FRACTIONAL \( q \)-DIFFERINTEGRAL OPERATOR RELATED TO UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper, we introduce a new subfamily of univalent functions defined in the open unit disk involving a fractional \( q \)-differintegral operator. Some results on coefficient estimates, weighted mean, convolution structure and convexity are discussed.

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1. Introduction

Let \( \mathcal{A}_n \) denote the family of analytic and univalent functions in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), expressed in the type:

\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0, \quad n \in \mathbb{N} = \{1, 2, \ldots\}).
\]
The $q$-shifted factorial is defined for $w, q \in \mathbb{C}$, by:

$$(w, q) = \begin{cases} 1 & , \quad n = 0, \\ (1 - w)(1 - wq) \cdots (1 - wq^{n-1}) & , \quad n \in \mathbb{N}, \end{cases}$$

and according to the basic analogue of the gamma function:

$$(q^w, q)_n = \frac{\Gamma_q(w + n)(1 - q)^n}{\Gamma_q(w)}, \quad (n > 0),$$

where the $q$-gamma function is given by:

$$\Gamma_q(y) = \frac{(q, q)_\infty(1 - q)^{1-y}}{(q^y, q)_\infty}, \quad (0 < q < 1).$$

If $|q| < 1$, the relation (2) is meaningful for $n = \infty$ as a convergent product defined by:

$$(w, q)_\infty = \prod_{j=0}^\infty (1 - wq^j).$$

The corresponding relation for $q$-gamma function is given by:

$$\Gamma_q(1 + y) = \frac{(1 - q^y)\Gamma_q(y)}{1 - q}.$$ 

Jackson’s $q$-derivative and $q$-integral of a function $f(z)$ defined on a subset of $\mathbb{C}$, respectively introduced by:

$$D_q, z f(z) = \frac{f(z) - f(qz)}{z(1 - q)}, \quad (z \neq 0, \quad q \neq 0),$$

and

$$\int_0^z f(t)d(t, q) = z(1 - q) \sum_{k=0}^\infty q^k f(zq^k).$$

See [3] and [4]. Also [1, 6] and [8] are useful.

According to the relation:

$$\lim_{q \to 1} \frac{(q^w, q)_n}{(1 - q)^n} = (w)_n,$$

we conclude that the $q$-shifted factorial (1) reduces to the familiar Pochhammer symbol:

$$(w)_n = w(w + 1) \cdots (w + n - 1).$$
The fractional $q$-integral operator $I_{q,z}^w f(z)$ of a function $f(z)$ of order $w$ is given by:

$$I_{q,z}^w f(z) = D_{q,z}^{-w} f(z)$$

$$= \frac{1}{\Gamma_q(w)} \int_0^z (z - tq)_{w-1} f(t) d(t, q), \quad (w > 0),$$

where $f(z)$ is holomorphic in a simply connected region around the origin. On the other hand, the $q$–binomial function $(z - tq)_{w-1}$ is single–valued when:

$$\left|\arg \left(\frac{-tq^w}{z}\right)\right| < \pi,$$

$$\left|\frac{tq^w}{z}\right| < 1,$$

and $|\arg z| < \pi.$

The fractional $q$-derivative operator $D_{q,z}^w f(z)$ of a function $f(z)$ of order $w$ is introduced as:

$$D_{q,z}^w f(z) = D_{q,z} I_{q,z}^{1-w} f(z)$$

$$= \frac{1}{\Gamma_q(1-w)} D_{q,z} \int_0^z (z - tq)_{-w} f(t) d(t, q), \quad (0 \leq w < 1).$$

The extended fractional $q$-derivative operator for a function $f(z)$ of order $w$ is given as follows, see [7]:

$$D_{q,z}^{-w} f(z) = D_{q,z} I_{q,z}^{m} I_{q,z}^{m-w} f(z), \quad (m - 1 \leq w < n, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Now, we consider a fractional $q$-differintegral operator $\Omega_{q,z}^w$ for a function $f(z)$ of the form (10) by:

$$\Omega_{q,z}^w f(z) = \frac{\Gamma_q(2-w)}{\Gamma_q(2)} z^{w-1} D_{q,z}^w f(z)$$

$$= 1 - \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)} a_k z^{k-1}.$$

For more details see [5], see also [2].

We say that $f \in \mathcal{A}_n$ is in the class $\Omega_{q,z}^w (\alpha, \beta, \gamma, t)$, if it satisfies the inequality:

$$\left|\frac{z^2(\Omega_{q,z}^w f(z))'}{(\gamma + (\alpha - \gamma)(1-\beta)) f_t(z) + \gamma z \Omega_{q,z}^w f(z)}\right| < 1,$$

where $0 \leq t \leq 1$, $-1 \leq \gamma \leq \alpha \leq 1$, $0 < \beta < 1$ and $f_t(z) = (1-t)z + tf(z)$. 
Let \( \Omega_{q,z}^w(\alpha, \beta, \gamma, t) \) be analytic in \( U \). Then \( f \in \Omega_{q,z}^w(\alpha, \beta, \gamma, t) \) if and only if:

\[
\sum_{k=n+1}^{\infty} \left[ \frac{\Gamma_q(2-w)\Gamma_q(k+1)(k-1-\gamma)}{\Gamma_q(2)\Gamma_q(k+1-w)(\alpha-\gamma)(1-\beta)} \right] a_k \leq 1.
\]

**Proof.** Let \( |z| = 1 \) and (17) holds true. So we have:

\[
\left| z^2(\Omega_{q,z}^w(f(z)))' - (\gamma + (\alpha - \gamma)(1-\beta))f_z(z) - \gamma\Omega_{q,z}^w(f(z)) \right| = -\sum_{k=n+1}^{\infty} \left| \frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)} (k-1)a_k z^k \right| \\
\leq \sum_{k=n+1}^{\infty} \left| \left( t(\alpha - \gamma)(1-\beta) - \frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)} \right) a_k z^k \right|.
\]

By (17), the above inequality is less than or equal to zero, so \( f(z) \in \Omega_{q,z}^w(\alpha, \beta, \gamma, t) \).

To prove the converse, let \( f(z) \in \Omega_{q,z}^w(\alpha, \beta, \gamma, t) \), thus:

\[
\left| \frac{z^2(\Omega_{q,z}^w(f(z)))'}{(\gamma + (\alpha - \gamma)(1-\beta))f_z(z) + \gamma\Omega_{q,z}^w(f(z))} \right| = \left| \sum_{k=n+1}^{\infty} \left( t(\alpha - \gamma)(1-\beta) - \frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)} \right) a_k z^k \right| < 1.
\]

Since for all \( z \in U \), \( \Re\{z\} \leq |z| \), so we get:

\[
\Re \left\{ \sum_{k=n+1}^{\infty} \frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)} (k-1)a_k z^k \right\} < 1.
\]
By letting $z \to 1$ through positive values and choosing the values of $z$ such that $z^2(\Omega_{q,z}^w f(z))'$, $z\Omega_{q,z}^w f(z)$ and $f_t(z)$ are real, we conclude:

$$\sum_{k=n+1}^{\infty} \left( (k - 1) \frac{\Gamma_q(2 - w)\Gamma_q(k + 1)}{\Gamma_q(2)\Gamma_q(k + 1 - w)} \right) a_k \leq (\alpha - \gamma)(1 - \beta) - \sum_{k=n+1}^{\infty} \left( t(\alpha - \gamma)(1 - \beta) - \frac{\Gamma_q(2 - w)\Gamma_q(k + 1)}{\Gamma_q(2)\Gamma_q(k + 1 - w)} \right) a_k.$$

So, we get:

$$\sum_{k=n+1}^{\infty} \left( \frac{\Gamma_q(2 - w)\Gamma_q(k + 1)}{\Gamma_q(2)\Gamma_q(k + 1 - w)} \right) (k - 1 - \gamma) + t(\alpha - \gamma)(1 - \beta) a_k \leq (\alpha - \gamma)(1 - \beta),$$

and this complete the proof. \hfill \Box

**Remark 2.2.** We note that the function:

$$\frac{\Gamma_q(2 - w)\Gamma_q(k + 1)}{\Gamma_q(2)\Gamma_q(k + 1 - w)} \left( k - 1 - \gamma \right) + t(\alpha - \gamma)(1 - \beta)$$

shows that the inequality (17) is sharp.

Also by applying Theorem 2.1, if $f(z) \in \Omega_{q,z}^w(\alpha, \beta, \gamma, t)$, then for $k \geq n + 1$:

$$a_k \leq \frac{\Gamma_q(2)\Gamma_q(k + 1 - w)(\alpha - \gamma)(1 - \beta)}{\Gamma_q(2 - w)\Gamma_q(k + 1)(k - 1 - \gamma) + t\Gamma_q(2)\Gamma_q(k + 1 - w)(\alpha - \gamma)(1 - \beta)}.$$

Now, we introduce weighted mean property.

**Theorem 2.3.** If $f$ and $g$ belong to $\Omega_{q,z}^w(\alpha, \beta, \gamma, t)$, then the weighted mean of $f$ and $g$ is also in the same class.

**Proof.** We have to prove that:

$$h_m(z) = \frac{1 - m}{2} f(z) + \frac{1 + m}{2} g(z),$$

is in the class $\Omega_{q,z}^w(\alpha, \beta, \gamma, t)$.

Since $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k$, so:

$$h_m(z) = z - \sum_{k=n+1}^{\infty} \left( \frac{(1 - m)a_k + (1 + m)b_k}{2} \right) z^k.$$

To prove $h_m(z) \in \Omega_{q,z}^w(\alpha, \beta, \gamma, t)$, by Theorem 2.1, we need to show that:

$$L = \sum_{k=n+1}^{\infty} \left( \frac{\Gamma_q(2 - w)\Gamma_q(k + 1 - \gamma) + t(\alpha - \gamma)(1 - \beta)}{\Gamma_q(2)\Gamma_q(k + 1 - w)(\alpha - \gamma)(1 - \beta)} \right) \left( \frac{(1 - m)a_k + (1 + m)b_k}{2} \right) < 1.$$
But, for this we have:

\[
L = \left(\frac{1-m}{2}\right) \sum_{k=n+1}^{\infty} \left( \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t \right) a_k \\
+ \left(\frac{1+m}{2}\right) \sum_{k=n+1}^{\infty} \left( \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t \right) b_k.
\]

By (17), we get:

\[
L < \frac{1-m}{2} + \frac{1+m}{2} = 1.
\]

Hence the result follows. ⪿

3. Convolution preserving and convexity

In this section, we show that the family \(\Omega_{w_q,z}^{w_q}(\alpha, \beta, \gamma, t)\) is closed under convolution. Also we conclude that this class is a convex set.

**Theorem 3.1.** Let \(f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k\) and \(g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k\) be in the class \(\Omega_{q,z}^{w}(\alpha, \beta, \gamma, t)\), then \((f * g)(z)\) defined by:

\[
(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k b_k z^k,
\]

belongs to \(\Omega_{q,z}^{w}(\alpha, \hat{\beta}, \gamma, t)\), where:

\[
\hat{\beta} \leq 1 - \frac{(k-1-\gamma)(\alpha-\gamma)(1-\beta)^2\Gamma_q(2-w)\Gamma_q(k+1)}{W^2 - t(\alpha-\gamma)^2(1-\beta)^2\Gamma_q(2)\Gamma_q(k+1-w)},
\]

and

\[
W = (k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w).
\]

**Proof.** By Theorem 2.1, it is sufficient to show that:

\[
\sum_{k=n+1}^{\infty} \left( \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + 1 \right) a_k b_k \leq 1.
\]

By applying Cauchy-Schwarz inequality, from (17), we get:

\[
\sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} \sqrt{a_k b_k} \leq 1.
\]
Hence, we find the largest \( \hat{\beta} \) such that:

\[
\sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)} a_k b_k \\
\leq \sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)} \sqrt{a_k b_k} \leq 1,
\]

or equivalently:

\[
\sqrt{a_k b_k} \leq \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)} \\
\times \frac{1-\hat{\beta}}{1-\beta} \\
\frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)}{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)}.
\]

This inequality holds if,

\[
\frac{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k + 1 - w)}{W} \leq \frac{1-\hat{\beta}}{1-\beta} \frac{1-\beta}{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k + 1 - w)},
\]

where \( W \) is given in (22). So

\[
\frac{1-\hat{\beta}}{W^2} \geq \frac{(\alpha-\gamma)(1-\beta)^2}{W^2}.
\]

After a simple calculation, we obtain the required result. \( \square \)

**Remark 3.2.** With the same assumptions of Theorem 3.1, \((f \ast g)(z)\) belongs to \( \Omega_{q,z}^w(\alpha, \beta, \hat{\gamma}, t) \), where:

\[
\hat{\gamma} \leq \frac{\alpha X - k + 1}{X - 1},
\]

\[
X = \frac{W^2 - t(1-\beta^2)(\alpha-\gamma)^2\Gamma_q(2)\Gamma_q(k + 1 - w)}{(\alpha-\gamma)^2(1-\beta)\Gamma_q(2-w)\Gamma_q(k + 1)},
\]

and \( W \) is given in (22).

**Theorem 3.3.** The class \( \Omega_{q,z}^w(\alpha, \beta, \hat{\gamma}, t) \) is a convex set.
Proof. It is enough to show that if \( f_j(z) \) \((j = 1, 2, \ldots, m)\) be in the class \( \Omega^w_{q,z}(\alpha, \beta, \gamma, t) \), then the function:

\[
H(z) = \sum_{j=1}^{m} \delta_j f_j(z),
\]

is also in \( \Omega^w_{q,z}(\alpha, \beta, \gamma, t) \), with \( \delta_j \geq 0 \) and \( \sum_{j=1}^{m} \delta_j = 1 \). By (23), we obtain:

\[
H(z) = \sum_{j=1}^{m} \delta_j \left( z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \right)
= z - \sum_{k=n+1}^{\infty} \left( \sum_{j=1}^{m} \delta_j a_{k,j} \right) z^k.
\]

But by Theorem 2.1, we have:

\[
\sum_{k=n+1}^{\infty} \left( \frac{(k-1-\gamma) \Gamma_q(z-w) \Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta) \Gamma_q(2) \Gamma_q(k+1-w)} + t \right) \left( \sum_{j=1}^{m} \delta_j a_{k,j} \right) \delta_j
\]

by (17), we have:

\[
< \sum_{j=1}^{m} \delta_j
= 1.
\]

So by Theorem 2.1, we get the required result. \( \square \)

References


