

# SOME CHARACTERIZATIONS OF HYPER $MV$ -ALGEBRAS

L. TORKZADEH AND SH. GHORBANI

DEPARTMENT OF MATHEMATICS, KERMAN BRANCH, ISLAMIC AZAD UNIVERSITY,  
KERMAN, IRAN

DEPARTMENT OF MATHEMATICS, BAM HIGHER EDUCATION COMPLEXES,  
KERMAN, IRAN

E-MAILS: LTORKZADEH@YAHOO.COM, SH.GHORBANI@MAIL.UK.AC.IR

(Received: 23 April 2012, Accepted: 1 October 2012)

ABSTRACT. In this paper we characterize hyper  $MV$ -algebras in which 0 or 1 are scalar elements. We prove that any finite hyper  $MV$ -algebra that 0 is a scalar element in it, is an  $MV$ -algebra. Finally we characterize hyper  $MV$ -algebras of order 2 and order 3.

**AMS Classification:** 06D35, 06F35, 03G25.

**Keywords:** Hyper  $MV$ -algebra,  $MV$ -algebra.

## 1. Introduction

The concept of an  $MV$ -algebra was introduced by C.C. Chang in 1958 [2] to prove the completeness theorem of infinite valued Łukasiewicz propositional calculus. The hyper structure theory was introduced by F. Marty at 8th congress of Scandinavian Mathematicians in 1934 [4]. Since then many researchers have worked in these areas[1,6]. Recently in [3], Sh. Ghorbani and et.al applied the hyper structure to  $MV$ -algebras and introduced the concept of a hyper  $MV$ -algebra which is a generalization of an  $MV$ -algebra and investigated some related results. In paper [5] authors defined (weak)hyper  $MV$ -ideals in hyper  $MV$ -algebras. In this paper we want to find conditions that a hyper  $MV$ -algebra becomes an  $MV$ -algebra. We

---

*JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER*

*VOL. 1, NUMBER 2 (2012) 147-161.*

©MAHANI MATHEMATICAL RESEARCH CENTER

prove that a hyper  $MV$ -algebra that 1 is a scalar element on it is an  $MV$ -algebra. Also we characterize hyper  $MV$ -algebras in which 0 is a scalar element.

In the next section, some preliminary theorems and definitions are stated from [3] and [5]. In section 3, we characterize hyper  $MV$ -algebras in which 0 or 1 are scalar. Also we obtain conditions under which a hyper  $MV$ -algebra is an  $MV$ -algebra. In section 4, we characterize hyper  $MV$ -algebras of order 2 and order 3 and we obtain three non-isomorphic hyper  $MV$ -algebras of order 2 and thirty-six non-isomorphic hyper  $MV$ -algebras of order 3.

## 2. Preliminaries

An  $MV$ -algebra is an algebra  $(A, \oplus, *, 0)$  of type  $(2, 1, 0)$  satisfying the following equations:

$$(MV1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

$$(MV2) \quad x \oplus y = y \oplus x,$$

$$(MV3) \quad x \oplus 0 = x,$$

$$(MV4) \quad x^{**} = x,$$

$$(MV5) \quad x \oplus 0^* = 0^*,$$

$$(MV6) \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x,$$

for all  $x, y, z \in A$ .

Consider the real unit interval  $[0, 1]$  and for all  $x, y \in [0, 1]$ , define  $x \oplus y = \min\{1, x + y\}$  and  $x^* = 1 - x$ . Then  $([0, 1], \oplus, *, 0)$  is an  $MV$ -algebra. The rational numbers in  $[0, 1]$  and the  $n$ -element set  $L_n = \{0, 1/(n-1), \dots, (n-2)/(n-1), 1\}$ , for each integer  $n \geq 2$ , yield examples of subalgebras of  $[0, 1]$ .

For any two elements  $x$  and  $y$  of an  $MV$ -algebra  $(A, \oplus, *, 0)$ , define  $x \leq y$  if and only if  $x^* \oplus y = 1$ , where  $1 := 0^*$ . Then  $\leq$  is a partial order, called the natural order of  $A$ .

**Definition 2.1.** [3] A hyper  $MV$ -algebra is a nonempty set  $M$  endowed with a hyper operation " $\oplus$ ", a unary operation " $*$ " and a constant " $0$ " satisfying the following axioms:

$$(hMV1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

$$(hMV2) \quad x \oplus y = y \oplus x,$$

$$(hMV3) \quad (x^*)^* = x,$$

- (hMV4)  $0^* \in x \oplus x^*$ ,
  - (hMV5)  $0^* \in x \oplus 0^*$ ,
  - (hMV6)  $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ ,
  - (hMV7) if  $0^* \in x^* \oplus y$  and  $0^* \in y^* \oplus x$ , then  $x = y$ .
- for all  $x, y, z \in M$ .

Let  $M$  be a hyper  $MV$ -algebra,  $x \ll y$  is defined by  $0^* \in x^* \oplus y$ , for all  $x, y \in M$  and for every  $A, B \subseteq M$ , we define  $A \ll B$  if and only if there exist  $a \in A$  and  $b \in B$  such that  $a \ll b$ . Also, we define  $1 := 0^*$ ,  $A^* := \{a^* | a \in A\}$  and  $A \oplus B := \bigcup_{\substack{a \in A \\ b \in B}} a \oplus b$ ,

for every  $A, B \subseteq M$ .

Let  $[0, 1]$  be real unit interval. We can see that  $([0, 1], \oplus, *, 0)$  is a hyper  $MV$ -algebra, where  $x \oplus y = [0, \min\{1, x + y\}]$  and  $x^* = 1 - x$ , for all  $x, y \in [0, 1]$ .

**Proposition 2.2.** [3] *Let  $(M, \oplus, *, 0)$  be a hyper  $MV$ -algebra. Then for all  $x, y, z \in M$  and for all nonempty subsets  $A, B$  and  $C$  of  $M$  the following statements hold:*

- (a<sub>1</sub>)  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ ,
- (a<sub>2</sub>)  $0 \ll x$ ,
- (a<sub>3</sub>)  $x \ll x$ ,
- (a<sub>4</sub>) if  $x \ll y$ , then  $y^* \ll x^*$  and  $A \ll B$  implies  $B^* \ll A^*$ ,
- (a<sub>5</sub>)  $x \ll 1$ ,
- (a<sub>6</sub>)  $A \ll A$ ,
- (a<sub>7</sub>)  $A \subseteq B$  implies  $A \ll B$ ,
- (a<sub>8</sub>)  $x \ll x \oplus y$  and  $A \ll A \oplus B$ ,
- (a<sub>9</sub>)  $(A^*)^* = A$ ,
- (a<sub>10</sub>)  $0 \oplus 0 = \{0\}$ ,
- (a<sub>11</sub>)  $x \in x \oplus 0$ .

A hyper  $MV$ -algebra  $(M, \oplus, *, 0)$  is called nontrivial if  $M \neq \{0\}$ . It is clear that a hyper  $MV$ -algebra is nontrivial if and only if  $0 \neq 1$ . In this paper, we consider nontrivial hyper  $MV$ -algebras. An element  $a \in M$  is called a scalar element, if  $|a \oplus x| = 1$ , for all  $x \in M$ .

Let  $(M_1, \oplus_1, *^1, 0_1)$  and  $(M_2, \oplus_2, *^2, 0_2)$  be two hyper  $MV$ -algebras. A bijection  $f : M_1 \rightarrow M_2$  is said to be an isomorphism, if for all  $x, y \in M$ : (i)  $f(0) = 0$ , (ii)  $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ , (iii)  $f(x^{*1}) = (f(x))^{*2}$ .

### 3. Hyper $MV$ -algebras in which 0 or 1 are scalar elements

In the rest of this paper, we denote a hyper  $MV$ -algebra  $(M, \oplus, *, 0)$  by  $M$ .

**Theorem 3.1.** *If  $x \ll y$  and  $y \in 0 \oplus a$ , for  $a, x, y \in M$ , then  $x \ll a$ .*

*Proof.* By  $x \ll y$  we have:

$$1 \in x^* \oplus y \subseteq x^* \oplus (0 \oplus a) = 0 \oplus (x^* \oplus a).$$

Thus there exists  $t \in x^* \oplus a$  such that  $1 \in 0 \oplus t = 1^* \oplus t$ , so we get that  $1 \ll t$ . Since  $t \ll 1$ , then  $t = 1$ . Thus  $1 \in x^* \oplus a$ , that is  $x \ll a$ .  $\square$

By the above theorem and  $y \ll y$  we have:

**Corollary 3.2.** *If  $y \in 0 \oplus a$ , for  $a, y \in M$ , then  $y \ll a$ .*

**Theorem 3.3.** *If  $x \in 0 \oplus a$  and  $x^* \ll x$ , for  $x, a \in M$ , then  $a^* \ll a$ .*

*Proof.* By hypothesis we have:

$$1 \in x \oplus x \subseteq x \oplus (0 \oplus a) = 0 \oplus (a \oplus x).$$

Then there is  $s \in a \oplus x$  such that  $1 \in 0 \oplus s = 1^* \oplus s$  and we have  $1 \ll s$ , by  $s \ll 1$ , we get that  $s = 1$ . Thus

$$1 = s \in x \oplus a \subseteq (0 \oplus a) \oplus a = 0 \oplus (a \oplus a).$$

Hence there exists  $l \in a \oplus a$  such that  $1 \in 0 \oplus l = 1^* \oplus l$ , that is  $l = 1$ . Therefore  $1 \in a \oplus a = (a^*)^* \oplus a$ , i.e.  $a^* \ll a$ .  $\square$

**Example 3.4.** Let  $(M, \oplus, *, 0)$  be an  $MV$ -algebra. Define a hyper operation  $\oplus'$  on  $M$  by  $x \oplus' y = \{x \oplus y\}$ , for all  $x, y \in M$ . Then  $(M, \oplus', *, 0)$  is a hyper  $MV$ -algebra.

The condition (hMV7) is necessary in any hyper  $MV$ -algebra. Since hyper  $MV$ -algebras are a generalization of  $MV$ -algebras, then hyper  $MV$ -algebras whose all elements are scalar should be  $MV$ -algebras. By the following example, we show that if the condition (hMV7) is deleted from the definition of a hyper  $MV$ -algebra, we obtain a hyper  $MV$ -algebra whose all elements are scalar, while it is not an  $MV$ -algebra.

**Example 3.5.** Consider the following tables on  $M = \{0, a, 1\}$ :

$\oplus$	0	$a$	1
0	{1}	{1}	{1}
$a$	{1}	{1}	{1}
1	{1}	{1}	{1}

$*$	0	$a$	1
	1	$a$	0

Then  $(M, \oplus, *)$  satisfies in the axioms (hMV1)-(hMV6) while it does not satisfy in the axiom (hMV7), since  $1 \in 0^* \oplus 1$  and  $1 \in 1^* \oplus 0$ , but  $0 \neq 1$ . Also  $M$  is not an MV-algebra, since  $a \oplus 0 = 1 \neq a$ .

Note that in a hyper MV-algebra  $M$ , if  $|x \oplus y| = 1$ , for  $x, y \in M$ , i.e.  $x \oplus y = \{a\}$ , for  $a \in M$ , we denote by  $x \oplus y = a$ .

**Lemma 3.6.** Let  $(M, \oplus, *)$  be a hyper MV-algebra . If  $|x \oplus y| = 1$ , for all  $x, y \in M$ , then  $(M, \oplus, *)$  is an MV-algebra.

*Proof.* The proof is easy. □

**Theorem 3.7.** Let 1 be a scalar element in a hyper MV-algebra  $M$ . Then  $|x \oplus y| = 1$ , for all  $x, y \in M$ . Hence  $M$  is an MV-algebra.

*Proof.* By (hMV6) and Proposition 2.2 for  $a \in M$  we have:

$$1 \in a^* \oplus a \subseteq (0 \oplus a)^* \oplus a = (1^* \oplus a)^* \oplus a = (a^* \oplus 1)^* \oplus 1 = 1^* \oplus 1 = 1.$$

Thus  $a^* \oplus a = 1$ , for all  $a \in M$ . Let  $x, y \in M$  and  $e, f \in x \oplus y$ . Then by (hMV1) and (hMV6) we have:

$$\begin{aligned} (x \oplus y)^* \oplus (x \oplus y) &= ((x \oplus y)^* \oplus x) \oplus y \\ &= ((x^* \oplus y^*)^* \oplus y^*) \oplus y \\ &= ((x^* \oplus y^*)^* \oplus (y^* \oplus y)) \\ &= (x^* \oplus y^*)^* \oplus 1 = 1. \end{aligned}$$

Since  $e^* \oplus f, f^* \oplus e \subseteq (x \oplus y)^* \oplus (x \oplus y) = 1$ , we obtain  $e^* \oplus f = f^* \oplus e = 1$ , that is  $e \ll f$  and  $f \ll e$ . Hence  $e = f$ . Thus  $|x \oplus y| = 1$ , for all  $x, y \in M$  and so by Lemma 3.6,  $M$  is an MV-algebra. □

**Theorem 3.8.** Let 0 be a scalar element in a hyper MV-algebra  $M$ . Then  $0, x \notin 1 \oplus x$ , for all  $x \in M - \{1\}$ .

*Proof.* Let  $x \in M - \{1\}$ . If  $x = 0$ , then  $0 \notin \{1\} = 1 \oplus 0$ . Now let  $x \neq 0$ . By (hMV6) we have:

$$(1 \oplus x)^* \oplus x = (0^* \oplus x)^* \oplus x = (x^* \oplus 0)^* \oplus 0 = x \oplus 0 = x. \quad (I)$$

If  $0 \in 1 \oplus x$ , then  $0 \in 1 \oplus x = 0^* \oplus x \subseteq (1 \oplus x)^* \oplus x = x$  and so  $x = 0$ , which is a contradiction. If  $x \in 1 \oplus x$ , then

$$1 \in x^* \oplus x \subseteq (1 \oplus x)^* \oplus x = x.$$

Hence  $x = 1$ , which is a contradiction. Therefore  $0, x \notin 1 \oplus x$ , for all  $x \in M - \{1\}$ .  $\square$

By putting  $x = 1$  in (I) of the proof of Theorem 3.8, we can obtain if 0 is a scalar, then  $0 \notin 1 \oplus 1$ .

**Theorem 3.9.** *Let 0 be a scalar element in a hyper MV-algebra  $M$  and  $x, y \in M$ .*

*If  $y \in 1 \oplus x$ , then:*

- (i)  $x \oplus y^* = x$ ,
- (ii)  $(1 \oplus y^*)^* \oplus x = x$ ,
- (iii)  $x \oplus y = x \oplus 1$ ,
- (v)  $x \neq 1$  implies  $x \notin 1 \oplus y$ ,
- (iv)  $x^* \ll y$  and  $y^* \ll x$ .

*Proof.* (i) Since 0 is a scalar element, by the proof of Theorem 3.8 we have  $(1 \oplus x)^* \oplus x = x$ . Thus  $y \in 1 \oplus x$  implies  $y^* \oplus x = x$ .

(ii) By (i) and (hMV6) we have:

$$x = y^* \oplus x = (y \oplus 0)^* \oplus (0 \oplus x) = (0^* \oplus y^*)^* \oplus y^* \oplus x = (1 \oplus y^*)^* \oplus x.$$

(iii) By hypothesis we get :

$$\begin{aligned} x \oplus y &= (y^* \oplus x) \oplus y \\ &= (y^* \oplus y) \oplus x \\ &= ((y \oplus 0)^* \oplus y) \oplus x \\ &= ((y^* \oplus 1)^* \oplus 1) \oplus x \\ &= ((y^* \oplus 1)^* \oplus x) \oplus 1 \\ &= x \oplus 1. \end{aligned}$$

(iv) Let  $x \neq 1$  and  $x \in 1 \oplus y$ . Then by (iii),  $x \oplus y = y \oplus 1$  and so  $x \oplus 1 = y \oplus 1$ . Hence  $x \in 1 \oplus y$ , implies that  $x \in 1 \oplus x$ , which is a contradiction by Theorem 3.8.

(v) By (iii) we obtain  $1 \in x \oplus y = (x^*)^* \oplus y = (y^*)^* \oplus x$ , that is  $x^* \ll y$  and  $y^* \ll x$ .  $\square$

**Theorem 3.10.** *Let  $0$  be a scalar element in a hyper MV-algebra  $M$ . Then  $1 \oplus 1 = 1$ .*

*Proof.* We have  $1 \in 1 \oplus 1$ . If  $1 \neq b \in 1 \oplus 1$ , then by Theorem 3.9 part (iii),  $b \oplus 1 = 1 \oplus 1$  and so  $b \in b \oplus 1$ , which is not true by Theorem 3.8. Therefore  $1 \oplus 1 = 1$ .  $\square$

**Theorem 3.11.** *Let  $0$  be a scalar element in a hyper MV-algebra  $M$  and  $x^* = x$ , for all  $x \in M$ . Then  $|x \oplus y| = 1$ , for all  $x, y \in M$ . Moreover  $M = L_3 = \{0, 1/2, 1\}$ .*

*Proof.* We show that  $1$  is a scalar element in  $M$  and so by Theorem 3.7  $|x \oplus y| = 1$ , for all  $x, y \in M$ . By Theorem 3.10 we have  $1 \oplus 1 = 1$ . Now let  $1 \neq y \in 1 \oplus x$  for  $x \in M - \{1\}$ . Then by Theorem 3.9 part (iv) we obtain  $x = x^* \ll y$  and  $y = y^* \ll x$  and so  $x = y$ . Hence  $x \in 1 \oplus x$  for  $x \neq 1$ , it is not true by Theorem 3.8. Therefore  $1 \oplus x = 1$ , that is  $1$  is a scalar element.  $\square$

**Theorem 3.12.** *Let  $M$  be a hyper MV-algebra and  $x \oplus x^* = 1$ , for all  $x \in M$ . Then  $M$  is an MV-algebra.*

*Proof.* First we show that  $0$  is a scalar element in  $M$ . Let  $a \in M$ . We have  $a \in 0 \oplus a$ . If  $x \in 0 \oplus a$ , then

$$x \oplus a^* \subseteq (0 \oplus a) \oplus a^* = 0 \oplus (a \oplus a^*) = 0 \oplus 1 = 0 \oplus 0^* = 1.$$

So  $x \oplus a^* = 1$ , that is  $a \ll x$ . Since  $x \in 0 \oplus a$ , by Corollary 3.2 we have  $x \ll a$  and so  $x = a$ . Therefore  $0 \oplus a = a$ , i.e.  $0$  is a scalar element, so by Theorem 3.10,  $1 \oplus 1 = 1$ . Now we show that  $1$  is a scalar element in  $M$ . Let  $x \in M - \{1\}$ . Then  $1 \in 1 \oplus x^*$  implies that  $1 \oplus x \subseteq (1 \oplus x^*) \oplus x = 1 \oplus (x^* \oplus x) = 1 \oplus 1 = 1$ . Hence  $1 \oplus x = 1$ , for  $x \neq 1$ , that is  $1$  is a scalar element. Therefore by Theorem 3.7,  $M$  is an MV-algebra.  $\square$

**Theorem 3.13.** *Let  $0$  be a scalar element in a hyper MV-algebra  $M$ . Then  $|1 \oplus x| = 1$  or  $x \oplus 1$  is an infinite subset of  $M$ , for all  $x \in M$ .*

*Proof.* Let  $x \in M$ . If  $x = 1$ , then  $1 \oplus 1 = 1$  and so  $|1 \oplus x| = 1$ . Now let  $x \neq 1$ . Since  $1 \in 1 \oplus x$ , then  $|1 \oplus x| \geq 1$ . Let  $|1 \oplus x| \neq 1$ . We show that  $|1 \oplus x| \neq n$ , for all  $n \in \mathbb{N}$ .

By induction, for  $n = 1$  we have  $|1 \oplus x| \neq 1$ . If  $|1 \oplus x| = 2$ , then  $1 \oplus x = \{1, y\}$ , for  $y \in M - \{1\}$ . So by (hMV1) we have

$$\{1, y\} = 1 \oplus x = (1 \oplus 1) \oplus x = 1 \oplus (1 \oplus x) = (1 \oplus 1) \cup (1 \oplus y) = \{1\} \cup (1 \oplus y).$$

Hence  $y \in 1 \oplus y$  which is a contradiction by Theorem 3.8. Now let induction is true for  $n$ . If  $|1 \oplus x| = n + 1$ , then  $1 \oplus x = \{1, a_1, a_2, \dots, a_n\}$ . So

$$\{1, a_1, a_2, \dots, a_n\} = 1 \oplus x = 1 \oplus (1 \oplus x) = (1 \oplus 1) \cup (1 \oplus a_1) \cup \dots \cup (1 \oplus a_n).$$

Thus there is  $a_k$ ,  $1 \leq k \leq n$  such that  $|1 \oplus a_k| \neq 1$ . Also if  $|1 \oplus a_j| = 1$ , for all  $1 \leq j \leq n$ ,  $j \neq k$ , then  $1 \oplus a_k = 1 \oplus x$  and so  $a_k \in 1 \oplus a_k$ , that is not true. Hence there exist  $a_k, a_t \in 1 \oplus x$ ,  $t \neq k$ , such that  $|1 \oplus a_k| \neq 1$  and  $|1 \oplus a_t| \neq 1$ . Since  $a_k \notin 1 \oplus a_k$  and  $a_t \notin 1 \oplus a_t$ . Then  $|1 \oplus a_k| \leq n$ , which is a contradiction by hypothesis of induction. Therefore  $|1 \oplus x| \neq n + 1$ .  $\square$

**Corollary 3.14.** *Let 0 be a scalar element in a finite hyper MV-algebra M. Then 1 is a scalar element.*

By the above corollary and Theorem 3.7 we have:

**Corollary 3.15.** *Let 0 be a scalar element in a finite hyper MV-algebra M. Then M is an MV-algebra.*

**Open problem.** If 0 is a scalar element in an infinite hyper MV-algebra. Is M an MV-algebra?

#### 4. Hyper MV-algebras of order 2 and order 3

**Theorem 4.1.** *There are three non-isomorphic hyper MV-algebras of order 2.*

*Proof.* Consider  $M = \{0, 1\}$ . Since M must be a hyper MV-algebra, we have  $0 \oplus 0 = 0$ . Also  $1 \in 0 \oplus 1$ , hence  $0 \oplus 1 = 1$  or  $0 \oplus 1 = \{0, 1\}$ . If  $0 \oplus 1 = 1$ , then 0 is a scalar and so  $1 \oplus 1 = 1$ , by Corollary 3.14. If  $0 \oplus 1 = \{0, 1\}$ , by  $1 \in 1 \oplus 1$ , we have  $1 \oplus 1 = 1$  or  $1 \oplus 1 = \{0, 1\}$ . By considering each case for  $1 \oplus 1$ , we can obtain a hyper MV-algebra. Therefore we have the following hyper MV-algebras and we can check that they are non-isomorphic.

$\oplus$	0	1	$\oplus$	0	1	$\oplus$	0	1
0	{0}	{1}	0	{0}	{0, 1}	0	{0}	{0, 1}
1	{1}	{1}	1	{0, 1}	{1}	1	{0, 1}	{0, 1}



$$\begin{array}{c|cc}
 * & 0 & 1 \\
 \hline
 & 1 & 0
 \end{array}$$

□

**Theorem 4.2.** *There are thirty-six non-isomorphic hyper MV-algebras of order 3.*

*Proof.* Let  $M = \{0, b, 1\}$ . Then the following tables show a probable hyper MV-algebra structure on  $M$ :

$$\begin{array}{c|ccc}
 \oplus & 0 & b & 1 \\
 \hline
 0 & 0 & a_{12} & a_{13} \\
 b & a_{21} & a_{22} & a_{23} \\
 1 & a_{31} & a_{32} & a_{33}
 \end{array}
 \qquad
 \begin{array}{c|ccc}
 * & 0 & b & 1 \\
 \hline
 & 1 & b & 0
 \end{array}$$

Since  $1 \not\leq b$ , then  $1 \notin 1^* \oplus b = 0 \oplus b = a_{12}$  and so  $1 \notin a_{12}$ . By (hMV5) and (hMV4) we have  $1 \in a_{13} \cap a_{23} \cap a_{33} \cap a_{22}$  and by (hMV2)  $a_{13} = a_{31}, a_{21} = a_{12}$  and  $a_{23} = a_{32}$ . Also by Proposition 2.2 we have  $b \in b \oplus 0$ , thus we consider two cases:

case 1:  $0 \oplus b = b$ , case 2:  $0 \oplus b = \{0, b\}$ .

In case 1, if  $|0 \oplus 1| = 1$ , then 0 is a scalar and so by Corollary 3.14,  $a_{22} = a_{23} = a_{31} = a_{32} = a_{33} = 1$ .

If  $|0 \oplus 1| > 1$ . Then by (hMV6) we have :

$$0 \oplus 1 \subseteq (b \oplus 1)^* \oplus 1 = (0 \oplus b)^* \oplus b = b \oplus b, \quad (1)$$

$$(1 \oplus b)^* \oplus b = (0^* \oplus b)^* \oplus b = (b \oplus 0)^* \oplus 0 = b \oplus 0 = b. \quad (2)$$

By (1) and (2) we get that  $0, b \notin 1 \oplus b$  and thus  $a_{32} = 1$ . Hence

$$(b \oplus b) \oplus 1 = b \oplus (b \oplus 1) = b \oplus 1 = 1, \quad (3)$$

it implies that  $0 \notin b \oplus b$ , so by (1),  $0 \notin 0 \oplus 1$ . Thus  $a_{13} = \{1, b\}$  and so by (1),  $a_{22} = \{1, b\}$ , also by (3), we get that  $a_{33} = 1$ . Hence by hypothesis we have :

$$(0 \oplus 1) \oplus 1 = \{1, b\} \oplus 1 = 1 \text{ and } 0 \oplus (1 \oplus 1) = 0 \oplus 1 = \{1, b\}$$

that is  $(0 \oplus 1) \oplus 1 \neq 0 \oplus (1 \oplus 1)$ , i.e. (hMV1) does not hold. Thus in case 1, we have only one the following hyper MV-algebra.

$$\begin{array}{c|ccc}
 \oplus & 0 & b & 1 \\
 \hline
 0 & 0 & b & 1 \\
 b & b & 1 & 1 \\
 1 & 1 & 1 & 1
 \end{array}$$

In case 2, we consider the following subcases :

(a)  $b \oplus b = 1$ , (b)  $b \oplus b = \{0, 1\}$ , (c)  $b \oplus b = \{1, b\}$ , (d)  $b \oplus b = \{0, b, 1\}$ .

(a) By (hMV1) and (hMV6) we have:

$$\begin{aligned} 1 \oplus 0 &= (b \oplus b) \oplus 0 = b \oplus (b \oplus 0) = b \oplus \{0, b\} = \{0, b, 1\}, \\ 1 \oplus b &= (b \oplus b) \cup (1 \oplus b) = (0 \oplus b)^* \oplus b = (b \oplus 1)^* \oplus 1 = \{0, b, 1\}, \\ 1 \oplus 1 &= (b \oplus b) \oplus 1 = b \oplus (b \oplus 1) = b \oplus \{0, b, 1\} = \{0, b, 1\}. \end{aligned}$$

Therefore in this case we have the following hyper  $MV$ -algebra

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1\}$	$\{0, b, 1\}$
1	$\{0, b, 1\}$	$\{0, b, 1\}$	$\{0, b, 1\}$

(b) By (hMV1) and (hMV6) we obtain:

$$\{0\} \cup (0 \oplus 1) = (b \oplus b) \oplus 0 = b \oplus (b \oplus 0) = \{0, b, 1\} \Rightarrow \{1, b\} \subseteq 0 \oplus 1, \quad (4)$$

$$b \in 0 \oplus 1 \subseteq (b \oplus 1)^* \oplus 1 = (0 \oplus b)^* \oplus b = \{1, b\} \oplus b = (1 \oplus b) \cup \{0, 1\}, \quad (5)$$

$$\{0, 1, b\} = (1 \oplus 0) \oplus b = 1 \oplus (0 \oplus b) = 1 \oplus \{0, b\} = (1 \oplus 0) \cup (1 \oplus b) \quad (6)$$

$$\{0, b\} = b \oplus 0 \subseteq (0 \oplus 1) \oplus 0 = (0 \oplus 0) \oplus 1 = 0 \oplus 1 \quad (7)$$

By (4) and (7) we get that  $0 \oplus 1 = \{0, b, 1\}$  and also by (5) we have  $\{1, b\} \subseteq 1 \oplus b$ . If  $1 \oplus b = \{1, b\}$ , since  $(1 \oplus 1) \oplus b = 1 \oplus (1 \oplus b) = 1 \oplus \{1, b\} = (1 \oplus 1) \cup \{1, b\}$ , we get that  $1 \oplus 1 \neq \{1, b\}$ . While we can check that  $1 \oplus 1 = \{1\}, \{0, 1\}$  or  $\{0, b, 1\}$ . Also for  $1 \oplus b = \{0, b, 1\}$ , we can obtain 4 cases for  $1 \oplus 1$ , that is  $1 \oplus 1 = \{1\}, 1 \oplus 1 = \{0, 1\}, 1 \oplus 1 = \{1, b\}$  or  $1 \oplus 1 = \{0, b, 1\}$ .

Therefore in this case we have 7 non-isomorphic hyper  $MV$ -algebras:

$\oplus$	0	$b$	1	$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$	0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$	$b$	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{1\}$	1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, 1\}$
$\oplus$	0	$b$	1	$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$	0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$	$b$	$\{0, b\}$	$\{0, 1\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, b, 1\}$	1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{0, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{b, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{0, b, 1\}$

(c) If  $0 \oplus 1 = \{1\}$ , then  $(1 \oplus 1)^* \oplus 1 = (0 \oplus 0)^* \oplus 0 = 1 \oplus 0 = \{1\}$ , so  $0 \notin 1 \oplus 1$ . By (hMV6) we have

$$\{1, b\} \cup (1 \oplus b) = (1^* \oplus b)^* \oplus b = (b \oplus 1)^* \oplus 1. \quad (8)$$

If  $b \in 1 \oplus 1$ , then  $b \oplus 1 = 1$ . So  $b \in 0 \oplus 1$ , which is not true. Thus  $b \notin 1 \oplus 1$  and so  $1 \oplus 1 = \{1\}$ . By (8) we can obtain  $b \oplus 1 \neq 1$  and  $b \oplus 1 \neq \{0, 1\}$ . Also  $1 \oplus b = (1 \oplus 0) \oplus b = (1 \oplus b) \oplus 0$  implies that if  $b \in 1 \oplus b$ , then  $1 \oplus b = \{0, b, 1\}$ . By considering  $1 \oplus b = \{0, b, 1\}$  and some manipulations we can obtain the following hyper MV-algebra:

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1, b\}$
1	$\{1\}$	$\{0, 1, b\}$	$\{1\}$

If  $0 \oplus 1 = \{0, 1\}$ , then by (8),  $0 \oplus 1 \subseteq (1 \oplus b) \cup \{1, b\}$  and so  $0 \in 1 \oplus b$ . Also  $\{0, b, 1\} = (1 \oplus 0) \oplus b = (1 \oplus b) \oplus 0$  implies that  $1 \oplus b = \{0, b, 1\}$ . Since  $(1 \oplus 1)^* \oplus 1 = 1 \oplus 0 = \{0, 1\}$ , so  $b \notin 1 \oplus 1$ . We obtain  $1 \oplus 1 = \{1\}$  or  $1 \oplus 1 = \{0, 1\}$ , in each case we can check that  $M$  is a hyper MV-algebra. So we have 2 the following non-isomorphic hyper MV-algebras.

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, b, 1\}$
1	$\{0, 1\}$	$\{0, b, 1\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, b, 1\}$
1	$\{0, 1\}$	$\{0, b, 1\}$	$\{0, 1\}$

If  $b \in 0 \oplus 1$ , then by  $(0 \oplus 0) \oplus 1 = (0 \oplus 1) \oplus 0$  we get that  $0 \oplus 1 = \{0, b, 1\}$ . By

$$(1 \oplus b)^* \oplus b = (b \oplus 0)^* \oplus 0 = \{0, b, 1\}, \quad (9)$$

we get that  $1 \oplus b \neq 1$ . Also by (8) we can obtain  $0 \in 1 \oplus b$  and so  $\{0, 1\} \subseteq 1 \oplus b$ . If  $1 \oplus b = \{0, 1\}$ , then by  $(1 \oplus 1) \oplus b = (1 \oplus b) \oplus 1 = \{0, b, 1\}$ , we have  $1 \oplus 1 \neq 1$ . Thus  $1 \oplus 1 = \{0, 1\}, \{1, b\}$  or  $\{0, b, 1\}$ . So by some manipulations we can obtain 3 hyper  $MV$ -algebras. If  $1 \oplus b = \{0, b, 1\}$ , there are four cases for  $1 \oplus 1$ . So in this case we have 7 non-isomorphic hyper  $MV$ -algebras.

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{0, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, b\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{b, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{0, 1, b\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{0, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, b, 1\}$
1	$\{0, b, 1\}$	$\{0, b, 1\}$	$\{1, b\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{0, b, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{1, b\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{1\}$

(d) by (9) we have  $1 \oplus b \neq 1$ . Also

$$\{0, b, 1\} = (1^* \oplus b)^* \oplus b = (1 \oplus b)^* \oplus 1. \quad (10)$$

If  $1 \oplus b = \{0, 1\}$ , then  $(1 \oplus 1) \cup (0 \oplus 1) = \{0, b, 1\}$ . If  $0 \oplus 1 = 1$ , then  $1 \oplus 1 = \{0, b, 1\}$ , it is not true, since the equality  $(1 \oplus 1)^* \oplus 1 = (0 \oplus 0)^* \oplus 0$  does not hold. If  $0 \oplus 1 = \{0, 1\}$ , then  $\{0, b, 1\} = (0 \oplus 1) \oplus b = (0 \oplus b) \oplus 1 = \{0, 1\}$ , it is not true. Similar to the above argument for (c), if  $b \in 0 \oplus 1$ , then  $0 \oplus 1 = \{0, b, 1\}$ . We can consider 4 cases for  $1 \oplus 1$ , i.e.  $1 \oplus 1 = 1, \{0, 1\}, \{1, b\}$  or  $\{0, b, 1\}$ . Therefore we have

4 the following non-isomorphic hyper *MV*-algebras.

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{0, 1\}$

  

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{b, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1\}$
1	$\{0, b, 1\}$	$\{0, 1\}$	$\{0, 1, b\}$

If  $1 \oplus b = \{1, b\}$ , then by (10),  $0 \in 0 \oplus 1$ . If  $0 \oplus 1 = \{0, 1\}$ , similar to the above argument  $b \notin 1 \oplus 1$ , so  $1 \oplus 1 = 1$  or  $\{0, 1\}$ . If  $0 \oplus 1 = \{0, 1, b\}$ , we can consider 4 cases for  $1 \oplus 1$  and so we have 6 the following non-isomorphic hyper *MV*-algebras.

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, 1\}$

  

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, 1\}$

  

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{1, b\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, 1, b\}$

If  $1 \oplus b = \{0, 1, b\}$ , we have the above cases for  $0 \oplus 1$  and additional case for  $0 \oplus 1$ , i.e.  $0 \oplus 1 = 1$ , in this case we have  $1 \oplus 1 = 1$ . So we obtain 7 the following

non-isomorphic hyper  $MV$ -algebras.

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$
1	$\{1\}$	$\{0, 1, b\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$
1	$\{0, 1\}$	$\{0, 1, b\}$	$\{1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1, b\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$
1	$\{0, 1, b\}$	$\{0, 1, b\}$	$\{0, 1\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1, b\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$
1	$\{0, 1, b\}$	$\{0, 1, b\}$	$\{1, b\}$

$\oplus$	0	$b$	1
0	0	$\{0, b\}$	$\{0, 1, b\}$
$b$	$\{0, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$
1	$\{0, 1, b\}$	$\{0, 1, b\}$	$\{0, 1, b\}$

By defining of isomorphism, it is easy to see that the above hyper  $MV$ -algebras are not isomorphic.  $\square$

## 5. Conclusion and Future Research

In this paper, we find some conditions which a hyper  $MV$ -algebra is an  $MV$ -algebra. For this purpose we consider hyper  $MV$ -algebras which 0 or 1 is scalar on them. Then hyper  $MV$ -algebras of order 2 and order 3 are determined. Application of this paper is to find more hyper  $MV$ -algebras which are not  $MV$ -algebras.

Important issues for future work are

- (i) Characterization of hyper  $MV$ -algebra with at least one scalar which this scalar is not 0 or 1.
- (ii) Characterization of hyper  $MV$ -algebras of order more than 3.
- (iii) Find an algorithm for computing hyper  $MV$ -algebra.

## REFERENCES

- [1] R.A. Boorzooei, A. Hasankhani, M.M. Zahedi and Y.B. Jun, *On hyper  $K$ -algebra*, Math. Japon, Vol. 1 (2000) 113-121.
- [2] C.C. Chang, *Algebraic analysis of many valued logic*, Transactions American Mathematical Society, Vol. 88 (1958) 476-490.
- [3] Sh. Ghorbani, A. Hasankhani and E. Eslami, *Hyper  $MV$ -algebras*, Set-Valued Mathematics and Applications, Vol. 1 (2008) 205-222.
- [4] F. Marty, *Sur une generalization de la notion de groups*, 8th congress Math. Scandinaves, Stockholm (1934) 45-49.
- [5] L. Torkzadeh and A. Ahadpanah, *Hyper  $MV$ -ideals in hyper  $MV$ -algebras*, Math. Logic Quart., Vol. 56 (2010) 51-62.
- [6] L. Torkzadeh, M.M. Zahedi and M. Abbasi, *(Weak) Dual hyper  $K$ -ideals*, Soft Comput, Vol. 11 (2007) 985-990.