Thus by hypothesis we get that $h^* \oplus 0 \subseteq I^*$, for all $h \in x \otimes y^*$. Hence $h \otimes 0^* \subseteq I$, and so $x \otimes y^* \otimes 0^* \subseteq I$. Also $y \in I$ implies that $(y \otimes 0^*) \cap I \neq \emptyset$. So similar to the above argument and hypothesis we get that $y \otimes 0^* \subseteq I$. Thus $x \in x \otimes 0^* \subseteq I$, because I is a positive implicative hyper MV-ideal of type 1. Therefore I is a weak hyper MV-deductive system. Since I^* is S-reflexive and I is a weak hyper MV-deductive system , we show that I is a hyper MV-deductive system . Assume that $y \otimes x^* \ll I$ and $x \in I$, then there are $t \in y \otimes x^*$ and $i \in I$ such that $t \ll i$. So $0 \in (t \otimes i^*) \cap I$, and hence $(t^* \oplus i) \cap I^* \neq \emptyset$. By S-reflexivity I^* we get that $t^* \oplus i \subseteq I^*$ and so $(t \otimes i^*) \subseteq I$. Thus by $i \in I$ and hypothesis we obtain that $t \in I$. Hence $(y \otimes x^*) \cap I \neq \emptyset$, by hypothesis we get that $(y \otimes x^*) \subseteq I$ and $x \in I$, and so $y \in I$. Therefore I is a hyper MV-deductive system. □

Theorem 4.7. *Let I be a nonempty subset of M and I^* be S-reflexive. If I is a positive implicative hyper MV-ideal of type 2(resp. type 1), then it is a positive implicative hyper MV-ideal of type 4(resp. type 3).*

Proof. Let $x \otimes y^* \otimes z^* \ll I$ and $y \otimes z^* \ll I$. Then there are $i \in I$ and $s \in y \otimes z^*$ such that $s \ll i$. By Theorem 4.3 we get that $s \in I$ and hence $(y \otimes z^*) \cap I \neq \emptyset$.

Thus $(y \otimes z^*)^* \cap I^* \neq \emptyset$ and so $(y \otimes z^*)^* \subseteq I^*$. Hence $y \otimes z^* \subseteq I$. Therefore by hypothesis we get that $x \otimes z^* \subseteq I$ and so I is a positive implicative hyper MV-ideal of type 4. $\square$

Theorem 4.8. *Let M, N be hyper MV-algebras and $f : M \to N$ be a homomorphism. Then the following assertions hold:*
(i) If I is a positive implicative hyper MV-ideal of type 2(resp. 1,3,4) of N, then $f^{-1}(I)$ is a positive implicative hyper MV-ideal of type 2(resp. 1,3,4) of M,
(ii) Let f be onto and $Ker(f) \subseteq I$. Then

(a) If I is a positive implicative hyper MV-ideal of type 1(resp. 2,3) of M and I is a hyper MV-deductive system of M, then $f(I)$ is a positive implicative hyper MV-ideal of type 1(resp. 2,3) of N,

(b) If I is a positive implicative hyper MV-ideal of type 4 of M, then $f(I)$ is a positive implicative hyper MV-ideal of type 4 of N.

Proof. (i) Let I be a positive implicative hyper MV-ideal of type 2, $(x \otimes y^*) \otimes z^* \ll f^{-1}(I)$ and $y \otimes z^* \subseteq f^{-1}(I)$. Then there are $t \in (x \otimes y^*) \otimes z^*$ and $h \in f^{-1}(I)$ such that $t \ll h$ and $f(y \otimes z^*) \subseteq I$. So we get that $f(t) \ll f(h)$, since $f(h) \in I$, then $f(t) \ll I$. Thus $(f(x) \otimes f(y)^*) \otimes f(z)^* \ll I$ and $f(y) \otimes f(z)^*) \subseteq I$ and hence by hypothesis we get that $f(x \otimes z^*) = f(x) \otimes f(z)^* \subseteq I$. Therefore $x \otimes z^* \subseteq f^{-1}(I)$, that is $f^{-1}(I)$ is a positive implicative hyper MV-ideal of type 2.
$(ii)(a)$ Let $x \otimes y^* \otimes z^* \subseteq f(I)$ and $y \otimes z^* \subseteq f(I)$. Since f is onto, then there are $x_1, y_1, z_1 \in M$ such that $f(x_1) = x$, $f(y_1) = y$ and $f(z_1) = z$. Thus

$$f(x_1 \otimes y_1^* \otimes z_1^*) = f(x_1) \otimes (f(y_1))^* \otimes (f(z_1))^* = x \otimes y^* \otimes z^* \subseteq f(I).$$

Let $a \in x_1 \otimes y_1^* \otimes z_1^*$. Then $f(a) \in f(I)$. Hence there is $i \in I$ such that $f(i) = f(a)$. So $a \otimes i^* \ll Ker(f) \subseteq I$. Since $i \in I$ and I is a hyper MV-deductive system, then $a \in I$. Therefore $x_1 \otimes y_1^* \otimes z_1^* \subseteq I$. Similarly, we can get that $y_1 \otimes z_1^* \subseteq I$. Since I is a positive implicative hyper MV-ideal of type 1, then $x_1 \otimes z_1^* \subseteq I$. So $x \otimes z^* = f(x_1) \otimes (f(z_1))^* = f(x_1 \otimes z_1^*) \subseteq f(I)$. Therefore $f(I)$ is a positive implicative hyper MV-ideal of type 1.
The proof of $(ii)(b)$ is similar to the proof of $(ii)(a)$. $\square$

Theorem 4.9. *Let I be a nonempty subset of M. Then*
(i) I is a positive implicative hyper MV-ideal of type 1 iff for all $z \in M$, $I_z = \{x \in$

$M|\ \ x \otimes z^* \subseteq I\}$ is a weak hyper MV-deductive system of M,
(ii) If I is a positive implicative hyper MV-ideal of type $2(4)$, then for all $z \in M$, I_z is a hyper MV-deductive system of M.
(iii) If I is a positive implicative hyper MV-ideal of type 3, then for all $z \in M$, I_z is a weak hyper MV-deductive system of M.

Proof. (i) Let $x, y, z \in M$, $x \otimes y^* \subseteq I_z$ and $y \in I_z$. Then $x \otimes y^* \otimes z^* \subseteq I$ and $y \otimes z^* \subseteq I$. Since I is a positive implicative hyper MV-ideal of type 1, then $x \otimes z^* \subseteq I$ and so $x \in I_z$. Therefor I_z is a weak hyper MV-deductive system of M .
Conversely, let $x \otimes y^* \otimes z^* \subseteq I$ and $y \otimes z^* \subseteq I$. Then $x \otimes y^* \subseteq I_z$ and $y \in I_z$, and so $x \in I_z$. Thus $x \otimes z^* \subseteq I$. Therefore I is a positive implicative hyper MV-ideal of type 1.
(ii) Let $x, y, z \in M$, $x \otimes y^* \ll I_z$ and $y \in I_z$. Then there are $t \in x \otimes y^*$ and $h \in I_z$, such that $0 \in t \otimes h^*$. Since $h \in I_z$, then $h \otimes z^* \subseteq I$. So $0 \in 0 \otimes z^* \subseteq t \otimes h^* \otimes z^*$, hence $t \otimes h^* \otimes z^* \ll I$. Since $h \otimes z^* \subseteq I$ and I is a positive implicative hyper MV-ideal of type 2, then $t \otimes z^* \subseteq I$. Therefore $x \otimes y^* \otimes z^* \ll I$ and since $y \otimes z^* \subseteq I$,we get that $x \otimes z^* \subseteq I$, and so $x \in I_z$. Thus I_z is a hyper MV-deductive system of M.
The proof of (iii) follows from (i) and Theorem 3.2 part (ii). □

Theorem 4.10. *Let I be a positive implicative hyper MV-ideal of type 4. Then for all $z \in M$, $I_z^{\ll} = \{x \in M|\ \ x \otimes z^* \ll I\}$ is a hyper MV-deductive system of M.*

Proof. Let $x \otimes y^* \ll I_z^{\ll}$ and $y \in I_z^{\ll}$. Then there is $h \in x \otimes y^*$ and $k \in I_z^{\ll}$ such that $h \ll k$, and so $0 \in h \otimes k^*$ and $k \otimes z^* \ll I$. Thus $0 \in (0 \otimes z^*) \subseteq h \otimes k^* \otimes z^*$ implies that $h \otimes k^* \otimes z^* \ll I$. Hence by $k \otimes z^* \ll I$ and I is a positive implicative hyper MV-ideal of type 4, we get that $h \otimes z^* \subseteq I$. So $x \otimes y^* \otimes z^* \ll I$ and $y \otimes z^* \ll I$ implies that $x \otimes z^* \subseteq I$. Therefore $x \otimes z^* \ll I$, i.e. $x \in I_z^{\ll}$. □

Let $a \in M$. We define the subset $< a >$ of M as follows:

$$< a > = \{x \in M|\ \ x \ll a\}.$$

It is clear that $\{0, a\} \subseteq < a >$.

Theorem 4.11. *$\{0\}$ is a positive implicative hyper MV-ideal of type 4 if and only if $< a >$ is a hyper MV-deductive system of M, for all $a \in M$.*

Proof. Let $\{0\}$ be a positive implicative hyper MV-ideal of type 4. Then by Theorem 4.10, $\{0\}_a^{\ll}$ is a hyper MV-deductive system of M. But

$$\{0\}_a^{\ll} = \{x \in M| \;\; x \otimes a^* \ll \{0\}\} = \{x \in M| \;\; x \ll a\}$$

therefore $< a >$ is a hyper MV-deductive system, for all $a \in M$. Conversely, let $x \otimes y^* \otimes z^* \ll \{0\}$ and $y \otimes z^* \ll \{0\}$. Then $x \otimes y^* \ll z$ and $y \ll z$, i.e. $x \otimes y^* \ll < z >$ and $y \in < z >$. So by hypothesis we get that $x \in < z >$, i.e. $x \ll z$. Therefore $x \otimes z^* \ll \{0\}$ and hence $\{0\}$ is a positive implicative hyper MV-ideal of type 4. $\square$

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