HERMITE-HADAMARD TYPE INEQUALITIES FOR m-CONVEX FUNCTIONS BY USING A NEW INEQUALITY FOR DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, we give some inequalities for differentiable convex functions which are connected with the Hermite-Hadamard's integral inequality holding for convex functions. Also, we obtain some estimates to the right-hand side of Hermite-Hadamard inequality for functions whose absolute values of fourth derivatives raised to positive real powers are m-convex. Finally, some natural applications to special means of real numbers are given.

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1. Introduction

Many of integral inequalities are based on a convexity assumption of a certain function and the theory of inequality is one of the most important field study of convex analysis and abstract analysis. The major inequality in these fields is Hermite-Hadamard which can be stated as follows.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Recently, the generalizations, improvements, variations and applications for convexity and the Hermite-Hadamard inequality have attracted the attention of many researchers.

For example, M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan studied this inequality for s-convex functions and Greens functions ([6],[5]). Also, Mihesan gave the definition of (α, m)-convexity and obtained some results related to Hermit-Hadamard inequality [1]. Moreover, there are more results in this field in some other studies [e.g. 4-16].

The goal of the paper is to study Hermite-Hadamard type inequalities for m-convex functions in view of new inequality which is introduced in the section 3.

The structure of the paper is as follows: In Section 2, we collect definitions, notations and preliminary results related to inequalities . In Section 3 we give a new inequality on differentiable function in order to obtain Hermit-Hadamard inequality for some functions.

2. Preliminaries

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

The following definitions can be found in [1–16]

Definition 2.1. *The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be quasi-convex function, if*

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in I$ *and* $t \in [0,1]$.

In [1] G.Toader defined the concept of m-convexity as the following:

Definition 2.2. *The function* $f : [0,c] \to \mathbb{R}$, *is said to be* m*-convex function, where* $c > 0, m \in [0,1]$ *if*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0,c]$.

3. Main results

In this section we shall state our main results.

In order to prove our main results we need to prove the following lemma.

Lemma 3.1. *Let* $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I^o *such that* $f^{(4)} \in L[a,b]$, *where* $a, b \in I^o$ *with* $a < b$. *Then the following equality holds:*

$$\begin{aligned} I(f) &= \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \\ &= \frac{b-a}{12}(f'(a)-f'(b)) + \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2 f^{(4)}(ta+(1-t)b)dt. \end{aligned}$$

Proof. It suffices to note that

$$\begin{aligned} J &= \int_0^1 t^2(1-t)^2 f^{(4)}(ta+(1-t)b)dt \\ &= \frac{t^2(1-t)^2}{(a-b)} f^{(3)}(ta+(1-t)b)|_0^1 \\ &- \frac{2(1-2t)(t-t^2)}{(a-b)^2} f''(ta+(1-t)b)|_0^1 \\ &+ \frac{(12t^2-12t+2)}{(a-b)^3} f'(ta+(1-t)b)|_0^1 \\ &- \frac{(24t-12)}{(a-b)^4} f(ta+(1-t)b)|_0^1 + \frac{24}{(a-b)^4}\int_0^1 f(ta+(1-t)b)dt. \end{aligned}$$

Now, by some calculus and set $x = ta + (1-t)b$ we have

$$\begin{aligned} J &= \int_0^1 t^2(1-t)^2 f^{(4)}(ta+(1-t)b)dt \\ &= \frac{2}{(a-b)^3}(f'(a)-f'(b)) - \frac{12}{(a-b)^4}[f(a)+f(b)] + \frac{24}{(a-b)^4}\frac{1}{b-a}\int_a^b f(x)dx. \end{aligned}$$

Multiplying the both sides in $\frac{(a-b)^4}{24}$, we have

$$\begin{aligned} I(f) &= \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \\ &= \frac{b-a}{12}(f'(a)-f'(b)) + \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2 f^{(4)}(ta+(1-t)b)dt, \end{aligned}$$

which completes the proof. □

We will start with the following theorem containing Hermite-Hadamard type inequality.

Theorem 3.2. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f^{(4)} \in L[a,b]$, where $a, b \in I^o$ with $a < b$. If $|f^{(4)}|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} |I(f)| &= |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{1440}\{|f^{(4)}(a)| + |f^{(4)}(b)|\}. \end{aligned}$$

Proof. Using Lemma 3.1 we have

$$\begin{aligned} |I(f)| &= |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2(t|f^{(4)}(a)| + (1-t)|f^{(4)}(b)|)dt \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + |f^{(4)}(a)|\frac{(b-a)^4}{24}\int_0^1 t^3(1-t)^2dt \\ &+ |f^{(4)}(b)|\frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^3dt \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{24}(|f^{(4)}(a)|B(4,3) + |f^{(4)}(b)|B(3,4)) \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{1440}\{|f^{(4)}(a)| + |f^{(4)}(b)|\}, \end{aligned}$$

where we have used the fact that

$$\int_0^1 t^3(1-t)^2dt = B(4,3) = \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = \frac{1}{60},$$

$$\int_0^1 t^2(1-t)^3dt = B(3,4) = B(4,3) = \frac{1}{60}.$$

Note that

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \Gamma(x)=\int_0^{+\infty} e^{-t}t^{x-1}dt,\ x>0,y>0$$
$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

□

A similar result for m-convex function is embodied in the following theorem.

Theorem 3.3. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f^{(4)} \in L[a,b]$, where $a,b \in I^o$ with $a<b$. If $|f^{(4)}|$ is m-convex on $[a,b], m \in (0,1]$, then the following equality holds:*

$$\begin{aligned}|I(f)| &= |\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{1440}\{|f^{(4)}(a)|, m|f^{(4)}(\frac{b}{m})|\}.\end{aligned}$$

Proof. In a similar to proof of Theorem 3.2 the result obtained. □

Another Hermite-Hadamard type inequality for powers in terms of the fourth derivatives is obtained as following:

Theorem 3.4. *Let $f : I \subseteq [0,+\infty] \to \mathbb{R}$ be a differentiable on I^o such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f^{(4)}|^q$ is convex on $[a,b]$, let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality*

$$\begin{aligned}|I(f)| &= |\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{384}(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\frac{|f^{(4)}(a)|+|f^{(4)}(b)|}{2}\}^{\frac{1}{q}} \\ &\leq \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{384}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\}^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 3.1 and Hölder's integral inequality, we find

$$\begin{aligned}
&|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt|-\frac{b-a}{12}|f'(a)-f'(b)| \\
&\le \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\
&\le \frac{(b-a)^4}{24}(\int_0^1 (t^2(1-t)^2)^p dt)^{\frac{1}{p}}(\int_0^1 |f^{(4)}(ta+(1-t)b)|^q dt)^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}(\int_0^1 t^{2p}(1-t)^{2p}dt)^{\frac{1}{p}}(|f^{(4)}(a)|^q\int_0^1 tdt+|f^{(4)}(b)|^q\int_0^1(1-t)dt)^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}(B(2p+1,2p+1))^{\frac{1}{p}}(\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2})^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}\{\frac{2^{1-2(2p+1)}\Gamma(\frac{1}{2})\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\}^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{384}(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\}^{\frac{1}{q}},
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\int_0^1 |f^{(4)}(ta+(1-t)b)|^q dt &\le |f^{(4)}(a)|^q\int_0^1 tdt+|f^{(4)}(b))|^q\int_0^1(1-t)dt \\
&= \frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}.
\end{aligned}$$

Also, By [13] we have

$$\begin{aligned}
&B(x,x)=2^{1-2x}B(\frac{1}{2},x), \\
&B(2p+1,2p+1)=2^{1-2(2p+1)}B(\frac{1}{2},2p+1)=\frac{2^{1-2(2p+1)}\Gamma(\frac{1}{2})\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}.
\end{aligned}$$

□

As Theorem 3.4 we have the following result for m-convex function.

Theorem 3.5. *Let $f: I\subseteq[0,+\infty]\to\mathbb{R}$ be a differentiable on I^o such that $f^{(4)}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f^{(4)}|^q$ is m-convex on $[a,b], m\in(0,1]$, let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality*

$$\begin{aligned}
|I(f)| &= |\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \\
&\le \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{384}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\frac{|f^{(4)}(a)|^q+m|f^{(4)}(\frac{b}{m})|^q}{2}\}^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3.6. *Let $f : I \subseteq [0,+\infty] \to \mathbb{R}$ be a differentiable on I^o such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f^{(4)}|^q$ is m-convex on $[a,b], m \in (0,1]$, let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following inequality*

$$|I(f)| = |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx|$$
$$\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{720}(\frac{|f^{(4)}(a)|^q + m|f^{(4)}(\frac{b}{m})|^q}{2})^{\frac{1}{q}}.$$

Proof. In view of m-convexity of f and Hölder inequality we obtain

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| - \frac{b-a}{12}|f'(a)-f'(b)|$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt$$
$$\leq \frac{(b-a)^4}{24}\int_0^1 [t^2(1-t)^2]^{1-\frac{1}{q}}[t^2(1-t^2)]^{\frac{1}{q}}|f^{(4)}(ta+(1-t)b)|^q dt$$
$$\leq \frac{(b-a)^4}{24}[\int_0^1 t^2(1-t)^2dt]^{1-\frac{1}{q}}[\int_0^1 t^2(1-t)^2|f^{(4)}(ta+m(1-t)\frac{b}{m})|^q dt]^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}[B(3,3)]^{1-\frac{1}{q}}[|f^{(4)}(a)|^q\int_0^1 t^3(1-t)^2dt + m|f^{(4)}(\frac{b}{m})|^q\int_0^1 t^2(1-t)^3dt]^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{24}(\frac{1}{30})^{1-\frac{1}{q}}\{|f^{(4)}(a)|^q B(4,3) + m|f^{(4)}(\frac{b}{m})|^q B(3,4)\}^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^4}{720}(\frac{|f^{(4)}(a)|^q + m|f^{(4)}(\frac{b}{m})|^q}{2})^{\frac{1}{q}}.$$

□

The following result also holds:

Theorem 3.7. *Let $f : I \subseteq [0,+\infty] \to \mathbb{R}$ be a differentiable on I^o such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f^{(4)}|^q$ is m-convex on $[a,b], m \in (0,1]$, let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following inequality*

$$|I(f)| = |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx|$$
$$\leq \frac{b-a}{12}|f'(a)-f'(b)| + \frac{(b-a)^4}{24}(|f^{(4)}(a)|^q + m(2q+1)|f^{(4)}(\frac{b}{m})|^q))^{\frac{1}{q}}.$$

Proof. In view of m-convexity of f and Hölder inequality we obtain

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx|-\frac{b-a}{12}|f'(a)-f'(b)| \\
&\le \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\
&\le \frac{(b-a)^4}{24}[\int_0^1 t^{2p}dt]^{\frac{1}{p}}[\int_0^1 (1-t)^{2q}|f^{(4)}(ta+m(1-t)\frac{b}{m})|^q dt]^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}[\frac{1}{2p+1}]^{\frac{1}{p}}[|f^{(4)}(a)|^q\int_0^1 (1-t)^{2q}tdt+m|f^{(4)}(\frac{b}{m})|^q\int_0^1 (1-t)^{2q+1}dt] \\
&\le \frac{(b-a)^4}{24}\{|f^{(4)}(a)|^q B(2q+1,2)+m|f^{(4)}(\frac{b}{m})|^q\frac{1}{2q+2}\}^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}\{|f^{(4)}(a)|^q\frac{1}{(2q+1)(2q+2)}+m|f^{(4)}(\frac{b}{m})|^q\frac{1}{2q+2}\}^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}\{\frac{|f^{(4)}(a)|^q+m(2q+1)|f^{(4)}(\frac{b}{m})|^q}{(2q+1)(2q+2)}\}^{\frac{1}{q}} \\
&\le \frac{(b-a)^4}{24}(|f^{(4)}(a)|^q+m(2q+1)|f^{(4)}(\frac{b}{m})|^q)^{\frac{1}{q}}.
\end{aligned}
$$

Note that it is easy to obtain $(\frac{1}{2p+1})^{\frac{1}{p}}<1$, for $q>1$. □

Theorem 3.8. *Let* $f: I\subseteq[0,+\infty]\to\mathbb{R}$ *be a differentiable on* I^o *such that* $f^{(4)}\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f^{(4)}|^q$ *is m-convex on* $[a,b], m\in(0,1]$, *let* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then we have the following inequality*

$$
\begin{aligned}
|I(f)| &= |\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \\
&\le \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{72}(\frac{3}{(2q+1)(2q+2)(2q+3)(2q+4)})^{\frac{1}{q}} \\
&\times\{6|f^{(4)}(a)|^q+2m(2q+1)|f^{(4)}(\frac{b}{m})|^q\}^{\frac{1}{q}}.
\end{aligned}
$$

Proof. In view of m-convexity of f and Hölder inequality we obtain

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx|-\frac{b-a}{12}|f'(a)-f'(b)| \\
&\leq \frac{(b-a)^4}{24}\int_0^1 (t^2)^{1-\frac{1}{q}}(t^2)^{\frac{1}{q}}(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\
&\leq \frac{(b-a)^4}{24}[\int_0^1 t^2dt]^{1-\frac{1}{q}}[\int_0^1 t^2(1-t)^{2q}|f^{(4)}(ta+m(1-t)\frac{b}{m})|^q dt]^{\frac{1}{q}} \\
&\leq \frac{(b-a)^4}{24}[\frac{1}{3}]^{1-\frac{1}{q}}[|f^{(4)}(a)|^q\int_0^1 (1-t)^{2q}t^3dt+m|f^{(4)}(\frac{b}{m})|^q\int_0^1 t^2(1-t)^{2q+1}dt] \\
&\leq \frac{(b-a)^4}{24}[\frac{1}{3}]^{1-\frac{1}{q}}\{|f^{(4)}(a)|^qB(4,2q+1)+m|f^{(4)}(\frac{b}{m})|^qB(3,2q+2)\}^{\frac{1}{q}} \\
&\leq \frac{(b-a)^4}{24}[\frac{1}{3}]^{1-\frac{1}{q}}\{|f^{(4)}(a)|^q\frac{6}{(2q+1)(2q+2)(2q+3)(2q+4)} \\
&+m|f^{(4)}(\frac{b}{m})|^q\frac{2}{(2q+2)(2q+3)(2q+4)}\}^{\frac{1}{q}} \\
&\leq \frac{(b-a)^4}{72}(\frac{3}{(2q+1)(2q+2)(2q+3)(2q+4)})^{\frac{1}{q}} \\
&\times\{6|f^{(4)}(a)|^q+2m(2q+1)|f^{(4)}(\frac{b}{m})|^q\}^{\frac{1}{q}}.
\end{aligned}
$$

Note that it is easy to obtain $(\frac{1}{2p+1})^{\frac{1}{p}}<1$, for $q>1$. □

A similar result for quasi-convex function is embodied in the following theorem.

Theorem 3.9. *Let $f: I\subseteq[0,+\infty]\to\mathbb{R}$ be a differentiable on I^o such that $f^{(4)}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f^{(4)}|$ is quasi- convex on $[a,b]$, then we have the following inequality*

$$
\begin{aligned}
|I(f)|&=|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \\
&\leq \frac{b-a}{12}|f'(a)-f'(b)|+\frac{(b-a)^4}{720}\sup\{|f^{(4)}(a)|,|f^{(4)}(b)|\}.
\end{aligned}
$$

Proof. From Lemma 3.1 and using the quasi-convexity of $|f^{(4)}|$, we obtain

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| - \frac{b-a}{12}|f'(a)-f'(b)| \\
&\leq \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\
&\leq \frac{(b-a)^4}{24}\sup\{|f^{(4)}(a)|,|f^{(4)}(b)|\}\int_0^1 t^2(1-t)^2 dt \\
&\leq \frac{(b-a)^4}{24}\sup\{|f^{(4)}(a)|,|f^{(4)}(b)|\}B(3,3) \\
&\leq \frac{(b-a)^4}{720}\sup\{|f^{(4)}(a)|,|f^{(4)}(b)|\}.
\end{aligned}
$$

□

Theorem 3.10. *Let $f : I \subseteq [0,+\infty] \to \mathbb{R}$ be a differentiable on I^o such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f^{(4)}|^q$ is quasi- convex on $[a,b]$ with $\frac{1}{p}+\frac{1}{q}=1, p>1$, then we have the following inequality*

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| - \frac{b-a}{12}|f'(a)-f'(b)| \\
&\leq \frac{(b-a)^4}{384}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\sup\{|f^{(4)}(a)|^q,|f^{(4)}(b)|^q\}\}^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 3.1 and using the quasi-convexity of $|f^{(4)}|$, we obtain

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| - \frac{b-a}{12}|f'(a)-f'(b)| \\
&\leq \frac{(b-a)^4}{24}\int_0^1 t^2(1-t)^2|f^{(4)}(ta+(1-t)b)|dt \\
&\leq \frac{(b-a)^4}{24}(\int_0^1 t^{2p}(1-t)^{2p}dt)^{\frac{1}{p}}(\sup\{|f^{(4)}(a)|^q,|f^{(4)}(b)|^q\})^{\frac{1}{q}} \\
&\leq \frac{(b-a)^4}{24}(B(2p+1,2p+1))^{\frac{1}{p}}(\sup\{|f^{(4)}(a)|^q,|f^{(4)}(b)|^q\})^{\frac{1}{q}} \\
&\leq \frac{(b-a)^4}{384}\{\frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2}+2p)}\}^{\frac{1}{p}}\{\sup\{|f^{(4)}(a)|^q,|f^{(4)}(b)|^q\}\}^{\frac{1}{q}}.
\end{aligned}
$$

□

4. Applications to special means

Consider the following special means for two nonnegative real numbers $\alpha, \beta, \alpha \neq \beta$ as follows:

(1): The arithmetic mean:

$$A = A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \ \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(2): The logarithmic mean:

$$\overline{L} = \overline{L}(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}, \ \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(3): The generalized logarithmic mean:

$$L_n(\alpha, \beta) = [\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}]^{\frac{1}{n}}, \ n \in \mathbb{Z} \backslash \{-1, 0\}, \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $a < b, a \neq 0$ and $n \in \mathbb{N}, \ n \geq 4$. Then, the following inequality holds:*

$$\begin{aligned} |A(a^n, b^n) - L_n(a, b)| \leq & \frac{n^2}{12} \ln |\frac{a}{b}| \overline{L}(|a|^{n-1}, |b|^{n-1}) \\ & + \frac{(b-a)^4}{720} n(n-1)(n-2)(n-3) A(|a|^{n-4}, |b|^{n-4}). \end{aligned}$$

Proof. The proof is immediate from Theorem 3.2 applied for $f(x) = x^n$ for all $x \in \mathbb{R}$. □

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $a < b, a \neq 0$ and $n \in \mathbb{N}, \ n \geq 4, p > 1, \ \frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequality holds:*

$$\begin{aligned} |A(a^n, b^n) - L_n(a, b)| \leq & \frac{n^2}{12} \ln |\frac{a}{b}| \overline{L}(|a|^{n-1}, |b|^{n-1}) \\ & + \frac{(b-a)^4}{24} \{ \frac{2^{1-2(2p+1)} \Gamma(\frac{1}{2}) \Gamma(2p+1)}{\Gamma(\frac{3}{2} + 2p)} \}^{\frac{1}{p}} \{ \frac{|f^{(4)}(a)| + |f^{(4)}(b)|}{2} \}^{\frac{1}{q}} \\ \leq & \frac{n^2}{12} \ln |\frac{a}{b}| \overline{L}(|a|^{n-1}, |b|^{n-1}) + \frac{(b-a)^4}{384} \{ \frac{\Gamma(2p+1)}{\Gamma(\frac{3}{2} + 2p)} \}^{\frac{1}{p}} \\ & \times \{ n(n-1)(n-2)(n-3) A(|a|^{n-4}, |b|^{n-4}) \}^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof is immediate from Theorem 3.4 applied for $f(x) = x^n$ for all $x \in \mathbb{R}$. □

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