# ANALYSIS OF COVARIANCE BASED ON FUZZY TEST STATISTIC

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(Received: 11 April 2013, Accepted: 1 August 2013)

Abstract. One-way analysis of covariance is a popular and common statistical method, wherein the equality of the means of several random variables which have a linear relationship with a random mathematical variable, is tested. In this study, a method is presented to improve the one-way analysis of covariance when there is an uncertainty in accepting the statistical hypotheses. The method deals with a fuzzy test statistic which is produced by a set of confidence intervals. Finally an example is provided for illustration.

**AMS Classification:62A86**.

**Keywords:Analysis of covariance; Confidence interval; Fuzzy test statistic.**

## 1. Introduction and background

Analysis of variance is a common and popular method used in the analysis of experimental designs. It has many applications in agricultural sciences and industrial engineering. Many authors have studied this topic from various aspects for fuzzy environments. For instance, in [1] one-way and two-way analysis of variance

*JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER*
*VOL. 2 , NUMBER (1) 29-41.*

using a set of confidence intervals for the variance parameter has been carried out. In [10] analysis of variance for fuzzy data is discussed by considering the $\alpha$-cuts of fuzzy data via introducing the pessimistic and optimistic degrees and solving an optimization problem. One-way analysis of variance is presented in [6] to a case where observed data are fuzzy observations.

One-way analysis of covariance is an extension of analysis of variance. In this analysis, the equality of the means of several random variables are investigated which have a linear relationship with a random mathematical variable. Sometimes, in one-way analysis of covariance, the observed value of test statistic is close to the quantiles of statistical distributions and there is uncertainty with regard to accepting the null hypothesis $H_0$. In this paper an approach is presented to deal with this problem.

Buckley [2] introduced a method for estimating the parameters in statistical models. His method produces a fuzzy estimator using a set of confidence intervals for the required parameter. Using this estimator, a fuzzy test statistic and, subsequently, fuzzy critical values are produced. This fuzzy test statistic is used to perform the statistical hypotheses test. This issue has been studied by several other authors in different ways. In [3] an explicit and unique membership function has been derived for fuzzy estimators. In [9] Buckley's method is extended to the case where the statistical hypotheses are fuzzy. In [4] it is shown that when the crisp test statistic distribution is not symmetric, Buckley's method results in producing a fuzzy estimation where the membership degree for the point estimation of the required parameter never equals one. A solution to overcome this weakness is provided in [4] and another solution is presented in [1]. In this article we use the solution presented in [4]. It has been shown that this solution reduces to the Buckley's method when crisp test statistic distribution is symmetric.

The rest of this paper is organized as follows. In section 2, the necessary concepts of fuzzy sets theory are discussed. In section 3, a brief review of one-way analysis of covariance is presented. In section 4, fuzzy test statistics and fuzzy critical values are produced and based on them decision rules are presented. In section 5, an example is provided to illustrate the method. Finally, a conclusion is provided in section 6.

## 2. Preliminaries

In this section we review some concepts of fuzzy sets theory. Let $X$ be a universal set and $F(X) = \{\tilde{A} | \tilde{A} : X \to [0,1]\}$. Any $\tilde{A} \in F(X)$ is called a fuzzy set on $X$. The $\alpha$-cut of $\tilde{A}$ is the crisp set $\tilde{A}[\alpha] = \{x \in X | \tilde{A}(x) \geq \alpha\}$, for $\alpha \in (0,1]$. Moreover, $\tilde{A}[0]$ is separately defined [2] as the closure of the union of all the $\tilde{A}[\alpha]$ for $0 < \alpha \leq 1$. $\tilde{A} \in F(\mathbb{R})$ is a fuzzy number if:

*(i)* there is a unique $x_0 \in \mathbb{R}$ with $\tilde{A}(x_0) = 1$,
*(ii)* the $\alpha$-cuts of $\tilde{A}$ are closed and bounded intervals on $\mathbb{R}$ for any $\alpha \in (0,1]$,

where $\mathbb{R}$ is the set of all real numbers. In other words for every fuzzy number $\tilde{A}$, we have $\tilde{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$ for all $\alpha$ which describes the closed, bounded, intervals as function of $\alpha$.

Buckley's method results in producing a fuzzy number to estimate the required parameter from a statistical distribution whose $\alpha$-cuts are $(1-\alpha)100\%$ confidence intervals, $\alpha \in [0.01, 1]$. The following definition is given, to clarify the discussion that is presented in this paper.

**Definition 2.1.** A fuzzy number $\tilde{\theta}$ is an unbiased fuzzy estimator for parameter $\theta$ from a statistical distribution if:

*(i)* the $\alpha$-cuts of $\tilde{\theta}$ are $(1-\alpha)100\%$ confidence intervals for $\theta$, with $\alpha \in [0.01, 1]$ and $\tilde{\theta}[\alpha] = \tilde{\theta}[0.01]$ for $\alpha \in [0, 0.01)$,
*(ii)* if $\hat{\theta}$ is an unbiased point estimator for $\theta$ then $\tilde{\theta}(\hat{\theta}) = 1$.

Similar to conventional statistics, a fuzzy estimator is a rule for calculating a fuzzy estimation of an unknown parameter based on observed data: thus the rule and its result (the fuzzy estimation) are distinguished. For a fuzzy estimation an explicit and unique membership function is given by the following theorem [3].

**Theorem 2.1.** Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size $n$ from a distribution with unknown parameter $\theta$. If, based on observation $x_1, x_2, \ldots, x_n$, we consider $[\theta_1(\alpha), \theta_2(\alpha)]$, as a $(1-\alpha)100\%$ confidence interval for $\theta$, then the fuzzy estimation of $\theta$ is a fuzzy set with the following unique membership function

$$\tilde{\theta}(u) = min\left\{\theta_1^{-1}(u), [-\theta_2]^{-1}(-u), 1\right\}.$$

## 3. One-way analysis of covariance

In this section one-way analysis of covariance is briefly reviewed, for more details see [5, 8]. For the linear model $y_{ij} = \mu_i + \beta(x_{ij} - \bar{x}_{..}) + \epsilon_{ij}$, where $\epsilon_{ij}$'s have a normal distribution with an unknown variance $\sigma^2$ and mean 0, $x_{ij}$'s are random variables which have a linear relationship with $y_{ij}$'s, $\bar{x}_{..} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} x_{ij} / \sum_{i=1}^{a} n_i$, $\beta$ and $\mu_i$'s are unknown parameters, for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, n_i$.

Taking into account the above linear model we are interested to test the following statistical hypotheses:

$$\begin{cases} H_0 : \beta = 0 \\ H_1 : \beta \neq 0 \end{cases} \tag{1}$$

and

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \ldots = \mu_a \\ H_1 : \text{not all } \mu_i\text{'s are equal.} \end{cases} \tag{2}$$

To simplify the discussion we use the following notations.

$$S_{yy} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2, \qquad S_{xx} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{..})^2,$$

$$E_{yy} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2, \qquad E_{xx} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i.})^2,$$

$$S_{xy} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..}), \qquad SSE = E_{yy} - (E_{xy}^2 / E_{xx}),$$

$$E_{xy} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})(x_{ij} - \bar{x}_{i.}), \qquad SSE' = S_{yy} - (S_{xy}^2 / S_{xx})$$

and

$$N = \sum_{i=1}^{a} n_i,$$

where " **.** " and " $-$ " represent the mean on one or two subscripts, for instance $\bar{x}_{i.} = \sum_{j=1}^{n_i} x_{ij} / n_i$.

Now the critical region based on generalized likelihood ratio (GLR) method [8] for testing hypotheses in (1) is $F1 \geq k$, where $k$ is a real number and

$$F1 = \frac{E_{xy}^2 / E_{xx}}{SSE/(N - a - 1)}$$

The pivotal quantity $SSE/\sigma^2$ has the distribution $\chi^2$ with $N - a - 1$ degree of freedom and $E_{xy}^2/(\sigma^2 E_{xx})$, under the null hypothesis $H_0$ in (1), has the distribution $\chi^2$ with 1 degree of freedom. So both of these pivotal quantities can be used to produce confidence intervals for $\sigma^2$. It can be shown that, under the null hypothesis $H_0$ in (1), $F1$ has the distribution $F$ with 1 and $N-a-1$ degrees of freedom. The null hypothesis $H_0 : \beta = 0$ is rejected if the observed value of $F1$ statistic is equal or greater than $F_{1-\gamma,1,N-a-1}$, where $F_{1-\gamma,1,N-a-1}$ is $(1-\gamma)$'th quantile of the distribution $F$ with with 1 and $N-a-1$ degrees of freedom and $\gamma \in (0,1)$ is the significance level of testing.

Also, the critical region based on GLR method for testing hypotheses in (2) is $F2 \geq k$, where $k$ is a real number and

$$F2 = \frac{(SSE' - SSE)/(a-1)}{SSE/(N-a-1)}$$

The mathematical term $(SSE' - SSE)/\sigma^2$, under the null hypothesis $H_0$ in (2), has the distribution $\chi^2$ with $a-1$ degree of freedom; and this pivotal quantity can be used to produce the confidence intervals for the parameter $\sigma^2$. It can be shown that, under the hypothesis $H_0$ in (2), $F2$ has the distribution $F$ with $a-1$ and $N-a-1$ degrees of freedom and the null hypothesis $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ is rejected if the observed value of $F2$ statistic is equal or greater than $F_{1-\gamma,a-1,N-a-1}$, $(1-\gamma)$'th quantile of the distribution $F$ with $a-1$ and $N-a-1$ degrees of freedom.

**Remark 3.1.** Note that $E^2_{xy}/E_{xx}$, under the hypothesis $H_0$ in (1), $(SSE' - SSE)/(a-1)$, under the hypothesis $H_0$ in (2) and $SSE/(N-a-1)$ are unbiased point estimators of the parameter $\sigma^2$.

## 4. One-way analysis of covariance based on fuzzy test statistic

In this section we first consider the issue of testing the statistical hypotheses in (1). Generally the symbols $\chi^2_{\alpha,\nu}$ and $F_{\alpha,\nu_1,\nu_2}$ will be used to represent the $\alpha$'the quantile of the distribution $\chi^2$ with $\nu$ degree of freedom and the distribution $F$ with $\nu_1$ and $\nu_2$ degrees of freedom, respectively.

**Theorem 4.1.** In one-way analysis of covariance model, if we consider $SSE/(N-a-1)$ as an unbiased point estimator for parameter $\sigma^2$, then an unbiased fuzzy estimator for $\sigma^2$ is $\widetilde{\sigma^2}$ with $\alpha$-cuts $\widetilde{\sigma^2}[\alpha]$, where

$$\widetilde{\sigma^2}[\alpha] = \begin{cases} \left[SSE/\chi^2_{1-\alpha+\alpha p',N-a-1}, SSE/\chi^2_{\alpha p',N-a-1}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2}[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

and $p'$ is obtained from the relation $\chi^2_{p',N-a-1} = N-a-1$.

*Proof.* Based on the pivotal quantity $SSE/\sigma^2$, a $(1-\alpha)100\%$ confidence interval for $\sigma^2$ is $\left[SSE/\chi^2_{1-\alpha+\alpha p,N-a-1}, SSE/\chi^2_{\alpha p,N-a-1}\right]$ for any $\alpha \in (0,1)$ and $p \in (0,1)$. When $\alpha = 1$ and $p = p'$, satisfying $\chi^2_{p',N-a-1} = N-a-1$, this interval becomes the point $SSE/(N-a-1)$ the unbiased point estimator for $\sigma^2$. Now fixing $p = p'$ and varying $\alpha$ from 0.01 to 1 we obtain nested intervals which are the $\alpha$-cuts of a fuzzy number, say $\widetilde{\sigma^2}$. Finally, $\widetilde{\sigma^2}[\alpha] = \widetilde{\sigma^2}[0.01]$ for $\alpha \in [0, 0.01)$, we have the unbiased fuzzy estimator $\widetilde{\sigma^2}$ for $\sigma^2$. $\square$

**Lemma 4.1.** The membership function of fuzzy estimator $\widetilde{\sigma^2}$ in Theorem 4.1 is as follows:

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(SSE/x)}{1-p'} & \frac{SSE}{\chi^2_{0.99+0.01p',N-a-1}} \leq x \leq \frac{SSE}{N-a-1} \\ \\ \frac{G(SSE/x)}{p'} & \frac{SSE}{N-a-1} \leq x \leq \frac{SSE}{\chi^2_{0.01p',N-a-1}} \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $G$ is the cumulative distribution function of a $\chi^2$ variable with $N-a-1$ degree of freedom.

*Proof.* By Theorem 4.1, we have $\theta_1(\alpha) = SSE/\chi^2_{1-\alpha+\alpha p',N-a-1}$ for $\alpha \in [0.01, 1]$. Hence, $\theta_1^{-1}(x) = [1-G(\frac{SSE}{x})]/(1-p')$. Also $\theta_2(\alpha) = SSE/\chi^2_{\alpha p',N-a-1}$, therefore $[-\theta_2]^{-1}(-x) = G(\frac{SSE}{x})/p'$. Based on Theorem 2.1, we have $\widetilde{\sigma^2}(x) = min\{\theta_1^{-1}(x), [-\theta_2]^{-1}(-x), 1\}$. So,

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(SSE/x)}{1-p'} & \frac{SSE}{\chi^2_{0.99+0.01p',N-a-1}} \leq x \leq \frac{SSE}{N-a-1} \\ \\ \frac{G(SSE/x)}{p'} & \frac{SSE}{N-a-1} \leq x \leq \frac{SSE}{\chi^2_{0.01p',N-a-1}} \\ \\ 0 & \text{otherwise,} \end{cases}$$

$\square$

**Theorem 4.2.** Under the null hypothesis $H_0 : \beta = 0$, if we consider $E^2_{xy}/E_{xx}$ as an unbiased point estimator for parameter $\sigma^2$, then an unbiased fuzzy estimator for

$\sigma^2$ is $\widetilde{\sigma^2_{H_0 1}}$ with $\alpha$-cuts $\widetilde{\sigma^2_{H_0 1}}[\alpha]$, where

$$\widetilde{\sigma^2_{H_0 1}}[\alpha] = \begin{cases} \left[E^2_{xy}/(E_{xx}\chi^2_{1-\alpha+\alpha p'',1}), E^2_{xy}/(E_{xx}\chi^2_{\alpha p'',1})\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{\sigma^2_{H_0 1}}[0.01] & 0 \le \alpha < 0.01 \end{cases}$$

and $p''$ is obtained from the relation $\chi^2_{p'',1} = 1$.

*Proof.* We use the pivotal quantity $E^2_{xy}/(E_{xx}\sigma^2)$. The proof is now similar to that of Theorem 4.1. □

**Theorem 4.3.** The fuzzy test statistic for testing $H_0 : \beta = 0$ against $H_1 : \beta \ne 0$ is $\widetilde{F1}$ with $\alpha$-cuts

$$\widetilde{F1}[\alpha] = \begin{cases} \left[(f1)_1(\alpha)F1,\ (f1)_2(\alpha)F1\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{F1}[0.01] & 0 \le \alpha < 0.01 \end{cases}$$

where

$$(f1)_1(\alpha) = \chi^2_{\alpha p', N-a-1} \big/ \left[(N-a-1)\chi^2_{1-\alpha+\alpha p'',1}\right]$$

and

$$(f1)_2(\alpha) = \chi^2_{1-\alpha+\alpha p', N-a-1} \big/ \left[(N-a-1)\chi^2_{\alpha p'',1}\right]$$

*Proof.* Using the equality $\widetilde{F1}[\alpha] = \widetilde{\sigma^2_{H_0 1}}[\alpha] / \widetilde{\sigma^2}[\alpha]$ and interval arithmetic, fuzzy test statistic follows from Buckley's method. □

**Note 4.1.** Since the test statistic is a fuzzy number, then critical value is also a fuzzy number with $\alpha$-cuts

$$\widetilde{CV1}[\alpha] = \begin{cases} \left[(cv1)_1(\alpha),\ (cv1)_2(\alpha)\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{CV_1}[0.01] & 0 \le \alpha < 0.01 \end{cases}$$

where

$$(cv1)_1(\alpha) = (f1)_1(\alpha)F_{1-\gamma,1,N-a-1}$$

is obtained from the relation $P\left[(f1)_1(\alpha)F1 > (cv1)_1(\alpha)\right] = \gamma$, where $\gamma \in (0, 1)$ is the significance level of the test. Similarly, we obtain $(cv1)_2(\alpha) = (f1)_2(\alpha)F_{1-\gamma,1,N-a-1}$.

**Decision rule 4.1.** The decision rule is considered as follows. After observing the data,

*(i)* if $F_{1-\gamma,1,N-a-1} \leq F1$, then the hypothesis $H_0 : \beta = 0$ is rejected,

*(ii)* if $F_{1-\gamma,1,N-a-1} > F1$, then the area $A_1$ (shown in Figure 1) and also $A_T$ the total area under the triangle $\widetilde{F1}$ are calculated. If $A_1/A_T \leq \phi$, then the null hypothesis $H_0 : \beta = 0$ is accepted. Otherwise it is rejected, where $\phi \in [0, 1]$, which depends on the decision maker desire. In this paper we set $\phi = 0.3$. Note that Figure 1 is not drawn to scale and only illustrates our decision rule.

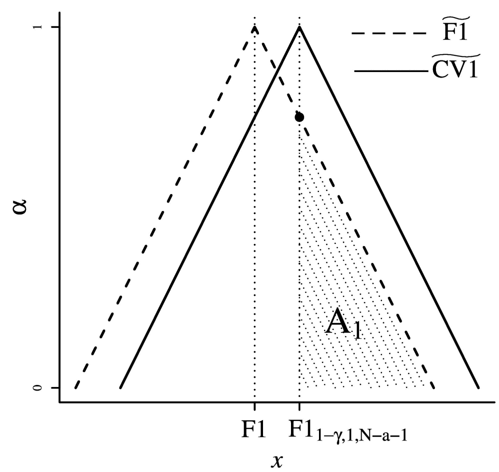



FIGURE 1. $\widetilde{F1}$, $\widetilde{CV1}$ and area $A_1$

In the sequel we consider testing the statistical hypotheses in (2) based on a fuzzy test statistic.

**Theorem 4.4.** Under the null hypothesis $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$, If we consider $(SSE^{'} - SSE)/(a-1)$ as an unbiased point estimator for parameter $\sigma^2$, then an unbiased fuzzy estimator for $\sigma^2$ is $\widetilde{\sigma^2_{H_02}}$ with $\alpha$-cuts $\widetilde{\sigma^2_{H_02}}[\alpha]$, where

$$\widetilde{\sigma^2_{H_02}}[\alpha] = \begin{cases} \left[\frac{SSE^{'}-SSE}{\chi^2_{1-\alpha+\alpha p''',a-1}}, \frac{SSE^{'}-SSE}{\chi^2_{\alpha p''',a-1}}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2_{H_02}}[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

and $p'''$ is obtained from the relation $\chi^2_{p''',a-1} = a-1$.

*Proof.* We use pivotal quantity $(SSE^{'} - SSE)/\sigma^2$. The proof is now similar to that of Theorem 4.1. □

**Theorem 4.5.** The fuzzy test statistic for testing $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ against $H_1$ : not all $\mu_i$'s are equal, is $\widetilde{F2}$ with $\alpha$-cuts

$$\widetilde{F2}[\alpha] = \begin{cases} \big[(f2)_1(\alpha)F2,\ (f2)_2(\alpha)F2\big] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{F2}[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

where

$$(f2)_1(\alpha) = \big[(a-1)\chi^2_{\alpha p',N-a-1}\big] \,/\, \big[(N-a-1)\chi^2_{1-\alpha+\alpha p''',a-1}\big]$$

and

$$(f2)_2(\alpha) = \big[(a-1)\chi^2_{1-\alpha+\alpha p',N-a-1}\big] \,/\, \big[(N-a-1)\chi^2_{\alpha p''',a-1}\big]$$

*Proof.* Using the equality $\widetilde{F2}[\alpha] = \widetilde{\sigma^2_{H_0 2}}[\alpha]/\widetilde{\sigma^2}[\alpha]$ and interval arithmetic, fuzzy test statistic follows from Buckley's method. □

**Note 4.2.** Similar to Note 4.1, the critical value is a fuzzy number with $\alpha$-cuts

$$\widetilde{CV}2[\alpha] = \begin{cases} \big[(f2)_1(\alpha)F_{1-\gamma,a-1,N-a-1},\ (f2)_2(\alpha)F_{1-\gamma,a-1,N-a-1}\big] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{CV}2[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

**Decision rule 4.2.** The final decision about accepting or rejecting $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ is considered as follows. After observing data,

*(i)* if $F_{1-\gamma,a-1,N-a-1} \leq F2$, then the hypothesis $H_0$ is rejected.

*(ii)* if $F_{1-\gamma,a-1,N-a-1} > F2$, then the area $A_1$ (similar to $A_1$ in Figure 1 for Decision rule 4.1) and also the total area $A_T$ under the graph $\widetilde{F2}$ are calculated. Now, if $A_1/A_T \leq \phi$ then the null hypothesis $H_0$ is accepted. Otherwise it is rejected, where $\phi = 0.3$.

The example presented in the next section is an appropriate example to illustrate this discussion, which is quoted from [5]. The software R is used to perform the calculation [7].

## 5. A numerical example

An example is quoted from [5] that includes an experiment performed to determine if there is a difference in the breaking strength of a monofilament fiber produced by three different machines for a textile company. Clearly the strength of the fiber is also affected by its thickness. However the strength of a fiber is related to its diameter, with thicker fibers being generally stronger than thinner ones. A random sample of five fiber specimens is selected from each machine. The fiber strength ($y$) and the corresponding diameter ($x$) for each specimen are shown in Table 1. The one-way analysis of covariance model is as follows:

$$y_{ij} = \mu_i + \beta(x_{ij} - \bar{x}..) + \varepsilon_{ij}, \text{ for } i = 1,2,3 \text{ and } j = 1,2,\dots,5.$$

Here, we have $F1 = 69.969$ and $F2 = 2.611$. Since, $F_{0.9,1,11} = 3.225$ for $\gamma = 0.1$. So by Decision rule 4.1, the null hypothesis $H_0 : \beta = 0$ is rejected.

Since $F_{0.9,2,11} = 2.860$, the values of $F_{1-\gamma,a-1,N-a-1}$ and $F2$, for $\gamma = 0.1$, are close to each other and in conventional statistics we are uncertain to accept the hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$. Therefore, we use the method presented in this paper for testing the statistical hypotheses in (2) based on a fuzzy test statistic.

We have $SSE^{'} = 41.270$ and $SSE = 27.986$. Therefore, based on Theorem 4.1 an unbiased fuzzy estimation for $\sigma^2$ is a fuzzy number with $\alpha$-cuts

$$\widetilde{\sigma^2}[\alpha] = \begin{cases} \left[27.986/\chi^2_{1-\alpha+\alpha 0.557,11}, 27.986/\chi^2_{\alpha 0.557,11}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2}[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

and $p' = 0.557$ is obtained from the relation $\chi^2_{p',11} = 11$.
So, by Lemma 4.1, the membership function of the unbiased fuzzy estimator is given as follows:

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(27.986/x)}{1-0.557} & \frac{27.986}{\chi^2_{0.99+0.01(0.557),11}} \leq x \leq \frac{27.986}{11} \\ \\ \frac{G(27.986/x)}{0.557} & \frac{27.986}{11} \leq x \leq \frac{27.986}{\chi^2_{0.01(0.557),11}} \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $G$ is the cumulative distribution function of the distribution $\chi^2$ with 11 degree of freedom as depicted in Figure 2.

TABLE 1. Breaking Strength Data ( $y$ =strength in pounds and $x$ = diameter in $10^{-3}$ inches)

| Machine 1 | | Machine 2 | | Machine 1 | |
|---|---|---|---|---|---|
| $y$ | $x$ | $y$ | $x$ | $y$ | $x$ |
| 36 | 20 | 40 | 22 | 35 | 21 |
| 41 | 25 | 48 | 28 | 37 | 23 |
| 39 | 24 | 39 | 22 | 42 | 26 |
| 42 | 25 | 45 | 30 | 34 | 21 |
| 49 | 32 | 44 | 28 | 32 | 15 |

Also, an unbiased fuzzy estimator for $\sigma^2$ based on Theorem 4.4, under the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$, is a fuzzy number with $\alpha$-cuts as follows:

$$\widetilde{\sigma^2_{H_0 2}}[\alpha] = \begin{cases} \left[13.284/\chi^2_{1-\alpha+\alpha 0.632,2}, 13.284/\chi^2_{\alpha 0.632,2}\right] & 0.01 \leq \alpha \leq 1 \\ \widetilde{\sigma^2_{H_0 2}}[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

and $p''' = 0.632$ is obtained from the relation $\chi^2_{p''',2} = 2$.

By Theorem 4.5 and Note 4.2, the fuzzy test statistic $\widetilde{F}2$ and the fuzzy critical value $\widetilde{CV}2$, with $\gamma = 0.1$ and $F2 = 2.611$, are fuzzy numbers with the following $\alpha$-cuts:

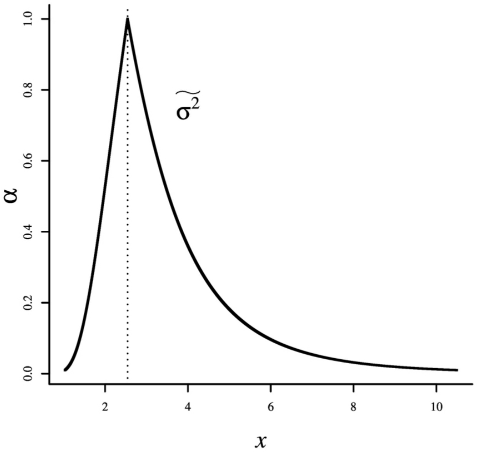
$$\widetilde{F}2[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.557,11}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.475, \ \frac{\chi^2_{1-\alpha+\alpha 0.557,11}}{\chi^2_{\alpha 0.632,2}} 0.475\right] & 0.01 \leq \alpha \leq 1 \\ \widetilde{F}2[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

and

$$\widetilde{CV}2[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.557,11}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.52, \ \frac{\chi^2_{1-\alpha+\alpha 0.557,11}}{\chi^2_{\alpha 0.632,2}} 0.52\right] & 0.01 \leq \alpha \leq 1 \\ \widetilde{CV}2[0.01] & 0 \leq \alpha < 0.01 \end{cases}$$

The graphs of these fuzzy numbers are shown in Figure 3.

The intersection between the vertical line $F_{0.9,2,11} = 2.860$ and the right-hand side $\widetilde{F}2$ is obtained as a point $\alpha^* = (2.86, 0.959)$ as shown in Figure 3. The area $A_1 \simeq 29.5208$ for $\alpha \in [0.01, 0.959]$ and $A_T \simeq 31.15$ for $\alpha \in [0.01, 1]$. Hence, $A_1/A_T \simeq 0.9478$. Since $A_1/A_T > \phi = 0.3$, the hypothesis $H_0$ is certainly rejected.



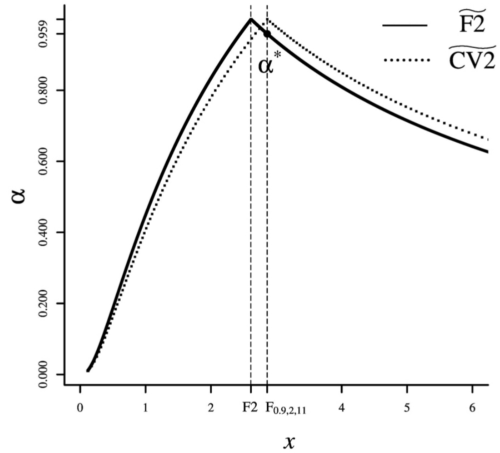




FIGURE 2. Fuzzy estimation for $\sigma^2$.

Figure 3: $\widetilde{F}2$, $\widetilde{CV}2$ and point $\alpha^*$

## 6. Conclusions

In this paper Buckley's method is applied to a one-way analysis of covariance and used for testing the statistical hypotheses when there is an uncertainty in accepting or rejecting the hypotheses.

This method can be used for other linear models; and an interesting topic for research is the study of this method on one-way analysis of covariance when the hypotheses are fuzzy.

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