MODULE GENERALIZED DERIVATIONS ON TRIANGULAUR BANACH ALGEBRAS

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ABSTRACT. Let A_1, A_2 be unital Banach algebras and X be an A_1-A_2- module. Applying the concept of module maps, (inner) module generalized derivations and generalized first cohomology groups, we present several results concerning the relations between module generalized derivations from A_i into the dual space A_i^* (for $i = 1, 2$) and such derivations from the triangular Banach algebra of the form $\mathcal{T} := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$ into the associated triangular $\mathcal{T}$- bimodule $\mathcal{T}^*$ of the form $\mathcal{T}^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$. In particular, we show that the so-called generalized first cohomology group from $\mathcal{T}$ to $\mathcal{T}^*$ is isomorphic to the directed sum of the generalized first cohomology group from A_1 to A_1^* and the generalized first cohomology group from A_2 to A_2^*.

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1. INTRODUCTION

Let A be a Banach algebra and M be a Banach A- bimodule. A *module derivation* $d : A \to M$ is a linear map which satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. The linear space of all bounded derivations from A into M is denoted by $Z^1(A, M)$. As an example, let $x \in M$ and define $d_x : A \to M$ by $d_x(a) := xa - ax$. Then d_x is a module derivation which is called inner. Denoting the linear space of inner derivations from

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A into M by $N^1(A,M)$, we may consider the quotient space $H^1(A,M) := Z^1(A,M)/N^1(A,M)$, called the first cohomology group from A into M.

A linear mapping $T : A \to M$ is called a *module map* if $T(ab) = T(a)b$. We denote by $\mathcal{B}(A,M)$ the set of all bounded linear module maps from A into M.

Recently, a number of analysts [1, 3, 9] have studied various extended notions of derivations in the context of Banach algebras. For instance, suppose that $T : A \to M$ is a module map and y is an arbitrary element of M. If we take $\delta : A \to M$ by $\delta(a) := T(a) - ay$, then it is easily seen that $\delta(ab) = \delta(a)b + ad_y(b)$ for every $a, b \in A$. Therefore considering the relation $d(ab) = d(a)b + ad(b)$ as an special case of $\delta(ab) = \delta(a)b + ad(b)$ for all $a, b \in A$, where $d : A \to M$ is a module derivation, leads the theory of derivations to be extensively developed.

The above consideration motivated the authors in [1] to generalize the notion of derivation as follows:

Let A be a Banach algebra and M be a Banach A- module. A linear mapping $\delta : A \to M$ is called *module generalized derivation* if there exist a module derivation $d : A \to M$ such that $\delta(ab) = \delta(a)b + ad(b)$ $(a, b \in A)$. For convenience, we say that a generalized derivation δ is a d- *derivation*. In order to construct a module generalized derivation, suppose that $x, y \in M$ and define $\delta_{x,y} : A \to M$ by $\delta_{x,y}(a) := xa - ay$. Then

$$\begin{aligned} \delta_{x,y}(ab) &= xab - aby \\ &= xab - ayb + ayb - aby \\ &= \delta_{x,y}(a)b + ad_y(b). \end{aligned}$$

Mathieu [8] called the map of the form $\delta_{x,y}$ an *inner generalized derivations*. A module generalized derivation $\delta : A \to M$ is called *approximately inner* (resp. *approximately bounded*) if there exists a sequence $\{\delta_n\}$ of inner (resp. *bounded*) generalized derivations from A into M such that $\{\delta_n\}$ converges to δ strongly. We denote by $GZ^1(A,M)$ (resp. $App.GZ^1(A,M)$) and $GN^1(A,M)$ (resp. $App.GN^1(A,M)$) the linear spaces of all (approximately) bounded module generalized derivations and (approximately) inner module generalized derivations from A into M, respectively. Also, we call the quotient space $GH^1(A,M) := GZ^1(A,M)/GN^1(A,M)$ (resp. $App.GH^1(A,M) := App.GZ^1(A,M)/App.GN^1(A,M)$) the (*approximate*) *generalized first cohomology group* from A to M.

We recall that the dual space M^* of M is a Banach A- module by regarding the module structure as follows

$$(a.f)(x) = f(xa), (f.a)(x) = f(ax).$$

The Banach algebra A is said to be (*approximately*) *weakly generalized amenable* if every generalized derivation $\delta : A \to A^*$ is (approximately) inner; i.e. $GH^1(A,A^*) = \{0\}$ (resp. $App.GH^1(A,A^*) = \{0\}$). The notion of an amenable Banach algebra was introduced by B. E. Johnson in [7]. Bade, Curtis and Dales [2], later defined the concept of weak amenability for commutative Banach algebras. More recently, Ghahramani and Loy [6] have defined the notion of approximate amenability of Banach algebras. The reader is referred to books [4, 10] for more information on this subject.

Let A_1, A_2 be unital Banach algebras and X be a unital A_1-A_2- module in the sense that $1_{A_1}x1_{A_2} = x$, for every $x \in X$. In this paper, we deal with the module generalized derivations from the triangular Banach algebra of the form $\mathcal{T} := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$ into the associated triangular $\mathcal{T}$- bimodule $\mathcal{T}^*$ of the form $\mathcal{T}^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$. Such algebras were introduced by Forrest and Marcoux in [5]. Applying several results concerning the relations between module generalized derivations from A_i into the dual space A_i^* (for $i = 1, 2$) and such derivations from $\mathcal{T}$ into $\mathcal{T}^*$, we show that the so-called generalized first cohomology group from $\mathcal{T}$ to $\mathcal{T}^*$ is isomorphic to the directed sum of the generalized first cohomology group from A_1 to A_1^* and the generalized first cohomology group from A_2 to A_2^*.

2. Module Derivations on triangular Banach algebras

Definition 2.1. $\mathcal{T} := \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; a \in A_1,\ x \in X,\ b \in A_2\}$ equipped with the usual 2×2 matrix addition and formal multiplication with the norm $\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| := \| a \| + \| x \| + \| b \|$ is a Banach algebra which is called the traingular Banach algebra associated to X. We define $\mathcal{T}^*$ as $\{\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}; f \in A_1^*,\ h \in X^*,\ g \in A_2^*\}$ and

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right] := f(a) + h(x) + g(b)$$

$\mathcal{T}^*$ is a triangular $\mathcal{T}$- bimodule with respect to the following module structure

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} := \begin{pmatrix} af + xh & bh \\ 0 & bg \end{pmatrix},$$

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} := \begin{pmatrix} fa & ha \\ 0 & hx + gb \end{pmatrix}.$$

The following results show some interesting relations between module generalized derivations from A_i to A_i^* (for $i = 1, 2$) and those from $\mathcal{T}$ to $\mathcal{T}^*$. Let $d_i : A_i \to A_i^*$ be a bounded module derivation and $\delta_i : A_i \to A_i^*$ be a bounded module d_i- derivation, for $i = 1, 2$. Define $\triangle_1, \triangle_2 : \mathcal{T} \to \mathcal{T}^*$ by

$$\triangle_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \triangle_2(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & 0 \\ 0 & \delta_2(b) \end{pmatrix}.$$

Theorem 2.2. *$\triangle_i$ is a bounded D_i- derivation (for $i = 1, 2$), where*

$$D_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} d_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D_2(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & 0 \\ 0 & d_2(b) \end{pmatrix}.$$

Moreover $\triangle_i$ (resp. D_i) is inner if and only if so is δ_i (resp. d_i).

Proof. By simple calculations, it can be observed that D_1 is a derivation and $\triangle_1$ is a D_1- derivation. Also

$$\| \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \| = \| \delta_1(a) \| \leq \| \delta_1 \| \{ \| a \| + \| x \| + \| b \| \}$$

Hence $\triangle_1$ (and similarly D_1) is bounded.

Suppose that $\triangle_1$ is inner. Then there exist $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}, \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \in \mathcal{T}^*$ such that

$$\begin{aligned}
\begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} &= \triangle_1(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \\
&= \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \\
&= \begin{pmatrix} f_1 a & h_1 a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a f_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f_1 a - a f_2 & h_1 a \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Hence $\delta_1(a) = f_1 a - a f_2$ and $h_1 a = 0$ for all $a \in A_1$. So δ_1 is an inner generalized derivation and $h_1 = 0$.

Conversely, if $\delta_1 : A_1 \to A_1^*$ is an inner module generalized derivation, then there exist $f_1, f_2 \in A_1^*$ such that $\delta_1(a) = f_1 a - a f_2$. Then

$$\begin{aligned}
\triangle_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) &= \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f_1 a - a f_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} f_2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore $\triangle_1$ is an inner generalized derivation. $\square$

Theorem 2.3. *Let $\mathcal{T}^*$ be the triangular bimodule $\begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$ associated to the triangular Banach algebra $\mathcal{T}$. Assume that $D : \mathcal{T} \to \mathcal{T}^*$ be a bounded derivation and $\triangle : \mathcal{T} \to \mathcal{T}^*$ be a bounded D- derivation. Then for $i = 1, 2$, there exist a continuous derivation $d_i : A_i \to A_i^*$, a continuous d_i- derivation $\delta_i : A_i \to A_i^*$, and $h_0, h_0' \in X^*$ such that*

$$\triangle(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & h_0'x + \delta_2(b) \end{pmatrix}.$$

Proof. First we show that there exist an element $h_0 \in X^*$, a continuous derivation $d_1 : A_1 \to A_1^*$, and a continuous derivation $d_2 : A_2 \to A_2^*$ such that

$$D(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} d_1(a) - xh_0 & h_0a - bh_0 \\ 0 & h_0x + d_2(b) \end{pmatrix}.$$

For this aim using some ideas of [5], we can verify that

(i) There exists $h_0 \in X^*$ such that $D(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}$.

(ii) There exists a bounded derivation $d_1 : A_1 \to A_1^*$ such that $D(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} d_1(a) & h_0a \\ 0 & 0 \end{pmatrix}$.

(iii) $D(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h_0x \end{pmatrix}$.

(iv) There exist a bounded derivation $d_2 : A_2 \to A_2^*$ such that $D(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -bh_0 \\ 0 & d_2(b) \end{pmatrix}$.

Now a similar calculation shows that

(i') There exist $f \in A_1^*$, $h_0' \in X^*$ such that $\triangle(\begin{pmatrix} 1_{A_1} & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f & h_0' \\ 0 & 0 \end{pmatrix}$.

(ii') There exists a bounded d_1- derivation $\delta_1 : A_1 \to A_1^*$ such that $\triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \delta_1(a) & h_0'a \\ 0 & 0 \end{pmatrix}$.

(iii') $\triangle(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h_0'x \end{pmatrix}$.

(iv') There exist a bounded d_2- derivation $\delta_2 : A_2 \to A_2^*$ such that $\triangle(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -bh_0 \\ 0 & \delta_2(b) \end{pmatrix}$.

and finally $\triangle(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & h_0'x + \delta_2(b) \end{pmatrix}$.

For this aim following the parts (i) and (ii), we only prove the parts (i') and (ii').The other parts are similar.

(i') There exist $f \in A_1^*$, $h \in X^*$, and $g \in A_2^*$ such that $\triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}$.

On the other hand

$$\begin{aligned}
\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} &= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & h \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & h \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

It follows that $g = 0$. Taking $h_0' := h$, completes the proof.

(ii') There exist $f_1 \in A_1^*$, $h_1 \in X^*$, and $g_1 \in A_2^*$ such that $\triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}$. On the other hand

$$\begin{aligned}
\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} &= \triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} f & h_0' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1(a) & h_0 a \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} fa & h_0' a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_1(a) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} fa + d_1(a) & h_0' a \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

It follows that $g_1 = 0$, $h_1 = h_0' a$, and $f_1 = fa + d_1(a)$.

Take $\delta_1(a) := f_1$. We show that δ_1 is a d_1- derivation. Trivially δ_1 is linear. Moreover

$$\begin{aligned}
\begin{pmatrix} \delta_1(a_1a_2) & h_0'a_1a_2 \\ 0 & 0 \end{pmatrix} &= \triangle(\begin{pmatrix} a_1a_2 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} \delta_1(a_1) & h_0'a_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1(a_2) & h_0a_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta_1(a_1)a_2 & h_0'a_1a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1d_1(a_2) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta_1(a_1)a_2 + a_1d_1(a_2) & h_0'a_1a_2 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

therefore δ_1 is a d_1- derivation. Further since $\triangle$ is bounded, so

$$\begin{aligned}
\| \delta_1(a) \| &\leq \| \delta_1(a) \| + \| h_0'a \| \\
&= \| \begin{pmatrix} \delta_1(a) & h_0'a \\ 0 & 0 \end{pmatrix} \| \\
&= \| \triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \| \\
&\leq \| \triangle \| \| a \| .
\end{aligned}$$

It follows that δ_1 is bounded and $\| \delta_1 \|\leq\| \triangle \|$. □

Theorem 2.4. *Let A_1, A_2 be unital Banach algebras, X be a unital A_1-A_2- module and let $\mathcal{T}^*$ be the triangular bimodule $\begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$ associated to the triangular Banach algebra $\mathcal{T}$. Then*

$$GH^1(\mathcal{T}, \mathcal{T}^*) \cong GH^1(A_1, A_1^*) \oplus GH^1(A_2, A_2^*).$$

Proof. Define $\pi : GZ^1(A_1, A_1^*)\oplus\, GZ^1(A_2, A_2^*) \to GH^1(\mathcal{T}, \mathcal{T}^*)$ by $\pi(\delta_1, \delta_2) := [\triangle_{\delta_2}^{\delta_1}]$, where $\triangle_{\delta_2}^{\delta_1} := \triangle_1 + \triangle_2$ (as we defined in Theorem 2.2) and $[\triangle_{\delta_2}^{\delta_1}]$ represents the equivalent class of $\triangle_{\delta_2}^{\delta_1}$ in $GH^1(\mathcal{T}, \mathcal{T}^*)$. Clearly π is linear. We are going to show that π is surjective. For, let $\triangle$ be a bounded

D- derivation from $\mathcal{T}$ to $\mathcal{T}^*$. Let δ_1, δ_2, h_0 and h_0' be as in Theorem 2.3. Then trivially

$$\begin{aligned}(\triangle - \triangle_{\delta_2}^{\delta_1})(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) &= \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & \delta_2(b) + h_0'x \end{pmatrix} - \begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix} \\ &= \begin{pmatrix} -xh_0 & h_0'a - bh_0 \\ 0 & h_0'x \end{pmatrix} \\ &= \begin{pmatrix} 0 & h_0' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

hence

$$\triangle - \triangle_{\delta_2}^{\delta_1} \in GN^1(\mathcal{T}, \mathcal{T}^*).$$

This implies that $[\triangle] = [\triangle_{\delta_2}^{\delta_1}]$ and π is surjective. Therefore

$$GH^1(\mathcal{T}, \mathcal{T}^*) \cong \; GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*)/ker\pi.$$

It is enough to show that $ker\pi = GN^1(A_1, A_1^*) \oplus \; GN^1(A_2, A_2^*)$. For this aim, note that if $(\delta_1, \delta_2) \in ker\pi$, then $\triangle_{\delta_2}^{\delta_1} : \mathcal{T} \to \mathcal{T}^*$ is an inner generalized derivation. So there exist $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}, \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \in \mathcal{T}^*$ such that

$$\begin{aligned}\begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix} &= \triangle_{\delta_2}^{\delta_1}(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) \\ &= \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \\ &= \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \\ &= \begin{pmatrix} f_1a & h_1a \\ 0 & g_1b \end{pmatrix} - \begin{pmatrix} af_2 & bh_2 \\ 0 & bg_2 \end{pmatrix} \\ &= \begin{pmatrix} f_1a - af_2 & h_1a - bh_2 \\ 0 & g_1b - bg_2 \end{pmatrix}.\end{aligned}$$

Hence $\delta_1(a) = f_1a - af_2$ and $\delta_2(b) = g_1b - bg_2$ for all $a \in A_1$, $b \in A_2$. So δ_1 and δ_2 are the inner d_{f_2}- and d_{g_2}- derivations, respectively. Hence

$$(\delta_1, \delta_2) \in GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*).$$

Conversely, if $(\delta_1,\delta_2)\in GN^1(A_1,A_1^*)\oplus GN^1(A_2,A_2^*)$, then δ_1 and δ_2 are inner. By theorem 2.2, $\triangle_i$ is an inner D_i- derivation, for $i=1,2$. Hence $\triangle_1+\triangle_2=\triangle_{\delta_2}^{\delta_1}$ is an inner (D_1+D_2)- derivation. Therefore

$$ker\pi = GN^1(A_1,A_1^*)\oplus GN^1(A_2,A_2^*).$$

Now we have

$$\begin{aligned} GH^1(\mathcal{T},\mathcal{T}^*) &= \frac{GZ^1(A_1,A_1^*)\oplus GZ^1(A_2,A_2^*)}{GN^1(A_1,A_1^*)\oplus GN^1(A_2,A_2^*)} \\ &\cong \frac{GZ^1(A_1,A_1^*)}{GN^1(A_1,A_1^*)}\oplus\frac{GZ^1(A_2,A_2^*)}{GN^1(A_2,A_2^*} \\ &= GH^1(A_1,A_1^*)\oplus GH^1(A_2,A_2^*). \end{aligned}$$

□

Remark 2.5. Let $d_i:A_i\to A_i^*$ be an approximately bounded derivation and $\delta_i:A_i\to A_i^*$ be an approximately bounded d_i- derivation, for $i=1,2$. Define $\triangle_1,\triangle_2:\mathcal{T}\to\mathcal{T}^*$ by

$$\triangle_1\begin{pmatrix} a & x\\ 0 & b\end{pmatrix}:=\begin{pmatrix}\delta_1(a) & 0\\ 0 & 0\end{pmatrix}\quad and\quad \triangle_2\begin{pmatrix} a & x\\ 0 & b\end{pmatrix}:=\begin{pmatrix} 0 & 0\\ 0 & \delta_2(b)\end{pmatrix}.$$

(i) Following exactly the method has been used in Theorem 2.2, shows that $\triangle_i$ is an approximately bounded D_i- derivation (for $i=1,2$), where
$$D_1(\begin{pmatrix} a & x\\ 0 & b\end{pmatrix}):=\begin{pmatrix} d_1(a) & 0\\ 0 & 0\end{pmatrix}\text{ and } D_2(\begin{pmatrix} a & x\\ 0 & b\end{pmatrix}):=\begin{pmatrix} 0 & 0\\ 0 & d_2(b)\end{pmatrix}.$$
Moreover, $\triangle_i$ (rep. D_i) is approximately inner if and only if so is δ_i (resp. d_i).

(ii) Assume that $D:\mathcal{T}\to\mathcal{T}^*$ be an approximately bounded derivation and $\triangle:\mathcal{T}\to\mathcal{T}^*$ be an approximately bounded D_i derivation. Then similar to the proof of Theorem 2.3, it can be shown that for $i=1,2$ there exist an approximately bounded derivation $d_i:A_i\to A_i^*$, an approximately bounded d_i- derivation $\delta_i:A_i\to A_i^*$, and $h_0,h_0'\in X^*$ such that

$$\triangle(\begin{pmatrix} a & x\\ 0 & b\end{pmatrix})=\begin{pmatrix}\delta_1(a)-xh_0 & h_0'a-bh_0\\ 0 & h_0'x+\delta_2(b)\end{pmatrix}.$$

(iii) As an immediate consequence of the part (i) and (ii), it is easily seen that

$$App.GH^1(\mathcal{T},\mathcal{T}^*)\cong App.GH^1(A_1,A_1^*)\oplus App.GH^1(A_2,A_2^*).$$

Corollary 2.6. *$\mathcal{T}$ is (approximately) weakly generalized amenable if and only if A_i is (approximately) weakly generalized amenable, for $i=1,2$.*

3. Acknowledgment

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