FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES FOR (I, J)-MINIMAX MODULES

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ABSTRACT. Let R be a commutative noetherian ring and let I and J be two ideals of R. In this paper, we introduce the concept of (I, J)-minimax R-module and it is shown that if M is an (I, J)-minimax R-module and t a non-negative integer such that $H^i_{I,J}(M)$ is (I, J)-minimax for all $i < t$, then for any (I, J)-minimax submodule N of $H^t_{I,J}(M)$, the R-module $\operatorname{Hom}_R(R/I, H^t_{I,J}(M)/N)$ is (I, J)-minimax. As a consequence, it follows that the Goldie dimension of $H^t_{I,J}(M)/N$ is finite and so the set of associated primes of $H^t_{I,J}(M)/N$ is finite. This generalizes the main result of Azami, Naghipour and Vakili [2, Theorem 4.2].

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity. The generalized local cohomology module with respect to a pair of ideals I and J of R is introduced in [12].

We are concerned with the subset

$$W(I,J) = \{p \in \operatorname{Spec}(R) \mid I^n \subseteq p + J \text{ for some integer } n \geq 1\}$$

of $\operatorname{Spec}(R)$. For an R-module M, we consider the (I,J)-torsion submodule $\Gamma_{I,J}(M)$ of M which consists of all elements x of M with $\operatorname{Supp}(Rx) \subseteq W(I,J)$. By [12, Corollary 1.8], we have $\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \text{ for some integer } n \geq 1\}$. Furthermore, for an integer i, we define the local cohomology functor $H^i_{I,J}(-)$ with respect to (I,J) to be the i-th right derived functor of $\Gamma_{I,J}(-)$. Note that if $J = 0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$, with the support in the closed subset $V(I)$. On the other hand, if J contains I, then $\Gamma_{I,J}(-)$ is the identity functor and $H^i_{I,J}(-) = 0$, for $i > 0$ [12].

In [3], Bordmann and Lashgari showed that if for a finitely generated R-module M and an integer t, the local cohomology modules $H^0_I(M), H^1_I(M), \cdots, H^{t-1}_I(M)$ are finitely generated, then the set $\operatorname{Ass}_R(H^t_I(M)/N)$ is finite for every finitely generated submodule N of $H^t_I(M)$.

In [2], Azami, Naghipour and Vakili showed that if M is an I-minimax R-module and t non-negative integer such that $H^i_I(M)$ is I-minimax for all $i < t$, then for any I-minimax submodule N of $H^t_I(M)$, the R-module $\operatorname{Hom}_R(R/I, H^t_I(M)/N)$ is I-minimax. It follows that the Goldie dimension of $H^t_I(M)/N$ is finite and so the associated primes of $H^t_a(M)/N$ are finite. This generalizes the main result of Brodmann and Lashgari [3]. One of the main tools for proving above mentioned result in [2] is the following statement which is the following proposition.

Proposition 1.1. *([2, Theorem 2.7]) Let R be a Noetherian ring and M be a finitely generated R-module and N an arbitrary R-module. Let t be a non-negative integer such that $\operatorname{Ext}^i_R(M,N)$ is I-minimax for all $i \leq t$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, $\operatorname{Ext}^i_R(L,N)$ is I-minimax for all $i \leq t$.*

This paper is concerned with what might be considered a generalization of the above-mentioned result of Azami, Naghipour and Vakili to the class of (I, J)-minimax modules. More precisely, we shall show that:

Theorem 1.2. *Let R be a Noetherian ring and let I and J be two ideas of R and M be an (I, J)-minimax R-module. Let t be a non-negative integer such that $H^i_{I,J}(M)$ is (I, J)-minimax for all $i < t$. Then for any (I, J)-minimax submodule N of $H^t_{I,J}(M)$ the R-module $\operatorname{Hom}(R/I, H^t_{I,J}(M)/N)$ is (I, J)-minimax. In particular, the Goldie dimension of $H^t_{I,J}(M)/N$ is finite and so the set $\operatorname{Ass}_R(H^t_{I,J}(M)/N)$ is finite.*

Recall that an R-module M is said to have finite Goldie dimension (written $G\dim M < \infty$) if M dose not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(M)$ of M decomposes as a finite direct sum of indecomposable (injective) submodules, see [9, Section A6], in particular, [9, Definition 6.2, Proposition 6.4 and 6.12]. One notices that [9] uses uniform dimension instead of Goldie dimension. Also, an R-module M is said to have finite I-relative Goldie dimension if the I-torsion submodule $\Gamma_I(M) := \bigcup_{n\geq 1}(o :_M I^n)$ of M is finitely generated.

An R-module M is said to have finite (I, J)-relative Goldie dimension if the Goldie dimension of the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M is finite.

We say that an R-module M is I-minimax if the I-relative Goldie dimension of any quotient module of M is finite. Also, an R-module M is (I, J)-minimax if the (I, J)-relative Goldie dimension of any quotient module of M is finite. One of our tools for proving Theorem 1.2 is the following proposition.

Proposition 1.3. *Let R be a Notherian ring and let I and J be ideals of R. Let M be a finitely generated R-module and N an arbitrary R-module. Let t be a non-negative integer such that $\operatorname{Ext}^i_R(M, N)$ is (I, J)-minimax for all $i \leq t$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, $\operatorname{Ext}^i_R(L, N)$ is (I, J)-minimax for all $i \leq t$.*

Let $\tilde{W}(I, J)$ denote the set of all ideals a of R such that $I^n \subseteq a + J$ for some non-negative integer n. We define a partial order on $\tilde{W}(I, J)$ by letting $a \leq b$ if $a \supseteq b$ for $a, b \in \tilde{W}(I, J)$. If $a \leq b$, we have $\Gamma_a(M) \subseteq \Gamma_b(M)$. The order relation on $\tilde{W}(I, J)$

and inclusion maps make $\{\Gamma_a(M)\}_{a\in\tilde{W}(I,J)}$ into a direct system of R-modules. By [12, Theorem 3.2] we have:

$$H^i_{I,J}(M) \cong \varinjlim_{a\in\tilde{W}(I,J)} H^i_a(M)$$

for any integer i, where

$$H^i_a(M) = \varinjlim_{n\geq 1} H^i_a(M)\,\mathrm{Ext}^i_R(R/a^n, M).$$

We refer the reader to [4] and [12] for the basic properties of local cohomology.

2. I-minimax, (I,J)-minimax and Goldie dimension

For an R-module M, the Goldie dimension of M is defined as the cardinal of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules [9, Proposition 6.12]. We shall use $G\dim M$ to denote the Goldie dimension of M. For a prime ideal p, let $\mu^0(p,M)$ denote the 0-th bass number of M with respect to the prime ideal p, that is, $\mu^0(p,M) = \dim_{R_p/pR_p} \mathrm{Hom}_{R_p}(Rp/pR_p, M_p)$. It is known that $\mu^0(p,M) > 0$ if and only if $p \in \mathrm{Ass}_R(M)$. Indeed, for a $p \in \mathrm{Spec}(R)$, let $\mathrm{Hom}_{R_p}(R_p/pR_p, M_p) \neq 0$. So $(\mathrm{Hom}_R(R/p,M))_p \neq 0$ and let $f \in \mathrm{Hom}_R(R/p,M)$ such that $f \neq 0$ in $(\mathrm{Hom}_R(R/p,M))_p$.We show that f is a monomorphism. Contrary, let $r \notin p$ and $f(r+p) = 0$. It follows that $rf(1+p) = 0$, then $rf = 0$. So $\frac{f}{1} = 0$ in $(\mathrm{Hom}_R(R/p,M))_p$. This contraction shows that f is a monomorphism. Hence, $p \in \mathrm{Ass}_R(M)$. Conversely, let $p \in \mathrm{Ass}_R(M)$. It follows that R/p is isomorphic to a submodule of M. Hence R_p/pR_p is isomorphic to a submodule of Mp. So that $\mathrm{Hom}_{R_p}(R_p/pR_p, M_p) \neq 0$.
It follows from [9, Proposition 6.12] and the decomposition $E(M) = \bigoplus_{p\in\mathrm{Ass}_R(M)} \mu^0(p,M)E(R/p)$ of [5, Theorem 3.2.8] that

$$G\dim M = \sum_{p\in\mathrm{Ass}_R(M)} \mu^0(p,M).$$

In view of this, for any ideal I of R and any R-module M, the I-relative Goldie dimension of M is defined as

$$G\dim_I M := \sum_{p\in V(I)} \mu^0(p,M).$$

The I-relative Goldie dimension of an R-module M has been studied in [6]. Motivating, for any two ideals I and J of R and any R-module M,we define the (I, J)-relative Goldie dimension of M as

$$G\dim_{(I,J)} M := \sum_{p\in W(I,J)} \mu^0(p, M).$$

In [15], H. Zöschinger introduced the interesting class of minimax modules and in [15] and [16] gave some equivalent conditions for a module to be minimax. The R-module M is said to be a minimax module if there is a finitely generated submodule N of M, such that M/N is Artinian. It was shown by T. Zink [14] and by E. Enochs [7] that a module over a complete local ring is minimax if and only if it is matlis reflexive. On the other hand, it is known that when R is a Noetherian ring, an R-module is minimax if and only if each of its quotient has finite Goldie dimension, [14] or [16]. This motivates the following definition:

Definition 2.1. Let I and J be two ideals of R. An R-module M is said to be minimax with respect to I or I-minimax if the I-relative Goldie dimension of any quotient module of M is finite, i.e., for any submodule N of M, $G\dim_I(M/N) < \infty$. Also, an R-module M is said to be minimax with respect to I and J or (I, J)-minimax if the (I, J)-relative Goldie dimention of any quotient module of M is finite, i.e., for any submodule N of M, $G\dim_{(I,J)}(M/N) < \infty$.

Lemma 2.2. *Let I and J be two ideals of R and M be an injective R-module. Then $\Gamma_{I,J}(M)$ is an injective R-module.*

Proof. By [12, Theorem 3.2], we have $H^i_{I,J}(M) \cong \varinjlim_{a\in\tilde{W}(I,J)} H^i_a(M)$. When $i = 0$, $\Gamma_{I,J}(M) \cong \varinjlim_{a\in\tilde{W}(I,J)} \Gamma_a(M)$, by [12, Theorem 3.2]. $\Gamma_a(M)$ is an injective R-module by [4, Proposition 2.1.4]. Since R is a Notherian ring, by [8, Theorem 3.1.17], $\Gamma_{I,J}(M)$ is an injective R-module. □

Proposition 2.3. *Let I and J be two ideals of R and M an R-module. Then $G\dim_{(I,J)} M = G\dim \Gamma_{I,J}(M)$.*

Proof. Let p be a prime ideal of R. By [12, Proposition 1.11], if $p \in W(I, J)$, then $\Gamma_{I,J}(E(R/p)) = E(R/p)$ and if $p \notin W(I, J)$, then $\Gamma_{I,J}(E(R/p)) = 0$. Hence, using

[5, Theorem 3.2.8], we have

$$\begin{aligned}\Gamma_{I,J}(E(M)) &= \Gamma_{I,J}(\bigoplus_{p\in \mathrm{Spec}(R)} \mu^0(p,M)E(R/p)\\ &= \bigoplus_{p\in \mathrm{Spec}(R)} \mu^0(p,M)\Gamma_{I,J}(E(R/p))\\ &= \bigoplus_{p\in W(I,J)} \mu^0(p,M)E(R/p)\end{aligned}$$

It is easy to see that $\Gamma_{I,J}(E(M))$ is an essential extension of $\Gamma_{I,J}(M)$. On the other hand $\Gamma_{I,J}(E(M))$ is an injective R-module by Lemma 2.2. Hence $\Gamma_{I,J}(E(M)) \cong E(\Gamma_{I,J}(M))$. Thus

$$G\dim_{(I,J)} M = \sum_{p\in W(I,J)} \mu^0(p,M) = G\dim \Gamma_{I,J}(M).$$

□

Corollary 2.4. *If M is (I,J)-torsion, then M is (I,J)-minimax if and only if M is minimax.*

Proof. The assertion follows from Proposition 2.3. □

Remark 2.5. Let I and J be two ideals of R and let M be an R-module.
(i) Assume that $I = 0$. Then M is $(0,J)$-minimax if and only if M is minimax.
(ii)If I' and J' be two ideals of R such that $I' \subseteq I$ and $J \subseteq J'$ and M is (I',J')-minimax, then M is (I,J)-minimax. In particular, every minimax module is (I,J)-minimax.
(iii) If M is Noethrian or Artinian, then M is (I,J)-minimax.

Proof. (i) Clearly $W(0,J) = \mathrm{Spec}(R)$. Hence $G\dim_{(0,J)} M/N = G\dim M/N$ for any submodule N of M. This complete the proof of (i).
(ii) Let I' and J' be two ideals of R such that $I' \subseteq I$ and $J \subseteq J'$.We then have $W(I,J) \subseteq W(I',J')$. So that

$$G\dim_{(I,J)} M/N = \sum_{p\in W(I,J)} \mu^0(p,M) \leq \sum_{p\in W(I',J')} \mu^0(p,M) = G\dim_{(I',J')} M/N$$

for any submodule N of M. This proves the assertion.
(iii) Assume that M is Noetherian or Artinian. Then M is minimax by definition. Hence, by (ii), M is (I,J)-minimax. □

The following proposition is needed in the proof of the main theorem of this paper.

Proposition 2.6. *Let I and J be two ideals of R and let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules. Then M is (I, J)-minimax if and only if M' and M'' are both (I, J)-minimax.

Proof. Assume that M' is a submodule of M and that $M'' = M/M'$. If M is (I, J)-minimax, then from the definition clearly that $M'$and M/M' are (I, J)-minimax. Now suppose that M' and M/M' are (I, J)-minimax. Let N be an arbitrary submodule of M and let $p \in \mathrm{Ass}(M/N) \cap W(I, J)$. Then the exact sequence

$$0 \to \frac{M' + N}{N} \to \frac{M}{N} \to \frac{M}{M' + N} \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}_{R_p}(k(p), \frac{M'_p}{M'_p \cap N_p}) \to \mathrm{Hom}_{R_p}(k(p), \frac{M_p}{N_p}) \to \mathrm{Hom}_{R_p}(k(p), \frac{M_p}{M'_p + N_p}),$$

where $k(p) = R_p/pR_p$. Moreover, since $\mathrm{Ass}_R(M/N) \subseteq \mathrm{Ass}_R(\frac{M'+N}{N}) \cup \mathrm{Ass}(\frac{M}{M'+N})$ and the sets $\mathrm{Ass}_R(\frac{M'+N}{N}) \cap W(I, J)$ and $\mathrm{Ass}_R(\frac{M}{M'+N}) \cap W(I, J)$ are finite, it follows that $G\dim_{(I,J)}(M/N) < \infty$ and so M is (I, J)-minimax. □

Corollary 2.7. *Let I and J be two ideals of R. Then any quotient and any finite direct sum of (I, J)-minimax modules, is (I, J)-minimax.*

Proof. The assertion follows from the definition and Proposition 2.6. □

Corollary 2.8. *Let I and J be two ideals of R and let M be a finitely generated R-module and N be an (I, J)-minimax R-module. Then $\mathrm{Ext}^i_R(M, N)$ and $\mathrm{Tor}^R_i(M, N)$ are (I, J)-minimax modules for all i. In particular, the R-modules $\mathrm{Ext}^i_R(R/I, N)$ and $\mathrm{Tor}^R_i(R/I, N)$ are (I, J)-minimax for all i.*

Proof. Since R is Noetherian and M is finitely generated, it follows that M possesses a free resolution

$$\mathbb{F}_\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

whose free modules have finite ranks.

Thus $\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(\mathbb{F}_\bullet, N))$ is subquotient of a direct sum of finitely many copies of N. Therefore, it follows from Corollary 2.7 that $\mathrm{Ext}^i_R(M,N)$ is (I,J)-minimax for all $i \geq 0$. By using a similar proof as above we can deduce that $\mathrm{Tor}^R_i(M,N)$ is (I,J)-minimax for all $i \geq 0$. □

Proposition 2.9. *Let I and J be two ideas of R and let M be an (I,J)-minimax R-module such that $\mathrm{Ass}_R(M) \subseteq W(I,J)$. Then $H^i_{I,J}(M)$ is (I,J)-minimax for all $i \geq 0$.*

Proof. If $i = 0$, then $H^0_{I,J}(M) = \Gamma_{I,J}(M)$ is a submodule of M and by Proposition 2.6, $\Gamma_{I,J}(M)$ is (I,J)-minimax. As $\mathrm{Ass}_R(M) \subseteq W(I,J)$, by [12, Proposition 1.7], M is an (I,J)-torsion R-module and so $M = \Gamma_{I,J}(M)$. Consequently, by [12, Corollary 1.13], $H^i_{I,J}(M) = 0$ for all $i > 0$ and so $H^i_{I,J}(M)$ is (I,J)-minimax for all $i \geq 0$, as required. □

Now we state Gruson's Theorem that will be needed.

Theorem 2.10. *[13, Theorem 4.1] (Gruson's Theorem) Let M be a finitely generated R-module. If L is a finitely generated R-module with $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, M$, then there exists a chain*

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of M

Theorem 2.11. *Let I and J be two ideals of R. Let M be a finitely generated R-module and N an arbitrary R-module. Let t be a non-negative integer such that $\mathrm{Ext}^i_R(M,N)$ is (I,J)-minimax for all $i \leq t$. Then for any finitely generated R-module L with $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, M$, the module $\mathrm{Ext}^i_R(L,N)$ is (I,J)-minimax for all $i \leq t$.*

Proof. Since $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, M$, according to Lemma 2.10 there exists a chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,$$

of R-module such that the modules L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of M. Now consider the exact sequences

$$0 \to K \to M^n \to L_1 \to 0$$
$$0 \to L_1 \to L_2 \to L_2/L_1 \to 0$$
$$\vdots$$
$$0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0$$

for some positive integer n.

Now from the long exact sequence

$$\cdots \to \mathrm{Ext}_R^{i-1}(L_{j-1}, N) \to \mathrm{Ext}_R^i(L_j/L_{j-1}, N) \to \mathrm{Ext}_R^i(L_j, N) \to \mathrm{Ext}_R^i(L_{j-1}, N) \to \cdots$$

and an easy induction on k, it suffices to prove the case when $k = 1$.

Thus there is an exact sequence

$$(*) \qquad 0 \to K \to M^n \to L \to 0$$

for some $n \in \mathbb{N}$ and some finitely generated R-module K.

Now, we use induction on t. First, $\mathrm{Hom}_R(L, N)$ is a submodule of $\mathrm{Hom}_R(M^n, N)$, hence in view of the assumption and Corollary 2.7 $\mathrm{Ext}_R^0(L, N)$ is (I, J)-minimax. So assume that $t > 0$ and that $\mathrm{Ext}_R^j(L', N)$ is (I, J)-minimax for every finitely generated R-module L' with $\mathrm{Supp}\, L' \subseteq \mathrm{Supp}\, M$ and for all $j \leq t - 1$. Now the exact sequence $(*)$ induces the long exact sequence

$$\cdots \to \mathrm{Ext}_R^{i-1}(K, N) \to \mathrm{Ext}_R^i(L, N) \to \mathrm{Ext}_R^i(M^n, N) \to \cdots .$$

Hence, by the inductive hypothesis, $\mathrm{Ext}_R^{i-1}(K, N)$ is (I, J)-minimax for all $i \leq t$. On the other hand, according to Corollary 2.7, since $\mathrm{Ext}_R^i(M^n, N) \cong \bigoplus^n \mathrm{Ext}_R^i(M, N)$, $\mathrm{Ext}_R^i(M^n, N)$ is (I, J)-minimax. Therefore, it follows from Proposition 2.6 that $\mathrm{Ext}_R^i(L, N)$ is (I, J)-minimax for all $i \leq t$ and the inductive step is complete. □

Corollary 2.12. *Let I and J be two ideals of R and let t be a non-negative integer. Then for any R-module M the following conditions are equivalent:*
(i) $\mathrm{Ext}_R^i(R/I, M)$ *is (I, J)-minimax for all $i \leq t$.*
(ii) For any ideal I' of R with $I' \supseteq I$, $\mathrm{Ext}_R^i(R/I', M)$ *is (I', J)-minimax for all $i \leq t$.*
(iii) For any finitely generated R-module N with $\mathrm{Supp}\, N \subseteq W(I, J)$*,* $\mathrm{Ext}_R^i(N, M)$

is (I,J)*-minimax for all* $i \leq t$.
(iv) For any minimal prime ideal p *over* I, $\operatorname{Ext}^i_R(R/p,M)$ *is* (I,J)*-minimax for all* $i \leq t$.

Proof. (i)⇒ (ii) Since $\operatorname{Supp}_R(R/I') = V(I') \subseteq V(I) = \operatorname{Supp}_R(R/I)$, we have $\operatorname{Ext}^i_R(R/I',M)$ is (I,J)-minimax for all $i \leq t$ by Theorem 2.11. Now it follows from remark 2.5 (ii) that $\operatorname{Ext}^i_R(R/I',M)$ is (I',J)-minimax for all $i \leq t$.
(ii)⇒ (iii) This parts follows from [1, Exercise 7.18] using induction.
(iii)⇒ (iv) Let p be a minimal prime ideal over I. Then $\operatorname{Supp}_R(R/P) = V(p) \subseteq V(I)$. Hence, $\operatorname{Ext}^i_R(R/p,M)$ is I-minimax for all $i \leq t$.
(iv)⇒ (i) Let $p_1, \cdots, p_n$ be the minimal primes of I. Then by assumption, the R-modules $\operatorname{Ext}^i_R(R/p_j,M)$ are (I,J)-minimax for each $j \in \{1,2,\cdots,n\}$. Hence by Corollary 2.7, $\bigoplus_{j=1}^{n} \operatorname{Ext}^i_R(R/p_j,M) \cong \operatorname{Ext}^i_R(\bigoplus_{j=1}^{n} R/p_j, M)$ is (I,J)-minimax. Since $\operatorname{Supp}(\bigoplus_{j=1}^{n} R/p_j) = \operatorname{Supp} R/I$, it follows from Theorem 2.11 that $\operatorname{Ext}^i_R(R/I,M)$ is (I,J)-minimax, as required. □

3. (I,J)-Cominimax modules and local cohomology

Let R be a Notherian ring and I and J be two ideals of R and M be an R-module. Recall that M is said to be (I,J)-cofinite if M has support in $W(I,J)$ and $\operatorname{Ext}^i_R(R/I,M)$ is finitely generated R-module for each i. This motivates the following definition:

Definition 3.1. Let R be a Notherian ring and let I and J be two ideals of R. We say that an R-module M is (I,J)-cominimax if $\operatorname{Supp} M \subseteq W(I,J)$ and $\operatorname{Ext}^i_R(R/I,M)$ is (I,J)-minimax for all $i \geq 0$.

Example 3.2. (i) Let I and J be two ideals of R and let M be an (I,J)-minimax R-module such that $\operatorname{Supp} M \subseteq W(I,J)$. Then it follows from Corollary 2.8 that M is (I,J)-cominimax. In particular, every minimax R-module with support in $W(I,J)$ is (I,J)-cominimax.
(ii) Let I and J be two ideals of R. Then every (I,J)-cofinite R-module is (I,J)-cominimax. In particular, any Noetherian or Arthinian R-module with support in $W(I,J)$ is (I,J)-cominimax.
(iii) Let I and J be two ideals of R and let N be a pure submodule of an R-module

M. Then M is (I, J)-cominimax if and only if N and M/N are (I, J)-cominimax. In fact, P. M. Cohn's characterization of purity (see [11, Theorem 3.56]) implies that the sequence

$$0 \to \mathrm{Ext}^i_R(R/I, N) \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M/N) \to 0$$

is exact for all i (see also the proof of [10, Proposition 2.7]). Hence, the result follows from Proposition 2.6.

Proposition 3.3. *Let I and J be two ideals of R. Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules such that two of the modules are (I, J)-cominimax. Then so is the third one.

Proof. The exact sequence

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M'') \to \mathrm{Ext}^{i+1}_R(R/I, M') \to \mathrm{Ext}^{i+1}_R(R/I, M) \to \cdots .$$

Now the result follows easily from Proposition 2.6. □

Corollary 3.4. *Let I and J be two ideals of R. Let $f : M \to N$ be a homomorphism between two (I, J)-cominimax modules such that one of three modules* $\mathrm{Ker}\, f$*,* $\mathrm{Im}\, f$ *and* $\mathrm{Coker}\, f$ *is (I, J)-cominimax. Then all of them are (I, J)-cominimax.*

Proof. The result follows from Proposition 3.3 and the following exact sequences.

$$0 \to \mathrm{Ker}\, f \to M \to \mathrm{Im}\, f \to 0,$$
$$0 \to \mathrm{Im}\, f \to N \to \mathrm{Coker}\, f \to 0.$$

□

Proposition 3.5. *Let I and J be two ideals of R and let M be an R-module such that* $\mathrm{Supp}\, M \subseteq W(I, J)$ *and $(0 :_M I)$ has finite Goldie dimension. Then M has finite Goldie dimension.*

Proof. Since $(0 :_M I)$ has finite Goldie dimension and $\operatorname{Supp} M \subseteq W(I, J)$, by [5, Exercise 1.2.27], $\operatorname{Ass}_R(M)$ is finite. On the other hand, for any $p \in \operatorname{Ass}_R(M)$, one easily has $0 :_{M_p} pR_p = 0 :_{(0:_{M_p} IR_p)} pR_p$ since $p \supseteq I$. Then we have

$$\begin{aligned} \operatorname{Hom}_{R_p}(k(p), M_p) &= \operatorname{Hom}_{R_p}(R_p/pR_p, M_p) \\ &\cong 0 :_{M_p} pR_p \\ &= 0 :_{(0:_{M_p} IR_p)} pR_p \\ &\cong \operatorname{Hom}_{R_p}(R_p/pR_p, 0 :_{M_p} IR_p) \\ &= \operatorname{Hom}_{R_p}(k(p), 0 :_{M_p} IR_p), \end{aligned}$$

as $k(p)$-vector spaces, where $k(p) = R_p/pR_p$. Therefore, $\mu^0(p, M)$ is finite and so $G \dim M < \infty$. □

Corollary 3.6. *Let I and J be two ideals of R and let M be an (I, J)-cominimax R-module. Then M has finite Goldie dimension. In particular the set of associated primes of M is finite.*

Proof. By Proposition 3.5. □

Proposition 3.7. *Let I and J be two ideals of R. Let M be an R-module such that $H^i_{I,J}(M)$ is (I, J)-cominimax for all i. Then $\operatorname{Ext}^i_R(R/I, M)$ is (I, J)-minimax for all i.*

Proof. It is well-known that $\operatorname{Hom}_R(R/I, M) \cong 0 :_M I$. Then we have

$$\begin{aligned} \operatorname{Hom}_R(R/I, M) &\cong 0 :_M I \\ &= 0 :_{\Gamma_{I,J}(M)} I \\ &\cong \operatorname{Hom}_R(R/I, \Gamma_{I,J}(M)) \\ &\cong \operatorname{Ext}^0_R(R/I, \Gamma_{I,J}(M)). \end{aligned}$$

Therefor for $i = 0$ the statement is true. Let $i > 0$ and do induction on i. We first reduce to the case $\Gamma_{I,J}(M) = 0$. To do this, let $\bar{M} = M/\Gamma_{I,J}(M)$. Then we have the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, \bar{M}) \to \cdots,$$

and the isomorphism $H^i_{I,J}(M) \cong H^i_{I,J}(\bar{M})$ for $i > 0$, by [12, Corollary 1.13]. So in view of Proposition 2.6, we may assume that M is (I, J)-torsion free. Let E be the injective envelop of M and set $L := E/M$. Since $\Gamma_{I,J}(M) = 0$, we have $\Gamma_{I,J}(E) \cap M = 0$. It follows that $\Gamma_{I,J}(E) = 0$. Then $\mathrm{Hom}_R(R/I, E) = 0$ and we therefore get the isomorphisms $H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M)$ and $\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Proposition 3.8. *Let I and J be two ideals of R and let M be an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ is (I, J)-minimax for all i. If t is non-negative integer such that $H^i_{I,J}(M)$ is (I, J)-cominimax for all $i \neq t$, then $H^t_{I,J}(M)$ is (I, J)-cominimax.*

Proof. We use induction on t. Let $\bar{M} := M/\Gamma_{I,J}(M)$. Then by [12, Corollary 1.13], if $i > 0$, then $H^i_{I,J}(\bar{M}) \cong H^i_{I,J}(M)$ and if $i = 0$, then $H^i_{I,J}(\bar{M}) = 0$. If $t = 0$, then $H^i_{I,J}(\bar{M})$ is (I, J)-cominimax for all i. Hence by Proposition 3.7, $\mathrm{Ext}^i_R(R/I, \bar{M})$ is (I, J)-minimax for all i. Therefor the exactness of $0 \to \Gamma_{I,J}(M) \to M \to \bar{M} \to 0$ implies that $\mathrm{Ext}^i_R(R, /I, \Gamma_{I,J}(M))$ is (I, J)-minimax for all i. It follows that $\Gamma_{I,J}(M)$ is (I, J)-cominimax. Let $t > 0$ and suppose that the result has been proved for $t - 1$. Since $\Gamma_{I,J}(M)$ is (I, J)-cominimax, the exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, \bar{M}) \to \cdots$$

allows us to assume that M is (I, J)-torsion free. Let E be the injective envelope of M and put $L = E/M$. Then $\Gamma_{I,J}(E) = 0$ and $\mathrm{Hom}_R(R/I, E) = 0$ and we therefore get the isomorphisms $H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M)$ and $\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Corollary 3.9. *Let I and J be two ideals of R and let M be an (I, J)-minimax R-module. If t is a non- negative integer such that $H^i_{I,J}(M)$ is (I, J)-cominimax for all $i \neq t$, then $H^t_{I,J}(M)$ is (I, J)-cominimax.*

Proof. This follows from Corollary 2.8 and Proposition 3.8. □

Proposition 3.10. *Let I and J be two ideals of R such that $I \subseteq J$ and M an (I, J)-minimax R-module. Then $H^i_{I,J}(M)$ is (I, J)-cominimax.*

Proof. Since $H^0_{I,J}(M)$ is a submodule of M, it turns out that $H^0_{I,J}(M)$ is (I, J)-cominimax by Proposition 2.6 and Example 3.2 (i). Since $I \subseteq J$, it is easy that $\Gamma_{I,J}(-)$ is the identity functor and $H^i_{I,J}(-) = 0$ for all $i > 0$. Therefore $H^i_{I,J}(M)$ is (I, J)-cominimax. □

4. Finiteness of associated primes

In this section, we show that the subjects of the previous sections can be used to prove a finiteness result about local cohomology modules. In fact, we generalize the main result about of Azami, Naghipour and Vakili to (I,J)-minimax modules. The main result is Theorem 4.2. The following theorem will serve to shorten the proof of the main theorem.

Theorem 4.1. *Let I and J be two ideals of R and let M be an R-module. Let t be a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-cominimax for all $i < t$ and $\operatorname{Ext}^i_R(R/I, M)$ is (I,J)-minimax. Then fot any (I,J)-minimax submodule N of $H^t_{I,J}(M)$ and for any finitely generated R-module L with $\operatorname{Supp} L \subseteq W(I,J)$, the R-module $\operatorname{Hom}_R(L, H^t_{I,J}(M)/N)$ is (I,J)-minimax.*

Proof. The exact sequence

$$0 \to N \to H^t_{I,J}(M) \to H^t_{I,J}(M)/N \to 0$$

provides the following exact sequence:

$$\operatorname{Hom}_R(L, H^t_{I,J}(M)) \to \operatorname{Hom}_R(L, H^t_{I,J}(M)/N) \to \operatorname{Ext}^1_R(L, N) \to \cdots .$$

By Corollary 2.8, $\operatorname{Ext}^1_R(L,N)$ is (I,J)-minimax,and so in view of Proposition 2.6 it is sufficient to show that the R-module $\operatorname{Hom}_R(L, H^t_{I,J}(M))$ is (I,J)-minimax. By Corollary 2.12, it is enough to show that the R-module $\operatorname{Hom}_R(R/I, H^t_{I,J}(M))$ is (I,J)-minimax.

We use induction on t. When $t = 0$, the R-module $\operatorname{Hom}_R(R/I, M)$ is (I,J)-minimax, by assumption. Since $0 :_M I = 0 :_{\Gamma_{I,J}(M)} I$, we have

$$\operatorname{Hom}_R(R/I, H^0_{I,J}(M)) \cong \operatorname{Hom}_R(R/I, \Gamma_{I,J}(M)) \cong \operatorname{Hom}_R(R/I, M),$$

it follows that $\operatorname{Hom}_R(R/I, H^0_{I,J}(M))$ is (I,J)-minimax.

Now suppose, inductively, that $t > 0$ and that the result is true for $t-1$. Since $\Gamma_{I,J}(M)$ is (I,J)-cominimax, it follows that $\operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is (I,J)-minimax for all $i \geq 0$. On the other hand, the exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to M/\Gamma_{I,J}(M) \to 0$$

induces the exact sequence

$$\operatorname{Ext}^t_R(R/I, M) \to \operatorname{Ext}^t_R(R/I, M/\Gamma_{I,J}(M)) \to \operatorname{Ext}^{t+1}_R(R/I, \Gamma_{I,J}(M)).$$

Hence, by Proposition 2.3 and the assumption, the R-module $\mathrm{Ext}^t_R(R/I, M/\Gamma_{I,J}(M))$ is (I,J)-minimax. Also since $H^0_{I,J}(M/\Gamma_{I,J}(M)) = 0$ and $H^i_{I,J}(M/\Gamma_{I,J}(M)) \cong H^i_{I,J}(M)$ for all $i > 0$, it follows that $H^i_{I,J}(M/\Gamma_{I,J}(M))$ is (I,J)-cominimax for all $i < t$. Therefor we may assume that M is (I,J)-torsion free. Let E be an injective envelope of M and put $M_1 := E/M$. Then $\Gamma_{I,J}(E) = 0$ and $\mathrm{Hom}_R(R/I, E) = 0$. Consequently, $\mathrm{Ext}^i_R(R/I, M_1) \cong \mathrm{Ext}^{i+1}_R(R/I, M)$ and $H^i_{I,J}(M_1) \cong H^{i+1}_{I,J}(M)$ for all $i \geq 0$ (including the case $i = 0$). The induction hypothesis applied to M_1 yields that $\mathrm{Hom}_R(R/I, H^{t-1}_{I,J}(M_1))$ is (I,J)-minimax. Hence $\mathrm{Hom}_R(R/I, H^t_{I,J}(M))$ is (I,J)-minimax. □

Now we are prepared to prove the main theorem of this section, which is a generalization of the main result of Azami, Naghipour and Vakili.

Theorem 4.2. *Let I and J be two ideals of R and let M be an (I,J)-minimax R-module. Let t be a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-minimax for all $i < t$. Then for any (I,J)-minimax submodule N of $H^t_{I,J}(M)$, the R-module $\mathrm{Hom}_R(R/I, H^t_{I,J}(M)/N)$ is (I,J)-minimax. In particular, the Goldie dimension of $H^t_{I,J}(M)/N$ is finite and so the set $\mathrm{Ass}_R(H^t_{I,J}(M)/N)$ is finite.*

Proof. Apply Theorem 4.1 and Corollary 2.8. □

Corollary 4.3. *Let R be a Notherian ring and let I, J be two ideals of R and M a finitely generated R-module. Let $\mathrm{Obj}(N)$ (resp. $\mathrm{Obj}(A)$) denote the category of all Noetherian (resp. Artinian) R-modules and R-homomorphisms. Let t be a non-negative integer such that $H^i_{I,J}(M) \in \mathrm{Obj}(N) \cup \mathrm{Obj}(A)$ for all $i < t$. Then the R-module $\mathrm{Hom}_R(R/I, H^t_{I,J}(M))$ is (I,J)-minimax and so the set $\mathrm{Ass}_R(H^t_{I,J}(M))$ is finite.*

Proof. Apply Theorem 4.1 and the fact that the class of (I,J)-minimax modules contains all Noetherian and Artinian modules. □

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