

RELATIVE INFORMATION FUNCTIONAL OF RELATIVE DYNAMICAL SYSTEMS

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ABSTRACT. In this paper by use of mathematical modeling of an observer [14, 15] the notion of relative information functional for relative dynamical systems on compact metric spaces is presented. We extract the information function of an ergodic dynamical system (X, T) from the relative information of T from the view point of observer χ_X , where X denotes the base space of the system. We also generalize the invariance of the information function of a dynamical system, under topological isomorphism, to the relative information functional.

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1. INTRODUCTION

Shannon [22] firstly introduced the concept of information function and investigated some properties of this function. Then, McMillan [12], Dumitrescu [6] and Tok [25] has proved some properties of the fuzzy information function. Recently, Guney, Tok and Yamankaradeniz defined fuzzy local information function and stated some properties of this function in [8]. The importance of information in ergodic theory arises from it's invariance under isomorphism. Therefore, systems with different information functions cannot be isomorphic. One of the main objects in physical phenomena is the "observer". A modeling for an observer of a set X is a fuzzy set

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$\Theta : X \rightarrow [0, 1]$ [14, 15, 16, 17, 18, 19]. In fact this kinds of fuzzy sets are called "one dimensional observes" .

In this paper we would like to use of the notion of observer to define the relative information functional for topological dynamical systems. The idea of the definition is based on the relation between "experiance " and "information" from the view point of an observer. We assign a weight factor $f(x)$ to any point $x \in X$, where X denotes the base space of the system. The weight factor can be cosidered as the local loss of information caused by the lack of experiance of any intelligent point. we also generalize the invariance of the information function of a system, under topological isomorphism, to the relative information functional.

In this article the set of all probability measures on X preserving T is denoted by $M(X, T)$. We also write $E(X, T)$ for the set of all ergodic measures of T .

2. PRELIMINARY FACTS

Definition 2.1. Let $\xi = \{A_1, \dots, A_n\}$ be a finite measurable partition of X and $\mu \in M(X, T)$. Then if $A_i \in \xi; i = 1, \dots, n$ is an observed event. The information $I(\xi, \mu)$ carried by ξ may be defined as;

$$I(\xi, \mu) = -\log \mu(A_i);$$

and the quantity

$$I(x, \xi, \mu) = -\sum_{i=1}^n \chi_{A_i}(x) \log \mu(A_i);$$

for each $x \in X$ is called information function. Where χ_A is the characteristic function of A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition 2.2. A partition ξ is a refinement of a partition η , if every element of η is a union of elements of ξ and it is denoted by $\eta \prec \xi$.

Definition 2.3. Given two partitions ξ, η their common refinement is defined as follows:

$$\xi \vee \eta = \{A_i \cap B_j; A_i \in \xi, B_j \in \eta\}.$$

Theorem 2.4. *Let ξ and η be two partitions of X with $I(x, \xi, \mu) < \infty$ and $I(x, \eta, \mu) < \infty$, for all $x \in X$. Then, for all $x \in X$*

- (i) $I(x, \xi, \mu) \geq 0$;
- (ii) $I(x, \xi \vee \eta, \mu) \geq I(x, \xi, \mu) + I(x, \eta, \mu)$.

Proof : See, [2] and [25].

Lemma 2.5. *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that is positive and sub additive. Then $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists and equal to $\inf_{n \in \mathbb{N}} \frac{a_n}{n}$.*

Proof : See, the Theorem 4-9 of [26]. □

Theorem 2.6. *If ξ is a finite measurable partition of X with $I(x, \xi, \mu) < \infty$, for each $x \in X$ then, $\lim_{n \rightarrow \infty} \frac{1}{n} I(x, \bigvee_{i=0}^{n-1} T^{-i} \xi, \mu)$, for each $x \in X$ exists and is equal to infimum.*

Proof : See [8].

Definition 2.7. *Let ξ is a finite measurable partition of the dynamical system (X, T) with $I(x, \xi, \mu) < \infty$, for each $x \in X$, the quantity*

$$I(x, T, \mu) = \sup_{\xi} I(x, T, \xi, \mu);$$

is called the information function of dynamical system (X, T) . Where the supremum is taken over all finite measurable partitions of X with the finite information functions.

In the following we recall some classical results that we need in the sequel.

Theorem 2.8. (Choquet) *Suppose that Y is a compact convex metrisable subset of a locally convex space E , and $x_0 \in Y$. Then, there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y , that is, $\Phi(x_0) = \int_Y \Phi d\tau$ for every continuous linear functional Φ on E , and $\tau(\text{ext}(Y)) = 1$.*

Proof : See [21]. □

Let $\mu \in M(X, T)$ and $f : X \rightarrow \mathbb{R}$ be a bounded measurable function. As we know that $E(X, T)$ equals the extreme points of $M(X, T)$, applying the Choquets Theorem for $E = M(X)$, the space of finite regular Borel measures on X , and $Y = M(X, T)$, and using the linear functional $\Phi : M(X) \rightarrow \mathbb{R}$ given by $\Phi(\mu) = \int_X f d\mu$, we have the following Corollary:

Corollary 2.9. *Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X . Then, for each $\mu \in M(X, T)$, there is a unique measure τ on the Borel subsets of the compact metrsable space $M(X, T)$, such that $\tau(E(X, T)) = 1$ and*

$$\int_X f(x) d\mu(x) = \int_{E(X, T)} \left(\int_X f(x) dm(x) \right) d\tau(m)$$

for every bounded measurable function $f : X \rightarrow \mathbb{R}$.

Under the assumptions of Corollary 2.9, we write $\mu = \int_{E(X, T)} m d\tau(m)$, called the ergodic decomposition of μ .

3. RELATIVE INFORMATION FUNCTIONAL OF RELATIVE DYNAMICAL SYSTEMS

This section is presenting the notion of information from the view point of different observers which describe a relative perspective of complexity and uncertainty in fuzzy systems. In this paper we assume that X is a compact metric space, and Θ is a one dimensional observer of X [14], that is, $\Theta : X \rightarrow [0, 1]$ is a fuzzy set [29]. Moreover we assume that $T : X \rightarrow X$ is a continuous map. In this case we say that (X, T, Θ) is a relative dynamical system. Infact if $E \subseteq X$, then the relative probability measure of E with respect to an observer Θ is the fuzzy set $m_\Theta^T(E) : X \rightarrow [0, 1]$ defined by

$$m_\Theta^T(E)(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)) \Theta(T^i(x)).$$

Where χ_E is the characteristic function of E [14].

Theorem 3.1. *Let (X, β, m) be a probability space, and let $\Theta : X \rightarrow [0, 1]$ be the characteristic function χ_X . Moreover let $T : X \rightarrow X$ be an ergodic map, then for each $x \in X$, $m_\Theta^T(E)(x)$ is almost everywhere equal to $m(E)$ where $E \in \beta$.*

Proof : See [15]. □

So relative probability measure is an extension of the notion of probability measure. In the rest of this paper m_x is a relative measure with respect to an observer Θ at $x \in X$, i.e. $m_x(E) = m_\Theta^T(E)(x)$ for any $E \subseteq X$.

Definition 3.2. Suppose that $T : X \rightarrow X$ is a continuous map on the topological space $X, x \in X$ and A a Borel subset of X . Then

$$m_x(A) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x)) \Theta(T^i(x)).$$

Now, let $x \in X$ and $\xi = \{A_1, A_2, \dots, A_n\}$ and $\eta = \{B_1, B_2, \dots, B_m\}$ be finite Borel partitions of X . We define

$$\rho_\Theta(x, T, \xi) := - \sum_{i=1}^n \chi_{A_i}(x) \log m_x(A_i);$$

and

$$\rho_\Theta(x, T, \xi|\eta) := - \sum_{i,j} \chi_{(A_i \cap B_j)}(x) \log \frac{m_x(A_i \cap B_j)}{m_x(B_j)}.$$

(We assume that $\log 0 = -\infty$ and $0 \times \infty = 0$).

Note that the quantity $\rho_\Theta(x, T, \xi|\eta)$ is the conditional version of $\rho_\Theta(x, T, \xi)$. It is clear $\rho_\Theta(x, T, \xi) \geq 0$.

Theorem 3.3. Suppose that $T : X \rightarrow X$ is a continuous map on the topological space $X, x \in X$ and ξ, η, ζ are finite Borel partitions and $x \in X$ then

- (i) $\rho_\Theta(x, T, \xi \vee \eta|\zeta) = \rho_\Theta(x, T, \xi|\zeta) + \rho_\Theta(x, T, \eta|\xi \vee \zeta)$;
- (ii) $\rho_\Theta(x, T, \xi \vee \eta) = \rho_\Theta(x, T, \xi) + \rho_\Theta(x, T, \eta|\xi)$;
- (iii) If $\xi \prec \eta$ then $\rho_\Theta(x, T, \xi|\zeta) \leq \rho_\Theta(x, T, \eta|\zeta)$;
- (iv) If $\xi \prec \eta$ then $\rho_\Theta(x, T, \xi) \leq \rho_\Theta(x, T, \eta)$.

Proof : Let $\xi = \{A_1, A_2, \dots, A_n\}, \eta = \{B_1, B_2, \dots, B_m\}, \zeta = \{C_1, C_2, \dots, C_k\}$ be finite Borel partitions of X and assume, without loss of generality, that all sets have the property that $m_x(A) \neq 0$.

(i) By definition we have

$$\rho_\Theta(x, T, \xi \vee \eta|\zeta) = - \sum_{i,j,k} \chi_{(A_i \cap B_j \cap C_k)}(x) \log \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(C_k)}.$$

But we may write

$$\frac{m_x(A_i \cap B_j \cap C_k)}{m_x(C_k)} = \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(A_i \cap C_k)} \cdot \frac{m_x(A_i \cap C_k)}{m_x(C_k)},$$

unless $m_x(A_i \cap C_k) = 0$, in latter case the left hand side is zero and we need not consider it; therefore

$$\begin{aligned} \rho_{\Theta}(x, T, \xi \vee \eta | \zeta) &= - \sum_{i,j,k} \chi_{(A_i \cap B_j \cap C_k)}(x) \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} - \sum_{i,j,k} \chi_{(A_i \cap B_j \cap C_k)} \log \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(A_i \cap C_k)} \\ &= - \sum_{i,j,k} \chi_{(A_i \cap B_j \cap C_k)} \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} + \rho_{\Theta}(x, T, \eta | \xi \vee \zeta) \end{aligned} \quad 3.1$$

But we have

$$\sum_j \chi_{(A_i \cap B_j \cap C_k)}(x) = \chi_{(A_i \cap C_k)}$$

Now multiplying both sides by $-\log \frac{m_x(A_i \cap C_k)}{m_x(C_k)}$ and summing over i and k we will obtain

$$- \sum_{i,j,k} \chi_{(A_i \cap B_j \cap C_k)}(x) \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} = \rho_{\Theta}(x, T, \xi | \zeta). \quad 3.2$$

Combining 3.1 and 3.2 we will have

$$\rho_{\Theta}(x, T, \xi \vee \eta | \zeta) = \rho_{\Theta}(x, T, \xi | \zeta) + \rho_{\Theta}(x, T, \eta | \xi \vee \zeta).$$

(ii) We can write

$$m_x(A_i \cap B_j) = \frac{m_x(A_i \cap B_j)}{m_x(A_i)} \cdot m_x(A_i).$$

So we have

$$\begin{aligned} \rho_{\Theta}(x, T, \xi \vee \eta) &= - \sum_{i,j} \chi_{(A_i \cap B_j)}(x) \log \frac{m_x(A_i \cap B_j)}{m_x(A_i)} - \sum_{i,j} \chi_{(A_i \cap B_j)}(x) \log m_x(A_i) \\ &= - \sum_{i,j} \chi_{(A_i \cap B_j)}(x) \log m_x(A_i) + \rho_{\Theta}(x, T, \eta | \xi) \\ &= \rho_{\Theta}(x, T, \xi) + \rho_{\Theta}(x, T, \eta | \xi). \end{aligned}$$

(iii) By (i) we have

$$\begin{aligned} \rho_{\Theta}(x, T, \eta | \zeta) &= \rho_{\Theta}(x, T, \xi \vee \eta | \zeta) \\ &= \rho_{\Theta}(x, T, \xi | \zeta) + \rho_{\Theta}(x, T, \eta | \xi \vee \zeta) \\ &\geq \rho_{\Theta}(x, T, \xi | \zeta). \end{aligned}$$

(iv) It follows from (ii) and (iii).

□

Definition 3.4. Suppose that $T : X \rightarrow X$ is a continuous map on the topological space $X, x \in X$ and ξ be a finite Borel partition of X . The map $I_{\Theta}(\cdot, T, \xi) : X \rightarrow [0, \infty]$ is defined as

$$I_{\Theta}(x, T, \xi) = \limsup_{l \rightarrow \infty} \frac{1}{l} \rho_{\Theta}(x, T, \bigvee_{i=0}^{l-1} T^{-i} \xi).$$

Theorem 3.5. Let ξ be a finite partition of X . Then for every $k \in \mathbb{N}$,

- (i) If $\xi \prec \eta$ then $I_{\Theta}(x, T, \xi) \leq I_{\Theta}(x, T, \eta)$;
- (ii) $I_{\Theta}(x, T, \xi) = I_{\Theta}(x, T, \bigvee_{j=0}^k T^{-j} \xi)$.

Proof :

- (i) If $\xi \prec \eta$ then $\bigvee_{j=0}^{n-1} T^{-j} \xi \prec \bigvee_{j=0}^{n-1} T^{-j} \eta$ for all $n \geq 1$. This easily leads to the result.
- (ii) We obtain immediately

$$\begin{aligned} I_{\Theta}(x, T, \bigvee_{j=0}^k T^{-j} \xi) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \rho_{\Theta}(x, T, \bigvee_{i=0}^{n-1} T^{-i} (\bigvee_{j=0}^k T^{-j} \xi)) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \rho_{\Theta}(x, T, \bigvee_{t=0}^{n+k-1} T^{-t} \xi) \\ &= \limsup_{p \rightarrow \infty} \frac{p}{p-k} \cdot \frac{1}{p} \rho_{\Theta}(x, T, \bigvee_{t=0}^{p-1} T^{-t} \xi) \\ &= I_{\Theta}(x, T, \xi). \end{aligned}$$

□

Definition 3.6. Let $T : X \rightarrow X$ is a continuous map on the topological space X . Then a partition ξ of X is called a relative generator of T if there exists an integer $k > 0$ such that

$$\eta \prec \bigvee_{i=0}^k T^{-i} \xi$$

for every partition η of X .

Theorem 3.7. Let ξ be a relative generator of T then $I_{\Theta}(x, T, \eta) \leq I_{\Theta}(x, T, \xi)$, for every partition η of X .

Proof : Since ξ is a relative generator of T , then for partition η , there exists an integer $k > 0$ such that

$$\eta \prec \bigvee_{i=0}^k T^{-i} \xi.$$

Hence

$$I_{\Theta}(x, T, \eta) \leq I_{\Theta}(x, T, \bigvee_{i=0}^k T^{-i} \xi) = I_{\Theta}(x, T, \xi).$$

□

Definition 3.8. Suppose that $T : X \rightarrow X$ is a continuous map on the topological space X , $x \in X$ and ξ be a finite Borel partition of X . We define the relative information of T at x by

$$I_{\Theta}(x, T, m_x) = \sup_{\xi} I_{\Theta}(x, T, \xi).$$

Theorem 3.9. Let ξ be a relative generator of T then $I_{\Theta}(x, T, \xi) = I_{\Theta}(x, T, m_x)$.

Proof : Theorem 3.7 implies that $I_{\Theta}(x, T, \eta) \leq I_{\Theta}(x, T, \xi)$ for each partition η of X . So $\sup_{\eta} I_{\Theta}(x, T, \eta) = I_{\Theta}(x, T, \xi)$. Thus $I_{\Theta}(x, T, \xi) = I_{\Theta}(x, T, m_x)$. □

In the following theorem, we extract information function for ergodic dynamical systems from the relative information of T as a special case.

Theorem 3.10. Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X . If $\Theta : X \rightarrow [0, 1]$ is the characteristic function χ_X , then for all $x \in X$ we have,

$$I_{\Theta}(x, T, m_x) = I(x, T, m).$$

Proof : Since $m \in E(X, T)$, for each Borel set A and $x \in X$, applying Theorem 3.1 we have $m_x(A) = m(A)$. So by replacing m_x by m we have,

$$I_{\Theta}(x, T, m_x) = I(x, T, m).$$

□

Definition 3.11. Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X , and ξ be a relative generator for the relative dynamical system (X, Θ, T) . Let $\mu \in M(X, T)$. The relative information functional of T (with respect to μ), $I_{\Theta}^T(\cdot, \mu, \xi) : C(X) \rightarrow \mathbb{R}$, is defined as

$$I_{\Theta}^T(f, \mu, \xi) = \int_X f(x) I_{\Theta}(x, T, \xi) d\mu(x)$$

for all $f \in C(X)$ (again $0 \times \infty := 0$).

In the following, we will prove the independence of weighted information functional from the selection of the generator.

Theorem 3.12. *Definition 3.11 is independent of the choice of relative generator i.e if ξ and η are two relative generators of T then,*

$$I_{\Theta}^T(f, \mu, \xi) = I_{\Theta}^T(f, \mu, \eta).$$

for all $f \in C(X)$.

Proof : Let ξ, η be relative generators of T . Then by Theorem 2.11 we have

$$I_{\Theta}(x, T, \xi) = I_{\Theta}(T, m_x) = I_{\Theta}(x, T, \eta).$$

So, if $f \in C(X)$, then,

$$f(x)I_{\Theta}(x, T, \xi) = f(x)I_{\Theta}(x, T, \eta)$$

for all $x \in X$. Therefore $I_{\Theta}^T(f, \mu, \xi) = I_{\Theta}^T(f, \mu, \eta)$. □

Remark 3.13. *By Theorem 3.12, we conclude that the definition of relative information functional is independent of the selection of generators. Therefore, given any invariant measure μ and any relative generator ξ , we have the unique relative information functional. So, we can write $I_{\Theta}^T(f, \mu)$ for $I_{\Theta}^T(f, \mu, \xi)$ without confusion.*

Definition 3.14. *we say that two relative dynamical systems (X, T_1, Θ_1) and (Y, T_2, Θ_2) are isomorphic if there exists a homeomorphism $\varphi : X \rightarrow Y$ such that $\varphi \circ T_1 = T_2 \circ \varphi$ and $\Theta_2(T_2 \circ \varphi(x)) = \Theta_1(T_1(x))$ for all $x \in X$.*

Theorem 3.15. *Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X . Then*

- (i) *Given any $\mu \in M(X, T)$, the relative information functional $f \rightarrow I_{\Theta}^T(f, \mu)$ is linear.*
- (ii) *Given any $f \in C(X)$, the map $\mu \rightarrow I_{\Theta}^T(f, \mu)$ is affine.*
- (iii) *If two relative dynamical systems (X, T_1, Θ_1) and (Y, T_2, Θ_2) are isomorphic, and $\mu \in M(X, T)$, then,*

$$I_{\Theta_1}^{T_1}(f, \mu) = I_{\Theta_2}^{T_2}(f \circ \varphi^{-1}, \mu \circ \varphi^{-1})$$

for all $f \in C(X)$.

Proof :

- (i) and (ii) are trivial.

- (iii) For $x \in X$ and the Borel set $A \subset X$, we have $m_{\Theta}^{T_1}(A)(x) = m_{\Theta}^{T_2}(\varphi(A))(\varphi(x))$. Therefore, $\rho_{\Theta}(x, T_1, \xi) = \rho_{\Theta}(\varphi(x), T_2, \varphi(\xi))$ for any finite Borel partition ξ . By definition of $I_{\Theta}(\cdot, T, \xi)$ we have $I_{\Theta_1}(\cdot, T_1, \xi) = I_{\Theta_2}(\cdot, T_2, \varphi(\xi)) \circ \varphi$. Note that $\varphi(\xi) = \{\varphi(A); A \in \xi\}$. Let $\mu \in M(X, T_1)$, and $f \in C(X)$. Then,

$$\begin{aligned} I_{\Theta_1}^{T_1}(f, \mu) &= \int_X f(x) I_{\Theta_1}(x, T_1, \xi) d\mu(x) \\ &= \int_X f(x) I_{\Theta_2}(\varphi(x), T_2, \varphi(\xi)) d\mu(x) \\ &= \int_Y f(\varphi^{-1}(x)) I_{\Theta_2}(x, T_2, \varphi(\xi)) d(\mu\varphi^{-1})(x) \\ &= I_{\Theta_2}^{T_2}(f\varphi^{-1}, \mu\varphi^{-1}). \end{aligned}$$

□

Theorem 3.16. *Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X . If $\mu \in M(X, T)$ and $\mu = \int_{E(X, T)} m d\tau(m)$ is the ergodic decomposition of μ , then,*

$$I_{\Theta}^T(f, \mu) = \int_{E(X, T)} I_{\Theta}^T(f, m) d\tau(m)$$

for all $f \in C(X)$.

Proof : Let ξ be a generator of relative dynamical system (X, Θ, T) . First, let $f \in C^+(X)$. Applying Corollary 2.9, we have

$$\begin{aligned} I_{\Theta}^T(f, \mu, \xi) &= \int_X f(x) I_{\Theta}(x, T, \xi) d\mu(x) \\ &= \int_{E(X, T)} \left(\int_X f(x) I_{\Theta}(x, T, \xi) dm(x) \right) d\tau(m) \\ &= \int_{E(X, T)} \left(\int_X I_{\Theta}^T(f, m, \xi) d\tau(m) \right). \end{aligned}$$

For $f \in C(X)$, write $f = f^+ - f^-$ where $f^+, f^- \in C^+(X)$. □

Theorem 3.17. *Suppose that $T : X \rightarrow X$ is a continuous map on the compact metric space X . Moreover let $x \in X$ and $\mu \in M(X, T)$. Then ,*

- (i) $I_{\Theta}^T(1, \mu) = I_{\Theta}(x, T, m_x)$.
- (ii) *The relative information functional $f \rightarrow I_{\Theta}^T(f, \mu)$ is a continuous linear function on $C(X)$, and $\|I_{\Theta}^T(\cdot, \mu)\| = I_{\Theta}(x, T, m_x)$.*

Proof : (i) Let ξ be a generator. Let $\mu \in M(X, T)$. By Theorem 2.11, we have

$$I_{\Theta}(x, T, \xi) = I_{\Theta}(x, T, m_x)$$

for arbitrary $x \in X$. Therefore,

$$I_{\Theta}^T(1, \mu) = \int_X I_{\Theta}(T, m_x) d\mu(x) = I_{\Theta}(x, T, m_x).$$

(ii) Let ξ be a relative generator. Let $f \in C(X)$, then,

$$\begin{aligned} |I_{\Theta}^T(f, \mu)| &= \left| \int_X f(x) I_{\Theta}(x, T, \xi) d\mu(x) \right| \leq \int_X |f(x)| I_{\Theta}(x, T, m) d\mu(x) \\ &\leq \|f\|_{\infty} \int_X I_{\Theta}(x, T, m) d\mu(x) = \|f\|_{\infty} I_{\Theta}^T(1, \mu) = \|f\|_{\infty} I_{\Theta}(x, T, m_x) \end{aligned}$$

Therefore, the relative information functional is a continuous function and $\|I_{\Theta}^T(\cdot, m)\| \leq I_{\Theta}^S(T, m_x)$. The equality holds by (i). \square

4. CONCLUSIONS

Constant observers appear in many physical systems. In this paper, we introduced the notion of relative information functional for relative dynamical systems on compact metric spaces from the view point of observer Θ . It is a continuous linear function on $C(X)$ such that its norm equals the relative information of T at each $x \in X$. We also generalized the invariance of the information function of a dynamical system, under topological isomorphism, to the relative information functional.

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