ON STRONGLY $H_v$-GROUPS

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Abstract. The largest class of hyperstructures is the one which satisfies the weak properties; these are called $H_v$-structures. In this paper we introduce a special product of elements in $H_v$-group and define a new class of $H_v$-groups called strongly $H_v$-groups. Then we show that in strongly $H_v$-groups $\beta = \beta^*$. Also we express $\theta$-hyperoperation and investigate some of its properties in connection with strongly $H_v$-groups.

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1. Introduction

The first definition of hyperoperation and hypergroup was adverted by Frederic Marty in the 8th Congress of Scandinavian Mathematicians in 1934. In 1990, in Greece, T. Vougiouklis introduced the concept of the weak hyperstructures which now are named $H_v$-structures. Over the last 28 years this class of hyperstructures, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics. Basically, the study of $H_v$-structures has
been continued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and some other mathematicians. We invite the readers for more study about hyperstructure theory and its applications to [1], [2], [3], [4], [7], [13] and [14]. We recall the following definitions from [1].

**Definition 1.1.** Let $H$ be a non-empty set and $* : H \times H \longrightarrow \mathcal{P}^r(H)$ be a hyperoperation. The couple $(H, *)$ is called a hypergroupoid. For any two non-empty subset $A$ and $B$ of $H$ and $x \in H$, we define

$$A \ast B = \bigcup_{a \in A, b \in B} a \ast b, \quad A \ast x = A \ast \{x\}.$$

**Definition 1.2.** A hypergroupoid $(H, *)$ is called hypergroup if for all $(x, y, z) \in H^3$, it satisfies the following conditions:

1. $(x \ast y) \ast z = x \ast (y \ast z)$, which means that

$$\bigcup_{u \in x \ast y} u \ast z = \bigcup_{v \in y \ast z} x \ast v,$$

2. $x \ast H = H = H \ast x$.

**Definition 1.3.** [2]. A hypergroupoid $(H, *)$ is called a $H_v$-group if the following axioms hold:

1. $x \ast (y \ast z) \cap (x \ast y) \ast z \neq \emptyset$ for all $(x, y, z) \in H^3$; (weak associativity)

2. $x \ast H = H = H \ast x$ for all $x$ in $H$. (reproduction)

In the following for $(x, y) \in (H^2, *)$, we write $xy$ instead of $x \ast y$.

**Example 1.4.** ([2], Example 6.1.2). Let $(G, \cdot)$ be a group and $R$ an equivalence relation on $G$. In $\overline{G}$ consider the hyperaction $\odot$ defined by

$$\overline{x} \odot \overline{y} = \{ z | z \in \overline{x \cdot y} \},$$

where $\overline{x}$ denotes the equivalence class of the element $x$. Then $(G, \odot)$ is an $H_v$-group which is not always a hypergroup.

2. **Strongly $H_v$-groups**

Consider a special product of elements of an $H_v$-group $H$. We introduce a notation as follows. Let $(x_1, x_2, ..., x_n) \in H^n$ and $V_{x,n} = V(x_1, x_2, ..., x_n)$ be the set
of all finite products of \(x_1, x_2, \ldots, x_n\), respectively. Also let \(V_n = \{V(x_1, x_2, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in H^n\}\) and \(V = \bigcup_{n \geq 1} V_n\). For example

\[
\begin{align*}
V_{x,2} &= \{x_1x_2\}, x = (x_1, x_2) \\
V_{x,3} &= \{x_1(x_2x_3), (x_1x_2)x_3\}, x = (x_1, x_2, x_3) \\
V_{x,4} &= \{x_1[x_2(x_3x_4)], x_1[(x_2x_3)x_4], [x_1x_2)x_3]x_4, [x_1(x_2x_3)]x_4, (x_1x_2)(x_3x_4)\}, \\
x &= (x_1, x_2, x_3, x_4)
\end{align*}
\]

Also let

\[
\begin{align*}
U_{x,2} &= U(x_1, x_2) = \{x_1x_2\} = x_1x_2 \\
U_{x,3} &= U(x_1, x_2, x_3) = \{(x_1x_2)x_3\} = (x_1x_2)x_3 \\
& \quad \vdots \\
& \quad \vdots \\
U_{x,n+1} &= U(x_1, x_2, \ldots, x_{n+1}) = U_{x,n}x_{n+1}.
\end{align*}
\]

Let \(U_n = \{U_{x,n} = U(x_1, x_2, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in H^n\}\) and \(U = \bigcup_{n \geq 1} U_n\). It is clear that \(U \subseteq V\). Now we define the class of strongly \(H_v\)-groups.

**Definition 2.1.** Let \(H\) be a \(H_v\)-group. We say that \(H\) is a strongly \(H_v\)-group if \(p \subseteq U_{x,n}\) for all \(p \in V_{x,n}\) and \(n \in \mathbb{N}\).

**Remark 2.2.** Due to the above notation if \(H\) is a strongly \(H_v\)-group then for all \(n, m \in \mathbb{N}\) we have \(U_nU_m \subseteq U_{n+m}\).

**Proposition 2.3.** The \(H_v\)-group \(H\) is strongly if and only if for all \((x, y, z) \in H^3\), \(x(yz) \subseteq (xy)z\).

**Proof.** Let \(H\) be strongly \(H_v\)-group, then by above notation it is clear that for all \((x, y, z) \in H^3\), \(x(yz) \subseteq (xy)z\). We prove the converse by induction on \(n\). Let \(x = (x_1, x_2, \ldots, x_n) \in H\) and \(v \in V_{x,n} = V(x_1, x_2, \ldots, x_n)\). If \(n = 3\), then by assumption we have \(x_1(x_2x_3) \subseteq (x_1x_2)x_3 = U_3\). Thus \(p \subseteq U_{x,3}\), for all \(p \in V_{x,3}\). Now let the the problem be true for all \(k < n\). We have \(v = wz\) such that \(w \in V(x_1, \ldots, x_r)\) and \(z \in V(x_{r+1}, \ldots, x_n)\), where \(r < n\). So there exist \(U_{x,r}, U_{x,n-r}\) such that \(w \subseteq\)
Thus \( v = wz \subseteq U_{x,r}U_{x,n-r} = (U_{x,r-1}x_r)(U_{x,n-r-1}x_n) \subseteq [(U_{x,r-1}x_r)U_{x,n-r-1}]x_n \subseteq U_{x,n-1}x_n = U_{x,n}. \) This completes the proof. \( \square \)

**Corollary 2.4.** The \( H_v \)-group \( H \) is strongly if and only if for all subsets \( A, B, C \) of \( H \), \( A(BC) \subseteq (AB)C \).

The main tools to study hyperstructures are the fundamental relations \( \beta^*, \gamma^* \) and \( \varepsilon^* \), which are defined, in \( H_v \)-groups, \( H_v \)-ring and \( H_v \)-vector spaces, as the smallest equivalences so that the quotients would be group, ring and vector space, respectively. These relations were introduced by T. Vougiouklis [6, 7]. A way to find the fundamental class is given by theorems as the following [8, 9, 10].

It is defined the relation \( \beta \) in \( H_v \)-group \( H \) by setting \( x \beta y \) if and only if \( \{x, y\} \subseteq p \), for some \( p \in V_{z,n} = V(z_1, ..., z_n) \) and \( n \in \mathbb{N} \) and \( (z_1, ..., z_n) \in H^n \).

**Theorem 2.5.** ([12], Theorem 1) Let \( H \) be an \( H_v \)-group. Then \( \beta^* \) is the transitive closure of \( \beta \).

It has been proved that if \( H \) is a hypergroup then \( \beta = \beta^* \) (see [5]). But in \( H_v \)-group so far not proven that \( \beta = \beta^* \). In the following we show that by considering special products to wit in strongly \( H_v \)-groups we can prove that \( \beta = \beta^* \). First we express the below definitions and explain the theorems such that their proofs in \( H_v \)-groups, due to the Proposition 2.3, are similar proving in hypergroups, that we avoid presenting their proofs. These theorems are expressed in [1] completely.

**Definition 2.6.** Let \( A \) be a subset of a \( H_v \)-group \( H \). \( A \) is called complete part if the following implication is valid:

\[
\forall n \in \mathbb{N}; \quad \forall p \in V_{x,n}; \quad p \cap A \neq \emptyset \Rightarrow p \subseteq A.
\]

**Definition 2.7.** Let \( A \) be a non-empty subset of \( H_v \)-group \( H \), the intersection of the subsets of \( H \) which are complete parts and contain \( A \) is called the complete closure of \( A \) in \( H \); it will be denoted by \( C(A) \).

**Theorem 2.8.** Let \( A \) be a non-empty subset of \( H_v \)-group \( H \), and let \( K_1(A) = A \), \( K_{n+1}(A) = \{x \in H \mid \exists m \in \mathbb{N}; \exists p \in V_{y,m}; \ x \in p, \ p \cap K_n(A) \neq \emptyset\}. \) Let \( K(A) = \cup_{n \geq 1} K_n(A) \). Then \( K(A) = C(A) \). (Note that \( V_{y,m} = V(y_1, ..., y_m) \)).

**Proof.** It is necessary to prove that i) \( K(A) \) is a complete part and ii) if \( A \subseteq B \) and \( B \) is complete part then \( K(A) \subseteq B \).
i) Let \( m \in \mathbb{N} \) and \( p \in V_{y,m} \) and \( p \cap K(A) \neq \emptyset \). Then there exists \( n \in \mathbb{N} \) such that \( p \cap K_n(A) \neq \emptyset \), from which follows \( p \subseteq K_{n+1}(A) \subseteq K(A) \).

ii) \( A \subseteq K_1(A) \). Suppose \( K_n(A) \subseteq B \), this implies that \( K_n(A) \subseteq B \), for every \( n \in \mathbb{N} \).

**Theorem 2.9.** The relation \( xKy \iff x \in C(y) = \bigcup_{n \geq 1} K_n(y) \) is an equivalence.

*Proof.* \( K \) is clearly reflexive. Now let \( xKy \) and \( yKz \). If \( P \) is a complete part and \( z \in P \), then \( C(z) \subseteq P \) thus \( y \in P \) and consequently \( x \in C(y) \subseteq P \) and so \( xKz \). □

**Theorem 2.10.** For all \( x, y \) in \( H_v \)-group \( H \), we have \( xKy \iff x\beta^*y \).

*Proof.* The proof is similar to the proof of Theorem 57 in [1]. □

**Definition 2.11.** Let \( H \) be a \( H_v \)-group and \( \varphi_H : H \rightarrow \beta^*H \) the canonical projection. The kernel of \( \varphi_H \) is called heart (or core) of \( H \) and denoted by \( \omega_H \), i.e. \( \omega_H = \{ x \in H \mid \varphi_H(x) = 1 \} \).

**Theorem 2.12.** \( \omega_H \) is the smallest subhypergroup of \( H \) that is complete part.

**Remark 2.13.** For all \( z \in \omega_H \) we have \( \omega_H = C(z) \), since \( x \in C(z) \iff xKz \iff x=\beta^*z \iff y \in \varphi_H(x) = \varphi_H(z) = 1 \iff x \in \omega_H \).

**Theorem 2.14.** If \( B \) is a non-empty subset of \( H_v \)-group \( H \), then \( \varphi_H^{-1}(B) = \omega_H B = B \omega_H = C(B) = \bigcup_{b \in B} C(b) \).

*Proof.* The structure of the proof is like to the proof of Theorem 66 in [1]. □

**Corollary 2.15.** By Theorems 2.9, 2.10, 2.14 for all \( x \in H \) we obtain \( \beta^*(x) = \{ y \mid y \beta^*x \} = \{ y \mid yKx \} = \{ y \mid y \in C(x) \} = \{ y \mid y \in x\omega_H \} \).

**Corollary 2.16.** For all \( (x, y) \in H^2 \), we have \( x\beta^*y \iff xKy \iff y \in C(x) = x\omega_H \).

Now let \( P(z) \) be as follows:

\[
P(z) = \{ U_{x,n} \in U_n \mid z \in U_{x,n}, n \in \mathbb{N} \}.
\]

We set \( M(z) \) is the union of \( P(z) \) that is \( M(z) = \cup_{U_{x,n} \in P(z)} U_{x,n} \).

**Theorem 2.17.** Let \( H \) be a strongly \( H_v \)-group and \( z \in H \). Then \( M(z) \) is a complete part.
Proof. Let \( p \in V_{x,n} \) be a finite products of \( x_1, \ldots, x_n \) and \( p \cap M(z) \neq \emptyset \). Assume \( a \in p \cap M(z) \), so there exists \( U_{y,m} \in P(z) \) such that \( a \in U_{y,m} \) and \( z \in U_{y,m} \). Because \( H \) is strongly \( H_v \)-group there exists \( U_{x,n} \in \mathcal{U} \) such that \( p \subseteq U_{x,n} \). By reproductivity property, there exists \( (w,b) \in H^2 \) such that \( x_n \in wz \) and \( z \in ab \). Therefore

\[
\begin{align*}
p \subseteq U_{x,n} = U_{x,n-1}x_n & \subseteq U_{x,n-1}(wz) \subseteq (U_{x,n-1}w)z \subseteq (U_{x,n-1}w)(ab) \\
& \subseteq (U_{x,n-1}w)(U_{y,m}b) \subseteq [(U_{x,n-1}w)U_{y,m}]b \\
& \subseteq U_{t,n+m}b = U_{t,k}b.
\end{align*}
\]

Also

\[
\begin{align*}
z & \in ab \subseteq pb \subseteq [U_{x,n-1}x_n]b \subseteq [U_{x,n-1}(wz)]b \\
& \subseteq [(U_{x,n-1}w)b] \subseteq [(U_{x,n-1}w)U_{y,m}]b \\
& \subseteq U_{t,n+m}b = U_{t,k}b.
\end{align*}
\]

So \( U_{t,k}b \in P(z) \) and \( p \subseteq U_{t,k}b \subseteq M(z) \) and this completes the proof. \( \square \)

Corollary 2.18. For all \( z \in \omega_H \), we have \( M(z) = \omega_H \).

Proof. Let \( U_{x,n} \in P(z) \), so \( z \in U_{x,n} \cap \omega_H \). Since \( \omega_H \) is complete thus \( U_{x,n} \subseteq \omega_H \) and \( M(z) = \cup_{U_{x,n} \in P(z)} U_{x,n} \subseteq \omega_H \). On the other hand since \( M(z) \) is complete, \( z \in M(z) \) and \( C(z) \) is the smallest complete part that contains \( z \) so \( C(z) \subseteq M(z) \) and by 2.13 \( \omega_H = C(z) \subseteq M(z) \). Therefore \( M(z) = \omega_H \) for all \( z \in \omega_H \). \( \square \)

Theorem 2.19. If \( H \) is a strongly \( H_v \)-group, then \( \beta = \beta^* \).

Proof. We know that \( \beta \subseteq \beta^* \). Let \( x,y \in H \) and \( x\beta^*y \). Then by Corollary 2.16 we have

\[ x\beta^*y \Leftrightarrow xKy \Leftrightarrow y \in C(x) = x\omega_H, \]

Therefore \( \exists v, w \in \omega_H \) such that \( x \in xv \) and \( y \in xw \). According to Corollary 2.18 there exist \( U_{t,n} \in P(v) \) such that \( w \in U_{t,n} \), where \( U_{t,n} = U_{t,n-1}U_n \). Also \( v \in U_{t,n} \) and thus \( v\beta w \). Therefore \( \{x,y\} \subseteq x(U_{t,n-1}x_n) \). Thus \( x\beta y \) and \( \beta^* \subseteq \beta \). \( \square \)

3. The \( \theta \)-hyperoperation

In this section we investigate a special class of hyperstructures called \( \theta \)-hyperoperation introduced by Vougiouklis in [11]. By using \( \theta \)-hyperoperation we obtain a \( H_v \)-structure. We investigate \( \theta \)-hyperoperation by adding strongly \( H_v \)-group condition. General definition of \( \theta \)-hyperoperation is as follows that we can see it in [11, 12].
Definition 3.1. Let \( H \) be a set equipped with \( n \) operations (or hyperoperations) \( \otimes_1, \otimes_2, \ldots, \otimes_n \) and a map (or multivalued map) \( f : H \rightarrow H \) (or \( f : H \rightarrow \mathcal{P}^*(H) \)). Then \( n \) hyperoperations \( \theta_1, \theta_2, \ldots, \theta_n \) on \( H \) can be defined, called \( \theta \)-hyperoperations by putting

\[
x \theta_i y = \{ f(x) \otimes_i y, x \otimes_i f(y) \}, \quad \forall (x, y) \in H^2, i \in \{1, 2, \ldots, n\}
\]

or, in case where \( \otimes_i \) is hyperoperation or \( f \) is multivalued map, we have

\[
x \theta_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \quad \forall (x, y) \in H^2, i \in \{1, 2, \ldots, n\}.
\]

If \( \otimes_i \) is associative then \( \theta_i \) is weak associative.

Similarly one can use several maps \( f \), instead than only one. We express a special case that way let \((H, \cdot)\) be a hypergroupoid and \( f : H \rightarrow H \), be a map on \( H \). It is defined in this case \( \bar{\theta} \)-hyperoperation as follows:

\[
x \bar{\theta} y = (x \cdot y) \cup (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall (x, y) \in H^2
\]

Proposition 3.2. Let \((H, \cdot)\) be an \( H_v \)-group, then \((H, \bar{\theta})\) is an \( H_v \)-group.

Proof. According to [11, 12], \((H, \bar{\theta})\) is weak associative. Since \((H, \cdot)\) is reproductive so for all \( x \in H \) we have \( x \cdot H = H = H \cdot x \). Therefore

\[
x \bar{\theta} H = \bigcup_{h \in H} x \bar{\theta} h = \bigcup_{h \in H} \{x \cdot h, f(x) \cdot h, x \cdot f(h)\} = H
\]

and \((H, \bar{\theta})\) is reproductive. Thus \((H, \bar{\theta})\) is an \( H_v \)-group.

Proposition 3.3. Let \((H, \cdot)\) be a strongly \( H_v \)-group and \( f \) be good homomorphism and projection \((f^2 = f)\), then \((H, \bar{\theta})\) is a strongly \( H_v \)-group.

Proof. By proposition 3.2, it is enough to show that \( H_v \)-group \((H, \bar{\theta})\) is strongly, i.e. \( x \bar{\theta} (y \bar{\theta} z) \subseteq (x \bar{\theta} y) \bar{\theta} z \), for all \((x, y, z) \in H^3\). We have

\[
x \bar{\theta} (y \bar{\theta} z) = \bigcup_{a \in y \bar{\theta} z} x \bar{\theta} a = \bigcup_{a \in y \bar{\theta} z} [x \cdot a \cup f(x) \cdot a \cup x \cdot f(a)]
\]
such that \( a \in (y \cdot z) \cup (f(y) \cdot z) \cup (y \cdot f(z)) \). Therefore
\[
\begin{align*}
\bar{x} \bar{\theta} (y \bar{\theta} z) & \subseteq \left[ x \cdot (y \cdot z) \right] \cup \left[ x \cdot (f(y) \cdot z) \right] \cup \left[ x \cdot (y \cdot f(z)) \right] \\
& \subseteq \left[ x \cdot (f(y) \cdot z) \right] \cup \left[ x \cdot (f(y) \cdot z) \right] \cup \left[ x \cdot (y \cdot f(z)) \right] \\
& \subseteq \left[ x \cdot (y \cdot z) \right] \cup \left[ (x \cdot f(y)) \cdot z \right] \cup \left[ (x \cdot y) \cdot f(z) \right] \\
& \subseteq \left[ (x \cdot f(y)) \cdot f(z) \right] \cup \left[ (x \cdot f(y)) \cdot f(z) \right] \cup \left[ (x \cdot f(y)) \cdot f(z) \right] \\
& \subseteq \bigcup_{b \in x \bar{\theta} y} \left[ b \cdot z, f(b) \cdot z, b \cdot f(z) \right] \\
& = \bigcup_{b \in x \bar{\theta} y} \bar{b} \bar{\theta} z = (x \bar{\theta} y) \bar{\theta} z.
\end{align*}
\]
Thus \((H, \bar{\theta})\) is a strongly \(H_v\)-group.

\[\square\]

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REFERENCES