A SPECTRAL CHELYSHKOV WAVELET METHOD TO SOLVE SYSTEMS OF NONLINEAR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. Based on Chelyshkov wavelet, an operational matrix of integration is extracted, and is applied for solving linear and nonlinear Abel integral equations and systems of Abel integral equations. Some numerical examples confirm the applicability, accuracy and efficiency of the method.

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1. Introduction

Singular and weakly singular integral equations have major importance in modeling phenomena in many branches of physics and engineering fields such as atomic scattering [33], radar ranging [25], optical fiber evaluation [19], seismology [14], X-ray radiography [9], plasma diagnostics [1] and microscopy [2].

Abel integral equation as a special case of singular integral equation can be derived directly from a physical or engineering phenomena, so many authors strived to find numerical methods for solving this class of integral equations. In [20] a mechanical quadrature method has been employed to solve some of the first Abel integral equations. Diago et al. have proposed a numerical method for solving a nonlinear Volterra integral equation of Abel type in [10]. The Homotopy perturbation method has been employed to solve a system of generalized Abel integral equations in [18]. An efficient algorithm has been used for numerical solving of singular integral equations of Abel type in [27]. Taylor expansion and Block–pulse functions using the collocation method have adopted to solve the second kind Volterra integral equations of Abel type by Shahsavaran in [32].

Wavelets as a family of functions constructed from dilation and translation of a single function due to orthogonality property are appropriate basis for numerical approximation of solutions of the integral equations. Orthogonal polynomials (such as Jacobi, Laguerre and etc), piecewise constant basis function (block pulse, Haar and Walsh), and sine - cosine functions in fourier series are other classes of orthogonal basis functions. In [6] Chelyshkov has introduced a new class of orthogonal basis functions.

The aim of this paper is to introduce a matrix method by using the orthogonal Chelyshkov wavelet for solving linear and nonlinear Abel and system of Abel integral equations in the following forms:

$$f(t) = \int_0^t \frac{f^s(x)}{(t-x)^\nu} dx, 0 < \nu < 1, s \geq 1, s \in \mathbb{N}, \tag{1}$$

$$f(t) = g(t) + \int_0^t \frac{f^s(x)}{(t-x)^\nu} dx, 0 < \nu < 1, s \geq 1, s \in \mathbb{N}, \tag{2}$$

$$\begin{cases} f_1(t) = g_1(t) + \int_0^t (\frac{a(x)}{(t-x)^{\nu_1}} f_1^{s_1}(x) + \frac{b(x)}{(t-x)^{\nu_2}} f_2^{s_2}(x))dx, \\ f_2(t) = g_2(t) + \int_0^t (\frac{d(x)}{(t-x)^{\nu_3}} f_1^{s_3}(x) + \frac{e(x)}{(t-x)^{\nu_4}} f_2^{s_4}(x))dx, \end{cases} \tag{3}$$

for $0 < \nu_k < 1, s_k \geq 1$ and $s_k \in \mathbb{N}$ $(k = 1, \ldots, 4)$, where these are known as the first kind, the second kind and the system of Abel integral equations, respectively. In order to describe the work, the outlines of the paper is as follows: In section 2 the Chelyshkov wavelets function is introduced. The operational matrices are extracted in section 3. Section 4 is related to the convergence analysis of the method. In section 5 the proposed methed is described. The validity and applicability of the method is demonstrated with some examples in section 6. Finally the last section is related to conclusions and some suggestions for further works.

2. Chelyshkov wavelets

Chelyshkov functions (ChFs), can be derived by the aid of Rodrigues formula:[34]

$$p_m(x) = \frac{1}{(N-m)!} \frac{1}{x^{N-m}} \frac{d^{N-m}}{dx^{N-m}} (x^{N+m+1}(1-x)^{N-m}), m = 0, 1, \ldots, N.$$

Also theses basis functions have the analytical forms:[26]

$$p_m(x) = \sum_{q=0}^{N-m} (-1)^q \binom{N-m}{q} \binom{N+m+q+1}{N-m} x^{m+q}, m = 0, 1, \ldots, N.$$

This class of polynomials is orthogonal functions in the interval [0, 1], with respect to the weight function $w(x) = 1$, as follows:

$$\int_0^1 p_n(x) p_m(x) dx = \frac{1}{2n+1} \delta_{nm}, \tag{4}$$

where δ_{nm} is the so called Kroneker deta function.

Now by using ChFs, Chelyshkov wavelet functions (ChWs) can be defined as follows:[24]

$$\psi_{n,m}(x) = \begin{cases} \sqrt{2^p(2m+1)} p_m(2^p x - n), \frac{n}{2^p} \leq x < \frac{n+1}{2^p}, \\ 0, o.w. \end{cases}$$

where $n = 0, 1, \ ..., 2^p - 1, m = 0, 1, \ ..., N$. The class of ChWs constitute an orthonormal system of functions on the interval $[0, 1]$, with respect to the weight function $w(x) = 1$, as follows:

$$\int_0^1 \psi_{n,m_1}(x)\psi_{n,m_2}(x)dx = \delta_{m_1 m_2}.$$

2.1. **Function approximation.** Define $L^2([0,1])$:=$\{y|y$ is measurable on $[0,1]$ and $\int_0^1 y^2(x)dx < \infty\}$, equipped with the inner product:

$$\langle y(x), \psi(x)\rangle = \int_0^1 y(x)\psi(x)dx.$$

Any function $f \in L^2([0,1])$ can be expanded via ChWs as follows:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \\ &\approx \sum_{n=0}^{2^p-1}\sum_{m=0}^{N} c_{n,m}\psi_{n,m}(x) \\ &= C^T\Psi(x), \end{aligned} \tag{6}$$

where:

$$c_{n,m} = (2m+1)\int_0^1 f(x)\psi_{n,m}(x)d(x).$$

Here C and $\Psi(x)$ are the following $2^p \times (N+1)$ –vectors:

$$C = [c_{00}, \ ...c_{0(N+1)}, \ ..., c_{(2^p-1)0}, \ ..., c_{(2^p-1)(N+1)}]^T,$$

$$\Psi(x) = [\psi_{00}(x), \ ..., \psi_{0(N+1)}(x), \ ...\psi_{(2^p-1)0}(x), \ ..., \psi_{(2^p-1)(N+1)}(x)]^T.$$

3. ChWs–Operational matrices of integration

In this section, based on ChWs function, we will construct an operational matrix of fractional integration. For this purpose, we apply the Rieman-Liouville integration operator as follows:

$$I^\nu u(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}u(t)dt, x > 0. \tag{7}$$

The fractional integration of order ν of ChW–vector, $\Psi(t)$, can be expanded into the ChW-series as follows:

$$\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\Psi(t)dt = P^\nu \Psi(x), \tag{8}$$

where P^ν is called the ChW–operational matrix of integration.

Now by the so called convolution operator of two functions we can write:

$$\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\psi_{n,m}(t)dt = \frac{1}{\Gamma(\nu)}\{x^{\nu-1} * \psi_{n,m}(x)\}.$$

By taking Laplace transform, we have:

$$\mathcal{L}\{\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\psi_{n,m}(t)dt\} = \frac{1}{\Gamma(\nu)}\{\mathcal{L}\{x^{\nu-1}\} * \mathcal{L}\{\psi_{n,m}(x)\}\}, \tag{9}$$

where

$$\mathcal{L}\{x^{\nu-1}\} = \frac{\Gamma(\nu)}{s^\nu}, \tag{10}$$

and

(11)

$$\mathcal{L}\{\psi_{i,j}\}(x) = \sqrt{(2j+1)}2^{\frac{p}{2}}[\sum_{q=0}^{N-j}(-1)^q \binom{N-j}{q}\binom{N+j+q+1}{N-j}(2^p)^{j+q} e^{\frac{-i}{2^p}s}\frac{\Gamma((j+q)+1)}{s^{(j+q)+1}}$$

$$-\sum_{q=0}^{N-j}(-1)^q \binom{N-j}{q}\binom{N+j+q+1}{N-j}(2^p)^{j+q}\sum_{z=0}^{j+q}\binom{j+q}{z}(\frac{1}{2^p})^{(j+q)-z} e^{\frac{-(i+1)}{2^p}s} \times \frac{\Gamma(z+1)}{s^{z+1}}].$$

By substituting (3.4) and (3.5) in (3.3), and using inverse Laplace transform, we get:

$$I^\nu \psi_{i,j}(x) = \sqrt{(2j+1)}2^{\frac{p}{2}}\sum_{q=0}^{N-j}(-1)^q \binom{N-j}{q}\binom{N+j+q+1}{N-j}(2^p)^{j+q}\frac{\Gamma((j+q)+1)}{\Gamma((j+q)+\nu+1)}$$

$$(x-\frac{i}{2^p})^{j+q+\nu} u(x-\frac{i}{2^p}) - \sqrt{(2j+1)}2^{\frac{p}{2}}\sum_{q=0}^{N-j}(-1)^q \binom{N-j}{q}\binom{N+j+q+1}{N-j}(2^p)^{j+q}$$

$$\times \sum_{z=0}^{j+q}\binom{j+q}{z}(\frac{1}{2^p})^{(j+q)-z} \times \frac{\Gamma(z+1)}{\Gamma z+\nu+1} \times (x-\frac{(i+1)}{2^p})^{z+\nu} u(x-\frac{(i+1)}{2^p}).$$

Now we can expand $I^{\nu}\psi_{i,j}(x)$ into ChW-basis as follows

$$I^{\nu}\psi_{i_1,j_1}(x)=\sum_{i_2=0}^{n}\sum_{j_2=0}^{m}c(i_1,j_1,i_2,j_2)\psi_{i_2,j_2}(x),$$

where

$$c(i_1,j_1,i_2,j_2)=\frac{\langle I^{\nu}\psi_{i_1,j_1}(x),\psi_{i_2,j_2}(x)\rangle}{\langle \psi_{i_2,j_2}(x),\psi_{i_2,j_2}(x)\rangle}.$$

and $\langle,\rangle$ denotes the inner product in $L^2[0,1], n=0,1,\dots,2^p-1, m=0,1,\dots,N$.

By the view of the orthogonality property of ChW functions, the entries of the $(2^p(M+1))\times(2^p(M+1))$ integration operational matrix P^{ν}, can be calculated as follows:

$$c(i_1,j_1,i_2,j_2)=\int_{\frac{i_2}{2^p}}^{\frac{i_2+1}{2^p}}[\sqrt{(2j_1+1)}2^{\frac{p}{2}}\sum_{q_1=0}^{N-j_1}(-1)^{q_1}\binom{N-j_1}{q_1}\binom{N+j_1+q_1+1}{N-j_1}(2^p)^{j_1+q_1}$$

$$\frac{\Gamma((j_1+q_1)+1)}{\Gamma((j_1+q_1)+\nu+1)}(x-\frac{i_1}{2^p})^{j_1+q_1+\nu}u(x-\frac{i_1}{2^p})-\sqrt{(2j_1+1)}2^{\frac{p}{2}}\sum_{q_1=0}^{N-j_1}(-1)^{q_1}\binom{N-j_1}{q_1}$$

$$\binom{N+j_1+q_1+1}{N-j_1}(2^p)^{j_1+q_1}\times\sum_{z=0}^{j_1+q_1}\binom{j_1+q_1}{z}(\frac{1}{2^p})^{(j_1+q_1)-z}\times\frac{\Gamma(z+1)}{\Gamma(z+\nu+1)}\times(x-\frac{(i_1+1)}{2^p})^{z+\nu}$$

$$u(x-\frac{(i_1+1)}{2^p})]\times\sqrt{2^p(2j_2+1)}\sum_{q_2=0}^{N-j_2}(-1)^{q_2}\binom{N-j_2}{q_2}\binom{N+j_2+q_2+1}{N-j_2}(2^px-i_2)^{i_2+q_2}dx.$$

where

$$i_1=0,1,\dots,2^p-1, j_1=0,1,\dots,N, i_2=0,1,\dots,2^p-1, j_2=0,1,\dots,N.$$

3.1. **Operational matrix for the product of two ChWs.** The product of $\Psi(x)$, $\Psi^T(x)$ and $2^p(N+1)$–vector C can be expanded into the ChW as:

$$\Psi(x)\Psi^T(x)C\approx U(C)\Psi(x), \tag{12}$$

where

$$U(C)=\langle\Psi(x)\Psi^T(x)C,\Psi(x)\rangle.$$

The elements of the $(2^p(N+1)\times 2^p(N+1))$–matrix $U(C)$, can be calculated as follows:

$$u_{i,j}=\int_0^1 \psi_i(x)\psi_r(x)c_r\psi_j(x), i=1,\dots,2^p(N+1), j=1,\dots,2^p(N+1),$$
$$r=1,\dots,2^p(N+1),$$

where c_r is the r-th entry of vector C.

4. Convergence analysis

This section is concerned with introducing the corresponding convergence theorem.

Theorem 4.1. *Suppose $f\in L^2[0,1]$, and $\Psi(x)$ is a ChWs–vector. A sequence $f_n(x)$ defined by $f_n(x)=\sum_{k=1}^{n} c_k\psi_k(x)$, $n\in\mathbb{N}$, with*

$$c_k=<f(x),\psi_k(x)>, k=1,2,\ \dots,n.$$

converges to $f(x)$ in the vector space of $\Psi(x)$'s components if and only if $\sum_{k=1}^{\infty}|c_k|^2<\infty$.

Proof. Let f_n be converged to f. Hence, for $n\in\mathbb{N}$, we have:

$$0\le ||f-\sum_{k=1}^{n}c_k\psi_k||_2^2$$
$$=\int_0^1 (f-\sum_{k=1}^{n}c_k\psi_k)^2dx,$$
$$=\int_0^1 f^2(x)dx-\int_0^1 |\,2f(x)\sum_{k=1}^{n}c_k\psi_k(x)|\,dx+\int_0^1(\sum_{k=1}^{n}c_k\psi_k(x))^2dx,$$
$$\le\int_0^1 f^2(x)dx-\int_0^1 |\,2f_n(x)\sum_{k=1}^{n}c_k\psi_k(x)\,|\,dx+\int_0^1(\sum_{k=1}^{n}c_k\psi_k(x))^2dx,$$
$$=\int_0^1 f^2(x)dx-\int_0^1(\sum_{k=1}^{n}c_k\psi_k(x))^2dx,$$
$$=\int_0^1 f^2(x)dx-\int_0^1(c_1^2\psi_1^2(x)+c_2^2\psi_2^2(x)+\cdots+c_n^2\psi_n^2(x))\,dx,$$

$$= ||f||_2^2 - \sum_{k=1}^{n} |c_k|^2,$$

Therefore:

$$0 \leq ||f||_2^2 - \sum_{k=1}^{n} |c_k|^2,$$

So for any $f \in L^2[0,1]$, we have:

$$\sum_{k=1}^{n} |c_k|^2 \leq ||f(x)||_2^2 < \infty,$$

Hence:

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty.$$

Now, suppose that $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. So if n$\in \mathbb{N}, then:$

$$0 \leq ||f - f_n||_2^2 = ||\sum_{k=1}^{\infty} c_k\psi_k - \sum_{k=1}^{n} c_k\psi_k||_2^2,$$

$$= ||\sum_{k=n+1}^{\infty} c_k\psi_k||_2^2,$$

$$= \int_0^1 (\sum_{k=n+1}^{\infty} c_k\psi_k(x))^2\, dx,$$

$$\leq (\sum_{k=n+1}^{\infty} c_k \int_0^1 \psi_k(x)\, dx)^2,$$

$$\leq \sum_{k=n+1}^{\infty} c_k^2 \int_0^1 \psi_k^2(x)dx,$$

$$= \sum_{k=n+1}^{\infty} c_k^2.$$

By the assumption, as n goes to infinity, the last term will go to zero, and the proof is completed. □

5. Implementation of the method

Suppose $f, g, a(x) \in L^2[0,1]$. So we can write:

$$f(x) \approx C^T\Psi(x), g(x) \approx G^T\Psi(x), a(x) \approx A^T\Psi(x)$$

Now we present some theorems to approximate the integral part of equations (1.1) and (1.2) and the system of integral equations (1.3) for linear case ($s = 1$) and nonlinear cases ($s > 1$).

Lemma 5.1. *If $f(t) \approx C^T\Psi(t)$ then:*

$$f^s(t) \approx C^T U^{s-1}(C)\Psi(t), s \in \mathbb{N},$$

where $U(C)$ is the operational matrix of product of two ChWs.

Proof. For $s = 2$, we have:

$$f^2(t) = f(t)f(t) \approx (C^T\Psi(t))(\Psi^T(t)C) = C^T U(C)\Psi(t),$$

and by using the induction on s, the proof is complete. □

Theorem 5.2. *The integral parts of the integral equations (1.1) and (1.2) for the case $s > 1$, can be expressed in terms of the ChWs–basis as follows:*

$$\int_0^t \frac{f^s(x)}{(t-x)^\nu}dx \approx \sum_{i=1}^{2^p N} M_i\psi_i(t) = M^T\Psi(t),$$

where

$$M_i \approx C^T U^{s-1}(C)P^\nu e_i.$$

and $e_i = [0, \dots, 0, 1, 0, \dots, 0]^T$ for $i = 1, \dots, 2^p(N+1)$.

Proof. Putting

$$M_i \simeq \int_0^1 \int_0^t \frac{f^s(x)}{(t-x)^\nu}\psi_i(t)dxdt,$$

Now, by using the Lemma (5.1), we have

$$M_i \approx \int_0^1 \int_0^t \frac{C^T U^{s-1}(C)\Psi(x)}{(t-x)^\nu}\psi_i(t)dxdt,$$

and so

$$M_i \approx \int_0^1 C^T U^{s-1}(C) \int_0^t \frac{\Psi(t)}{(t-x)^\nu} dx \psi_i(t) dt.$$

Using the operatioal matrix of integration, we write:

$$M_i \approx C^T U^{s-1}(C) P^\nu \int_0^1 \Psi(t)\psi_i(t) dt.$$

So we have:

$$M_i \approx C^T U^{s-1}(C) P^\nu e_i.$$

□

Theorem 5.3. *The integral parts of the system of integral equations (1.3) for the case $s = 1$, can be expressed in terms of the ChWs–basis as follows:*

$$\int_0^t \frac{a(x)f(x)}{(t-x)^\nu} dx \simeq \sum_{i=1}^{2^p N} O_i \psi_i(t) = O^T \Psi(t),$$

where

$$O_i \approx C^T U(A) P^\nu e_i.$$

$i = 1, \ldots, 2^p(N+1)$.

Proof. Putting

$$O_i \approx \int_0^1 \int_0^t \frac{f(x)a(x)}{(t-x)^\nu} \psi_i(t) dx dt,$$

we have

$$O_i \approx \int_0^1 \int_0^t \frac{(C^T\Psi(x))(\Psi^T(x)A)}{(t-x)^\nu} \psi_i(t) dx dt,$$

and so we obtain:

$$O_i \approx \int_0^1 C^T U(A) \int_0^t \frac{\Psi(x)}{(t-x)^\nu} dx \psi_i(t) dt.$$

So that we write:

$$O_i \approx C^T U(A) P^\nu \int_0^1 \Psi(x)\psi_i(t) dt.$$

Considering the orthogonality property of ChWs basis, the result will be obtained.

□

Theorem 5.4. *The integral parts of the system of integral equations (1.3) for the case $s = 1$, can be expressed in terms of the ChWs–basis as follows:*

$$\int_0^t \frac{a(x)f^s(x)}{(t-x)^\nu}dx \approx \sum_{i=1}^{2^p N} Q_i\psi_i(t) = Q_s(A)^T\Psi(t),$$

where

$$Q_i \approx C^T U^{s-1}(C)U(A)P^\nu e_i.$$

and $e_i = [0,\dots,0,1,0,\dots,0]^T$ *with* $i = 1,\dots,2^p(N+1)$.

Proof. Putting

$$Q_i \approx \int_0^1\int_0^t \frac{f^s(x)a(x)}{(t-x)^\nu}dx\psi_i(t)dxdt,$$

we have

$$Q_i \approx \int_0^1\int_0^t \frac{C^T U^{s-1}(C)\Psi(x)\Psi^T(x)A}{(t-x)^\nu}dx\psi_i(t)dxdt,$$

and we get:

$$Q_i \approx \int_0^1 C^T U^{s-1}(C)U(A)\int_0^t \frac{\Psi(x)}{(t-x)^\nu}dx\psi_i(t)dt.$$

Finally we obtain:

$$Q_i \approx C^T U^{s-1}(C)U(A)P^\nu \int_0^1 \Psi(t)\psi_i(t)dt.$$

Thus by using the orthogonality property of ChWs basis, the result will be followed. □

Now, by using the above theorems, the system of integral equations (1.3), for $s_k > 1, (k = 1,\dots,4)$ can be rewritten as follows:

$$\begin{cases} C_1 = G_1 + Q_{s_1}(A) + Q_{s_2}(B), \\ C_2 = G_2 + Q_{s_3}(D) + Q_{s_4}(E). \end{cases} \quad (13)$$

where the vectors Q_{s_k}, for $k = 1,2,3,4$ are obtained by using theorems (5.3) and (5.4). Eq (5.1) is a linear or nonlinear system of $2(2^p(N+1)$ equations for

$2(2^p(N+1))$ unknown coefficient vectors C_1 and C_2 that can be solved by using Newton-Raphson method in MATLAB software.

6. Numerical results

In this section, some examples are considererd to show the efficiency and applicability of our method. The error function is defined by $e_m(t) = f(t) - f_m(t)$, where $f(t)$ is the exact solution, and $f_m(t)$ is the approximate solution of the considering equation.

Example1. Consider the following Abel integral equation of the second kind:

$$f(t) = g(t) - \int_0^t \frac{f(x)}{\sqrt{t-x}} dx, 0 \le t \le 1, \tag{14}$$

where

$$g(t) = cos(t) + sin(t) + \sqrt{2\pi}(fresnels(\sqrt{\frac{2t}{\pi}}(-cos(t) + sin(t)) + fresnelc(\sqrt{\frac{2t}{\pi}}(cos(t) + sin(t)))),$$

$$fresnels(v) = \int_0^v sin(\tfrac{\pi u^2}{2})du, \text{ and, } fresnelc(v) = \int_0^v cos(\tfrac{\pi u^2}{2})du.$$

and the exact solution is $f(t) = sin(t) + cos(t)$.

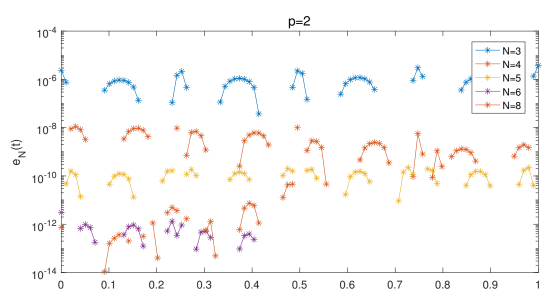
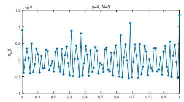
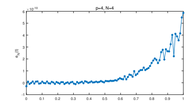


Figure 1. The error of approximate solutions for different values of N (Example 1).

We have applied our method to solve the Abel integral equation in Example 1. Figure 1 shows by increasing the number of the ChW basis vectors elements we will have better approximations. For more details, Figure 2 compares the error functions for $N = 3$ and $N = 4$.



(a)



(b)

FIGURE 2. The error of the numerical results by using ChW basis with $p = 4$ (Example 1).

Example2. Consider the following Abel integral equation:

$$f(t) = 2\sqrt{t} - \int_0^t \frac{f(x)}{\sqrt{t-x}} dx, 0 \le t \le 1, \tag{15}$$

with the exact solution:

$$f(t) = 1 - e^{\pi t} erfc(\sqrt{\pi t}),$$

where:

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx.$$

is the famous error function.

TABLE 1. Comparing the errors of approximate solutions for some p (Example 2).

	$p=1$	$p=2$	$p=3$	$p=4$
x=0.1	8.732e -03	-8.373 e-03	5.3333 e-03	-5.659e-06
x=0.2	-1.129e -02	7.7367e-03	-4.407 e-06	9.341e-07
x=0.3	-9.673e -06	1.087 e-04	-3.999 e-06	4.462e-07
x=0.4	9.909e -03	1.25 e-06	1.113 e-06	-9.976e-08
x=0.5	1.059e -03	-1.342 e-06	-1.316 e-06	1.281 e-08
x=0.6	1.738e -04	-4.845 e-06	5.196 e-07	8.488 e-09
x=0.7	-1.36 e -04	6.459 e-06	-2.225 e-09	1.264 e-09
x=0.8	-8.437e -06	1.22 e-06	-6.449 e-08	-4.758 e-10
x=0.9	1.452e -04	-1.927 e-07	3.219 e-08	-4.098 e-09

By using our method, some approximations of solution of the equation has been extracted. Table 1 shows that by increasing p of the Chelyshkov wavelet basis, the accuracy of the method has increased.

Example 3. Consider a nonlinear Abel integral equation of the first kind in the following form:

$$\int_0^t \frac{f^2(x)+f^3(x)}{(t-x)^{\frac{1}{2}}}dx = g(t), 0 \le t \le 1, \tag{16}$$

where $g(t) = \frac{16}{105}t^{\frac{5}{2}}(6x+7)$ and the exact solution is $f(t) = t$.

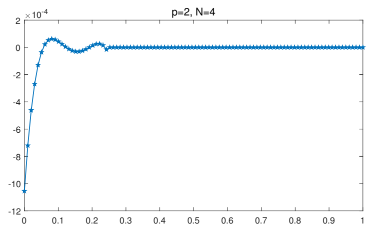


FIGURE 3. The error of approximate solutions (Example 3).

TABLE 2. Comparing the errors of approximate solutions of the present method with Ref[30] (Example 3).

	$e_{20}(t)$ (Ref [30])	$e_4(t)$ (The present method)
x=0.1	2.7026e -05	4.242 e-05
x=0.2	6.2518e -06	1.394 e-05
x=0.3	1.0315e -05	-8.769 e-13
x=0.4	1.9049e -05	6.617 e-13
x=0.5	2.4287e -05	-1.324 e-12
x=0.6	2.2919e -05	1.719 e-12
x=0.7	1.7833e -05	3.245 e-12
x=0.8	1.6220e -05	4.784 e-13
x=0.9	2.2109e -05	-4.877 e-12

Using our method, the nonlinear Abel integral equation of the first kind in Example 3, has been solved. Figure 3 shows the results. Also Table 2 compares the obtained results with the results in Ref[30]. By applying only four elements of the vector basis much more accurate results will be obtained.

Example4. Consider the following nonlinear system of Abel integral equations:

$$\begin{cases} f_1(t) - 2f_2(t) + \displaystyle\int_0^t \frac{{f_1}^2(x) + f_2^2(x)}{(t-x)^{\frac{1}{5}}} dx = g_1(t), \\ f_2(t) - f_1(t) + \displaystyle\int_0^t \frac{f_1(x) f_2(x)}{(t-x)^{\frac{1}{3}}} dx = g_2(t), \end{cases} \tag{17}$$

where

$$g_1(t) = t^2 - 2t^3 + \frac{390625}{1573656} t^{\frac{34}{5}} + \frac{3125}{9576} t^{\frac{24}{5}},$$
$$g_2(t) = t^3 - t^2 - \frac{2187}{5236} t^{\frac{17}{3}},$$

and $(f_1(t), f_2(t)) = (t^2, t^3)$ is the exact solution.

(a) (b)

FIGURE 4. Comparing the approximate errors by changing p in the used ChW-basis with $N = 3$ (Example 4).

(a) (b)

FIGURE 5. Comparing the approximate errors by changing p in the used ChW-basis with $N = 4$ (Example 4).

The solution of the system of equations has been approximated by applying the numerical ChW method. Figures 4 and 5 show that by increasing p, much more accurate results will be obtained.

Example 5. Consider the following system of linear Abel integral equations:

$$\begin{cases} f(t) = t + t^2 - \dfrac{25}{6552} t^{\frac{8}{5}} (130t^{\frac{6}{5}} + 182t^{\frac{1}{5}} - 210x + 273) \\ \qquad\qquad + \displaystyle\int_0^t \Big(\frac{1}{(t-x)^{\frac{1}{5}}} f(x) + \frac{1}{(t-x)^{\frac{2}{5}}} g(x)\Big)dx, \\ g(t) = t - t^2 - \dfrac{25}{924} t^{\frac{6}{5}} (55t^{\frac{6}{5}} + 66t^{\frac{1}{5}} - 140x + 154) \\ \qquad\qquad + \displaystyle\int_0^t \Big(\frac{1}{(t-x)^{\frac{3}{5}}} f(x) + \frac{1}{(t-x)^{\frac{4}{5}}} g(x)\Big)dx, \end{cases} \tag{18}$$

where $(f(t), g(t)) = (t + t^2, t - t^2)$ is the exact solution.

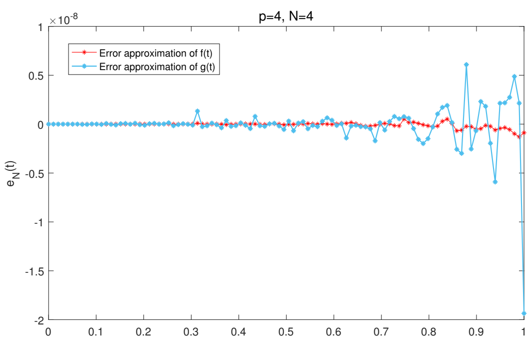


FIGURE 6. The error of approximate solution of the system in Example 5.

Figure 6 shows the error of the results for solving the system of Abel integral equations in Example 5, with $N = 4$ and $p = 4$.

Example 6. Consider the following system of Abel integral equations of the first kind with variable coefficients:

$$\begin{cases} \displaystyle\int_0^t \Big(\frac{(x^2-1)f_1(x)}{(t-x)^{\frac{1}{2}}} + \frac{xf_2(x)}{(t-x)^{\frac{1}{3}}}\Big)dx = g_1(t), \\ \displaystyle\int_0^t \Big(\frac{x^3 f_1(x)}{(t-x)^{\frac{1}{4}}} + \frac{(1-x)f_2(x)}{(t-x)^{\frac{1}{5}}}\Big)dx = g_2(t), \end{cases} \tag{19}$$

where

$$g_1(t) = \frac{16}{15}t^{\frac{9}{2}} - \frac{16}{15}t^{\frac{5}{2}} + \frac{27}{40}t^{\frac{11}{3}} + \frac{243}{440}t^{\frac{14}{3}},$$
$$g_2(t) = \frac{128}{231}t^{\frac{23}{4}} + \frac{125}{252}t^{\frac{14}{5}} - \frac{125}{1197}t^{\frac{19}{5}} - \frac{625}{1596}t^{\frac{24}{5}},$$

and $(f_1(t), f_2(t)) = (t^2, t^2 + t^3)$ is the exact solution.

(a) (b)

FIGURE 7. Comparing of the approximate errors by incremental changing in p in the used ChW-basis with $N = 8$ (Example 6).

Increasing p causes to much more accurate results which are shown in Figure 7.

Example7. Consider the following nonlinear system of Abel integral equations:

$$\begin{cases} 2f(t) + \displaystyle\int_0^t \frac{f^2(x) + g^2(x)}{(t-x)^{\frac{1}{2}}}dx = 2\sqrt{t} + \frac{4}{3}t^{\frac{3}{2}}, \\ g(t) + \displaystyle\int_0^t \frac{f(x)g(x)}{(t-x)^{\frac{1}{3}}}dx = \sqrt{t} + \frac{9}{10}t^{\frac{5}{3}}, \end{cases} \tag{20}$$

where $(f(t), g(t)) = (\sqrt{t}, \sqrt{t})$ is the exact solution.

TABLE 3. The error of approximate solutions of $f(t)$ and $g(t)$ (Example 7).

	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
$e_N(t)$	$8.197e-4$	$1.696e-7$	$1.731e-7$	$1.483e-7$	$1.195e-7$

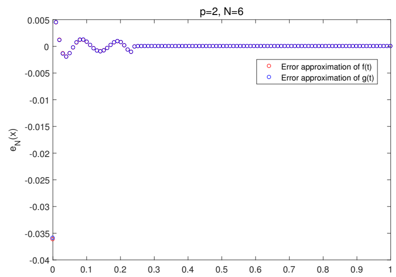


FIGURE 8. The error of approximate solutions of the system in Example 7.

The solution of the system of equations has been approximated by applying the numerical ChW method. Figure 8 and Table 3 show the results.

Example 8. Consider the following nonlinear system of singular integral equations:

$$\begin{cases} f_1(t) - 2f_2(t) + \displaystyle\int_0^t \frac{f_1^2(x)}{(t-x)^{\frac{1}{2}}}dx + \int_0^t \frac{f_1(x)f_2^2(x)}{(t-x)^{\frac{1}{3}}}dx = g_1(t), \\ f_2(t) + 2f_1(t) + \displaystyle\int_0^t \frac{f_1(x)f_2(x)}{(t-x)^{\frac{1}{4}}}dx + \int_0^t \frac{f_2^2(x)}{(t-x)^{\frac{1}{5}}}dx = g_2(t), \end{cases} \tag{21}$$

where $g_1(t) = 3t^2 - t + \frac{16}{315}t^{\frac{5}{2}}(16t^2 + 36t + 21) - \frac{243}{52360}t^{\frac{11}{3}}(-81t^3 + 90t^2 + 102t - 119)$

$g_2(t) = t^2 + 3t - \frac{128}{21945}t^{\frac{11}{4}}(64t^2 - 95) + \frac{125}{9576}t^{\frac{14}{5}}(25t^2 - 60t + 38)$

and $(f_1(t), f_2(t)) = (t + t^2, t - t^2)$ is the exact solution.

Figure 9 and Table 4 show the error of numerical solutions for the system of nonlinear singular integral equations in Example 8, with $N = 8$ and $p = 2$.

7. CONCLUSION

In this paper, the authors employed a spectral method for solving generalized linear or nonlinear integral equation and system of integral equations of Abel type.

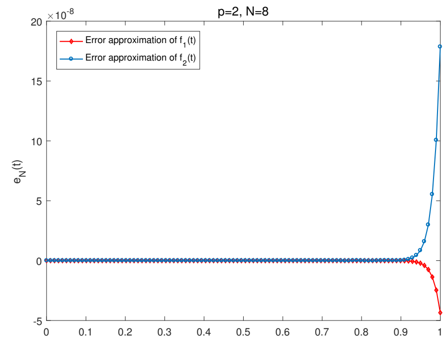


FIGURE 9. The error of approximate solutions of the system in Example 8.

TABLE 4. The error of approximate solutions of $f_1(t)$ and $f_2(t)$ (Example 8).

	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
$e_N(t)$	$2.245e-14$	$1.479e-13$	$1.103e-12$	$-2.328e-12$	$4.369e-10$

The presented operational method, based on Chelyshkov wavelet functions, converts the problem to a system of linear and nonlinear equations. Finally, we have presented some numerical examples to show the accuracy and applicability of our method. This method can be used for other classes of integral equations as well.

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