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ON ERGODIC SHADOWING AND SPECIFICATION PROPERTIES OF NONAUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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Abstract. We show that a nonautonomous discrete-time dynamical system (NDS) with the ergodic shadowing property is chain mixing. As a result, it is obtained that a k-periodic NDS with the ergodic shadowing property has the shadowing property. In particular, any k-periodic NDS on intervals having the ergodic shadowing is Devaney chaotic. Additionally, we prove that for an equicontinuous NDS with the shadowing property, the notions of topologically mixing, pseudo-orbital specification, weak specification property, and ergodic shadowing property are equivalent.

Keywords: Nonautonomous discrete systems, Ergodic shadowing, Specifiaction property, topologically mixing.
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1. Introduction

The pseudo orbit tracing property is one of the most important notions in the dynamical system, which is closely related to the chaos and stability of systems; see, for example, [2, 16, 18]. From the numerical point of view if a mapping f has the pseudo-orbit tracing property, then the orbits obtained in the process of numerical computation reflect the real dynamical behavior of f. Though the most evident application of the shadowing is related to numerical methods, the first result involving the concept of pseudo trajectories was obtained by Anosov [1] and Bowen [3] as a tool to study the qualitative property of dynamical systems. Shadowing together with topological mixing and specification are popular formal ways to get chaos in a global sense. In the recent work [6], Fakhari and Ghane introduced the notions of ergodic shadowing and pseudo orbital specification properties and showed that for a continuous onto map f, these properties are equivalent to the map being topologically mixing and has the shadowing property.

The notion of nonautonomous discrete-time dynamical system (NDS) was introduced by Kolyada and Snoha [7]. Instead, by the iteration of one map,

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a discrete-time nonautonomous dynamical system is defined by a sequence of maps that are composed in the given order, that is, at each time instance the dynamical law can be different. Thus, the study of dynamics of NDSs is usually more complex than the same studies in the setting of autonomous systems (that is, a system is given by a pair (X, f), where f is a continuous map). However many complex systems occurring in the real-world problems such as physical, biological, and economical are necessarily described by NDSs; see, for example, [4, 5]. Devaney Chaos for the nonautonomous discrete system was introduced by Tian and Chen [15]. Later, In [14], the author studied the shadowing, expansiveness, and topological stability of nonautonomous discrete dynamical systems induced by a sequence of continuous maps.

The author in [9], introduced the concept of weak specification property and studied its relation to topologically mixing, shadowing, and distality in nonautonomous systems. Several other kinds of specification were defined and studied including strong specification and quasi-weak specification properties in [13], and the authors showed that any k-periodic NDS on intervals having weak specification property is Devaney chaotic. Recently, in 2019, the authors generalized the notion of ergodic shadowing and pseudo orbital specification properties to NDSs and proved that every uniformly equicontinuous NDS with the shadowing and topologically mixing properties has the ergodic shadowing property; see [10]. Moreover, they showed that any NDS with the ergodic shadowing property is chain transitive.

This note proves that any uniformly equicontinuous NDS with the ergodic shadowing property is chain mixing. As an application, it is obtained that, for a k-periodic NDS, the ergodic shadowing implies that the shadowing property, in particular, any k-periodic NDS on intervals having the ergodic shadowing property, is Devaney chaotic. Further, we prove the following result.

Theorem 1.1. *Let $(X, f_{1,\infty})$ be a uniformly equicontinuous NDS given by a sequence of continuous surjective maps $\{f_i\}_{i=1}^{\infty}$ on a compact metric space X. If the NDS $(X, f_{1,\infty})$ has the shadowing property, then the following conditions are equivalent:*

(1) *Topologically mixing,*
(2) *Chain mixing*
(3) *Weak specification property,*
(4) *Pseudo-orbital specification property,*
(5) *Ergodic shadowing property.*

The present paper is organized as follows: Sec. 2 contains main definitions and background information. In Sec. 3, we show that any uniformly equicontinuous NDS with the ergodic shadowing property, is chain mixing. Moreover, we prove that any k-periodic nonautonomous discrete system with ergodic shadowing property, enjoys the shadowing property. In Sec. 4, we prove the invariance of ergodic shadowing property under iterations and the proof of Theorem 1.1

is stated. Also, we build an example that shows that the shadowing property does not result in the ergodic shadowing property, in general.

2. **Preliminaries**

Throughout this paper, (X, d) will denote a compact metric space, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$, and $B_\epsilon(x)$ denotes the open ball of radius $\epsilon > 0$ and center x.

Let $f_n : X \to X$ be a sequence of continuous maps, $n = 1, 2, \ldots$. Denote $f_{1,\infty} := \{f_n\}_{n=1}^{\infty}$ and for all positive integer i and n,

$$f_n^i := f_{n+i-1} \circ \cdots \circ f_n, \quad f_n^0 := id,$$

and the kth iterate by $f_{1,\infty}^{[k]} = \{f_{k(n-1)+1}^k\}_{n=1}^{\infty}$ for any $k \in \mathbb{N}$. We call the pair $(X, f_{1,\infty})$ a nonautonomous discrete-time dynamical system (on X) or simply NDS. The orbit of a point $x \in X$ under the NDS $(X, f_{1,\infty})$ is

$$\mathcal{O}_{f_{1,\infty}}(x) := \{x, f_1^1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\}.$$

We say that the NDS $(X, f_{1,\infty})$ is *uniformly equicontinuous* whenever for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $i \in \mathbb{N}$ and any two points $x, y \in X$, if $d(x, y) < \delta$, then $d(f_i(x), f_i(y)) < \epsilon$. For any two nonempty open subsets U and V of X, we have

$$N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}.$$

Following Tian and Chen [15], an NDS $(X, f_{1,\infty})$ is said to be *topologically transitive*, if $N_{f_{1,\infty}}(U, V) \neq \emptyset$ for any pair of nonempty open sets U, V of X, and it is said to be *topologically mixing* if there exists $N \in \mathbb{N}$ such that $N_{f_{1,\infty}}(U, V) \supset [N, \infty)$.

According to Thakkar and Das [14], for $\delta > 0$, the sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be a δ-*pseudo orbit* of the NDS $(X, f_{1,\infty})$ if $d(f_{i+1}(x_i), x_{i+1}) < \delta$ for all $i \geq 0$. The NDS $(X, f_{1,\infty})$ is said to have the *shadowing property* if every δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-traced by some point $z \in X$, that is,

$$d(f_1^i(z), x_i) < \epsilon, \quad \text{for all} \quad i \geq 0.$$

An ϵ-*chain* of NDS $(X, f_{1,\infty})$ from x to y of length n is a finite pseudo orbit $x_0 = x, x_1, \ldots, x_{n-1} = y$. An NDS $(X, f_{1,\infty})$ is *chain transitive* if, for every $\epsilon > 0$, there exists an ϵ-chain from x to y. The NDS $(X, f_{1,\infty})$ is called *chain mixing* if for any two points $x, y \in X$ and any $\epsilon > 0$, there is a positive integer N such that for any integer $n \geq N$, there is an ϵ-chain from x to y of length n. The NDS $(X, f_{1,\infty})$ is *totally chain transitive* if $f_{1,\infty}^{[k]}$ is chain transitive for every $k \geq 1$.

For any $A \subset \mathbb{Z}^+$, the *upper density* of A is defined by

$$d(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}| \tag{1}$$

Replacing lim sup with lim inf in (1) gives the definition of $\underline{d}(A)$, the lower density of A. If there exists a number $d(A)$ such that $d(A) = \underline{d}(A) = \overline{d}(A)$, then we say that the set A has density $d(A)$.

In [10], the authors generalized the notions of ergodic shadowing and pseudo orbital specification properties to the nonautonomous discrete system under the following set of terminology. Given a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ and $\delta > 0$, put

$$Npo(\xi, \delta) = \{i \in \mathbb{Z}^+ : d(f_{i+1}(x_i), x_{i+1}) \geq \delta\}$$

and

$$Npo_n(\xi, \delta) = Npo(\xi, \delta) \cap \{0, 1, \ldots, n-1\}.$$

For a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ and a point $z \in X$, consider

$$Ns(\xi, z, \delta) = \{i \in \mathbb{Z}^+ : d(f_1^i(z), x_i) \geq \delta\}$$

and

$$Ns_n(\xi, z, \delta) = Ns(\xi, z, \delta) \cap \{0, 1, \ldots, n-1\}.$$

Definition 2.1. [10] Let $\delta > 0$ and let $\xi = \{x_i\}_{i=0}^{\infty} \subset X$. We say that ξ is a *δ-ergodic pseudo orbit* of NDS $(X, f_{1,\infty})$ if $Npo(\xi, \delta)$ has density zero, that is,

$$\lim_{n\to\infty} \frac{1}{n}|Npo_n(\xi, \delta)| = 0.$$

An NDS $(X, f_{1,\infty})$ has the *ergodic shadowing property* if for any $\epsilon > 0$, there exists $\delta > 0$ such that any δ-ergodic pseudo orbit of $(X, f_{1,\infty})$ can be ϵ-ergodic shadowed by some point $z \in X$, that is,

$$\lim_{n\to\infty} \frac{1}{n}|Ns_n(\xi, z, \epsilon)| = 0.$$

Now, we recall the concept of pseudo-orbital specification property of NDSs.

Definition 2.2. [10] We say that an NDS $(X, f_{1,\infty})$ has the *pseudo-orbital specification property* if for any $\epsilon > 0$, there exist $\delta(\epsilon)$ and $N(\epsilon) > 0$ such that for any choice of points $x_1, \ldots, x_s \in X$, any sequence of nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ with $a_{j+1} - b_j > N(\epsilon)$, and the δ-pseudo orbit $\xi_j = \{x_{(j,i)}\}$, $i \in [a_j, b_j]$ and $1 \leq j \leq s$, there is a point $z \in X$ such that

$$d(f_1^i(z), x_{(j,i)}) < \epsilon, \quad \text{for any} \quad a_j \leq i \leq b_j, \quad 1 \leq j \leq s.$$

Here, we define some kind of specification property that is weaker than the pseudo-orbital specification property.

Definition 2.3. [13] We say that an NDS $(X, f_{1,\infty})$ has the *weak specification property* if for any $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that for any choice of points $x_1, \ldots, x_s \in X$ and any sequence of nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ with $a_{j+1} - b_j > N(\epsilon)$, $(1 \leq j \leq s)$, there is a point $z \in X$ satisfying

$$d(f_1^i(z), f_1^i(x_j)) < \epsilon, \quad \text{for any} \quad a_j \leq i \leq b_j, \quad 1 \leq j \leq s.$$

3. Shadowing and chain properties of NDS

In this section, we investigate the relation between ergodic shadowing and pseudo orbital specification properties with the shadowing property. First, we show that any NDS with ergodic shadowing property, is chain mixing.

Proposition 3.1. *Let $(X, f_{1,\infty})$ be an NDS given by the sequence $\{f_n\}_{n=1}^{\infty}$ of surjective maps on the compact metric space X. If the NDS $(X, f_{1,\infty})$ has the ergodic shadowing property, then it is chain mixing.*

Proof. We use the approach used in the proof of Lemma 3 in [17]. Let $x, y \in X$ and $\epsilon > 0$ be given. Choose $\delta > 0$ by the ergodic shadowing property of $(X, f_{1,\infty})$. Let $a_0 = A_0 = 0$, let $a_1 = A_1 = 2$, and let $a_n = n^{\Sigma_{i=1}^{n-1} a_i}$ and $A_n = \Sigma_{i=1}^{n} a_i$ for any $n \geq 2$. Take a sequence $\{x_i\}_{i=0}^{\infty}$ by $x_0 = x, x_1 = y$ and for any $n \geq 1$,

$$x_i := \begin{cases} f_1^i(x), & i \in [C_{n,k}, C_{n,k} + [\frac{n+1}{2}]), k \in [0, \frac{a_{n+1}}{n+1}), \\ f_{C_{n,k}+1}^{i-C_{n,k}-[\frac{n+1}{2}]}(z_{n,k}), & i \in [C_{n,k} + [\frac{n+1}{2}], C_{n,k} + n + 1), k \in [0, \frac{a_{n+1}}{n+1}), \end{cases}$$

where $C_{n,k} := A_n + k(n+1)$ and $z_{n,k} \in f_{C_{n,k}+1}^{-(n-[\frac{n+1}{2}])}(y)$. We can see that $\{x_i\}_{i=0}^{\infty}$ is a δ-ergodic pseudo orbit; therefore there exists a point $z \in X$ such that ϵ-ergodic shadows the sequence $\{x_i\}_{i=0}^{\infty}$.

Claim. There exists an integer $N \in \mathbb{N}$ such that for any $n \geq N$, there exists $k \in [0, \frac{a_{n+1}}{n+1})$ such that $Ns(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_n + k(n+1), A_n + k(n+1) + [\frac{n+1}{2}]) \neq \emptyset$ and $Ns(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_n + k(n+1) + [\frac{n+1}{2}], A_n + (k+1)(n+1)) \neq \emptyset$.

Proof of claim. By contrary, suppose that for any $N \in \mathbb{N}$, there exists $n(N) \geq N$ such that for any $k \in [0, \frac{a_{n(N)+1}}{n(N)+1})$, $Ns(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_{n(N)} + k(n(N)+1), A_{n(N)} + k(n(N)+1) + [\frac{n(N)+1}{2}]) = \emptyset$ or $Ns(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_{n(N)} + k(n(N)+1) + [\frac{n(N)+1}{2}], A_{n(N)} + (k+1)(n(N)+1)) = \emptyset$. This yields that

$$\begin{aligned} |Ns^c(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_{n(N)}, A_{n(N)+1})| &\geq \frac{A_{n(N)+1} - A_{n(N)}}{2} - \frac{a_{n(N)+1}}{n(N)+1} \\ &= \frac{n(N)-1}{2(n(N)+1)} a_{n(N)+1}. \end{aligned}$$

Therefore

$$\begin{aligned} \overline{d}(Ns^c(\{x_i\}_{i=0}^{\infty}, z, \epsilon)) &\geq \limsup_{N\to\infty} \frac{1}{A_{n(N)+1}} |Ns^c(\{x_i\}_{i=0}^{\infty}, z, \epsilon) \cap [A_{n(N)}, A_{n(N)+1})| \\ &\geq \limsup \frac{n(N)-1}{2(n(N)+1)} a_{n(N)+1} \frac{1}{A_{n(N)+1}} = \frac{1}{2}, \end{aligned}$$

which contradicts to $d(Ns^c(\{x_i\}_{i=0}^{\infty}, z, \epsilon)) = 0$. Thus for any $n \geq N$, there exist $k \in [0, \frac{a_{n+1}}{n+1})$, $i \in [A_n + k(n+1), A_n + k(n+1) + [\frac{n+1}{2}])$, $j \in [A_n + k(n+1) + [\frac{n+1}{2}], A_n + (k+1)(n+1))$, and $s \in [0, n+1-[\frac{n+1}{2}])$ such that

$$d(f_1^i(z), f_1^i(x)) < \epsilon, \quad d(f_1^j(z), t) < \epsilon, \quad t \in f_{j+1}^{-s}(y).$$

Hence

$$\{x, f_1(x), \ldots, f_1^{i-1}(x), f_1^i(z), \ldots, f_1^{j-1}(z), t, f_{j+1}(t), \ldots, f_{j+1}^{s-1}(t), y\}$$

is a δ-chain from x to y. □

Lemma 3.2. *Any NDS $(X, f_{1,\infty})$ with the pseudo orbital specification property, has the shadowing property.*

Proof. Let $\epsilon > 0$ be given and let $\delta = \delta(\epsilon)$ be an ϵ-modulus of pseudo-orbital specification property. By [11, Lemma 2.1], it is enough to show that any finite δ-pseudo orbit of $(X, f_{1,\infty})$, is ϵ-shadowed by a true orbit. Let $\xi = \{x_i\}_{i=0}^n$ be a finite δ-pseudo orbit of $(X, f_{1,\infty})$, that is,

$$d(f_{i+1}(x_i), x_{i+1}) < \delta, \quad 0 \le i \le n-1.$$

Put $a_1 = 0, b_1 = n$, and $x_{(1,i)} := x_i, 0 \le i \le n$. Then $\xi_1 = \{x_{(1,i)}\}_{i=0}^n$ is a δ-pseudo orbits for $(X, f_{1,\infty})$. By the pseudo-orbital specification property, there exists a point $z \in X$ such that $d(f_1^i(z), x_i) < \epsilon$ for any $0 \le i \le n$. □

Definition 3.3. [8] An NDS $(X, f_{1,\infty})$ is called *k-periodic* if there exists $k \in \mathbb{N}$ such that $f_{i+k}(x) = f_i(x)$ for any $x \in X$ and $i \in \mathbb{N}$.

Using Proposition 3.1, we show that any k-periodic NDS with the ergodic shadowing property, has the shadowing property, but the converse is not true; see Example 4.6.

Proposition 3.4. *If an NDS $(X, f_{1,\infty})$ is k-periodic and has the ergodic shadowing property, then it has the shadowing property.*

Proof. Suppose that $(X, f_{1,\infty})$ is k-periodic for some $k \in \mathbb{N}$. By [11, Lemma 2.1], it is enough to show that any finite pseudo orbit of $(X, f_{1,\infty})$, is ϵ-shadowed by a true orbit. Let $\epsilon > 0$ and let $\delta > 0$ be as in the definition of the ergodic shadowing. We will show that any finite δ-pseudo orbit $\xi = \{x_i\}_{i=0}^n$ with $n = mk$ for some $m \in \mathbb{N}$ is ϵ-shadowed by the orbit of some point X. By Proposition 3.1, we can choose a δ-chain $y_0 = x_n, y_1, \ldots, y_\ell = x_0$ from x_n to x_0, with $n+\ell = sk$ for some $s \in \mathbb{N}$. Thus $d(f_{i+1}(x_i), x_{i+1}) < \delta, i = 0, 1, \ldots, n-1$ and

$$d(f_{n+j+1}(y_j), y_{j+1}) = d(f_{j+1}(y_j), y_{j+1}) < \delta \quad \text{for any} \quad j = 0, \ldots, \ell-1.$$

Therefore, by the construction, the sequence $\{\xi_j\}_{j=0}^\infty$ is given by

$$\xi_j := \begin{cases} x_t, & t \in \{0, 1, \ldots, n-1\}, \\ y_{t-n}, & t \in \{n, n+1, \ldots, n+\ell-1\}, \end{cases}$$

where $t := j \bmod n+\ell$, is a δ-pseudo orbit of $(X, f_{1,\infty})$. By the ergodic shadowing property of the NDS $(X, f_{1,\infty})$, the sequence $\{\xi_j\}_{j=0}^\infty$ is ϵ-ergodic shadowed by some point $z \in X$. Since the point z ergodic shadows $\{\xi_j\}_{j=0}^\infty$, the set $Ns(\{\xi_j\}_{j=0}^\infty, z, \epsilon)$ cannot meet every interval $[r(n+\ell), r(n+\ell)+n-1] \subset \mathbb{N}, r \ge 0$, otherwise it would have positive density. Hence, at least one n interval is entirely shadowed by a piece of the z orbit. □

4. **Proof of main theorem**

The aim of this section is to show the invariance of ergodic showing property under iterations, also we prove some lemmas that lead to prove Theorem 1.1.

Lemma 4.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous self maps on X. If a uniformly equicontinuous NDS $(X, f_{1,\infty})$ given by this sequence has the ergodic shadowing property, then $(X, f_{1,\infty}^{[k]})$, for any $k > 1$, has the ergodic shadowing property.*

Proof. Fix $k > 1$, and let $\epsilon > 0$ be given. Let $\delta > 0$ be given by the definition of the ergodic shadowing property of $(X, f_{1,\infty})$. Let $\xi = \{y_j\}_{j=0}^{\infty}$ be a δ-ergodic pseudo orbit of $(X, f_{1,\infty}^{[k]})$. Put $x_{ik+j} = f_{ik+1}^{j}(y_i), 0 \leq j < k, i \geq 0$. Then $\eta = \{x_i\}_{i=0}^{\infty}$ is a δ-ergodic pseudo orbit of $(X, f_{1,\infty})$, so it can be ϵ-ergodic shadowed by some point $z \in X$, that is, $d(Ns(\eta, z, \epsilon)) = 0$. Since the NDS $(X, f_{1,\infty})$ is uniformly equicontinuous, we can choose $0 < \epsilon_0 < \epsilon/k$ such that for any $x, y \in X$ with $d(x, y) < \epsilon_0$, we have $d(f_j^i(x), f_j^i(y)) < \epsilon/k$ for any $0 \leq i < k$ and $j \geq 1$. This implies that if $d(f_1^i(z), x_i) < \epsilon_0$ for some $i \geq 0$, then $d(f_1^{i+j}(z), x_{i+j}) < \epsilon$ for any $0 \leq j \leq k-1$. This yields that

$$
\begin{aligned}
d(Ns(\eta, z, \epsilon_0)) &= \lim_{n\to\infty} \frac{1}{n}|Ns(\eta, z, \epsilon_0) \cap \{0, 1, \ldots, n-1\}| \\
&\geq \lim_{n\to\infty} \frac{k}{n}|Ns(\xi, z, \epsilon) \cap \{0, 1, \ldots, [n/k]-1\}| \\
&\geq \lim_{n\to\infty} \frac{1}{[n/k]+1}|Ns(\xi, z, \epsilon) \cap \{0, 1, \ldots, [n/k]-1\}| \\
&= \lim_{n\to\infty} \frac{1}{[n/k]}|Ns(\xi, z, \epsilon) \cap \{0, 1, \ldots, [n/k]-1\}|.\frac{[n/k]}{[n/k]+1} \\
&= d(Ns(\xi, z, \epsilon)).
\end{aligned}
$$

By choosing $\delta \leq \delta_0 < \epsilon_0$, where δ_0 is an ϵ_0-modulus of the ergodic shadowing property of $(X, f_{1,\infty})$ and the fact that $d(Ns(\eta, z, \epsilon_0)) = 0$, we have $d(Ns(\xi, z, \epsilon)) = 0$. □

Corollary 4.2. *If the NDS $(X, f_{1,\infty})$ has the ergodic shadowing property, then it is totally chain transitive.*

Proof. If the NDS $(X, f_{1,\infty})$ has the ergodic shadowing property, then using Lemma 4.1 implies that $(X, f_{1,\infty}^{[k]})$ has the ergodic shadowing property for any natural number k, so by [10, Lemma 5], $(X, f_{1,\infty}^{[k]})$ is chain transitive for any $k \in \mathbb{N}$. □

Lemma 4.3. *If the NDS $(X, f_{1,\infty})$ has the shadowing property, then the NDS $(X, f_{1,\infty})$ is chain mixing if and only if it is topologically mixing.*

Proof. Suppose that $(X, f_{1,\infty})$ is chain mixing. Let $U, V \subset X$ be a pair of nonempty open subsets of X, let $x \in U$, and let $y \in V$. Choose $\epsilon > 0$ such that $B_\epsilon(x) \subset U$ and $B_\epsilon(y) \subset V$. Let $\delta > 0$ be given by the shadowing property of $(X, f_{1,\infty})$. Since $(X, f_{1,\infty})$ is chain mixing, for any sufficiently large integer $n > 0$, there exists a δ-chain $\{x_i\}_{i=0}^n$ with $x_0 = x$ and $x_n = y$. Now, the sequence

$$\{x_0, x_1, \ldots, x_n, f_{n+1}(x_n), f_{n+1}^2(x_n), \ldots\}$$

is a δ-pseudo orbit for $(X, f_{1,\infty})$. Therefore, there exists a point $z \in X$ such that $d(z, x) < \epsilon$ and $d(f_1^n(z), x_n) < \epsilon$, that is, $z \in B_\epsilon(x) \subset U$ and $f_1^n(z) \in B_\epsilon(y) \subset V$. Thus the NDS $(X, f_{1,\infty})$ is topologically mixing.

Conversely, Let $x, y \in X$ and let $\epsilon > 0$ be given. Let $0 < \delta < \epsilon$ be such that $d(x, y) < \delta$ implies that $d(f_1(x), f_1(y)) < \epsilon$. Choose $N > 0$ such that $f_1^n(B_\delta(x)) \cap B_\delta(y) \neq \emptyset$ for any $n \geq N$. It yields that there is $z \in B_\delta(x)$ such that $f_1^n(z) \in B_\delta(y)$. Therefore, $\{x, f_1(z), f_1^2(z), \ldots, f_1^{n-1}(z), y\}$ is an ϵ-chain from x to y. □

Corollary 4.4. *Let the NDS $(I, f_{1,\infty})$ be a k-periodic NDS on the interval I. If $(I, f_{1,\infty})$ has the ergodic shadowing, then it is Devaney chaotic.*

Proof. By Proposition 3.4, the NDS $(I, f_{1,\infty})$ has the shadowing property. Hence applying Proposition 3.1 and Lemma 4.3, implies that $(I, f_{1,\infty})$ is topologically mixing. Therefore by [13, Corollary 4.3], $(I, f_{1,\infty})$ is Devaney chaotic. □

It is clear from the definition that if the NDS $(X, f_{1,\infty})$ has the pseudo-orbital specification property, then it has the weak specification property. In the following, we show that the converse is true for an NDS with the shadowing property.

Lemma 4.5. *If the NDS $(X, f_{1,\infty})$ has the shadowing and weak specification properties, then it has the pseudo-orbital specification property.*

Proof. Let $\epsilon > 0$ be arbitrary and let $\epsilon_0 < \epsilon/2$. Let $\delta > 0$ be an ϵ_0-modulus of the shadowing property and let $N(\epsilon_0)$ be an ϵ_0-modulus of the specification property of the NDS $(X, f_{1,\infty})$. For any nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \ldots a_s \leq b_s$ with $a_{j+1} - b_j > N(\epsilon_0), 1 \leq j \leq s$ and any δ-pseudo orbit $\xi_j = \{x_{(j,i)}\}, i \in I_j = [a_j, b_j] \subset \mathbb{N}, 1 \leq j \leq s$, since $f_i, i \geq 1$ are surjective, any finite δ-chain ξ_j can be extended to a δ-pseudo orbit for $(X, f_{1,\infty})$, which is denoted by ξ'_j. Since the NDS $(X, f_{1,\infty})$ has the shadowing property, there exists a point $z_j \in X, j = 1, \ldots, s$, that ϵ-shadows the sequence ξ'_j, in particular,

$$d(f_1^i(z_j), x_{(j,i)}) < \epsilon_0, \quad \text{for any} \quad i \in I_j = [a_j, b_j], \quad 1 \leq j \leq s. \tag{2}$$

Since $(X, f_{1,\infty})$ has the specification property, there exists $z \in X$ satisfying

$$d(f_1^i(z), f_1^i(z_j)) < \epsilon_0, \quad \text{for any} \quad i \in I_j = [a_j, b_j], 1 \leq j \leq s. \tag{3}$$

This together with (2) implies that

$$d(f_1^i(z), x_{(j,i)}) \leq d(f_1^i(y), f_1^i(z_j)) + d(f_1^i(z_j), x_{(j,i)}) < 2\epsilon_0 = \epsilon.$$

□

Now, we prove our main result.

Proof of Theorem 1.1. Lemma 4.3 shows (1) $\Leftrightarrow$ (2). (3) $\Leftrightarrow$ (4) follows from Lemma 4.5 and the fact that any NDS with the pseudo-orbital specification property has the weak specification property. (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obtained from [10, Lemma 3] and [10, Lemma 4], respectively. Proposition 3.1 implies (5) $\Rightarrow$ (2). □

Now, we construct a k-periodic NDS with the shadowing property, which does not have the ergodic shadowing property.

Example 4.6. Let Σ^2 be the space of two-sided sequences of 0 and 1, that is , $\Sigma^2 = \{0,1\}^{\mathbb{Z}}$ and let the shift map $\sigma : \Sigma^2 \to \Sigma^2$, be defined by

$$\sigma(\ldots\omega_{-1}, \omega_0\omega_1\omega_2\ldots) = (\ldots\omega_{-1}\omega_0, \omega_1\omega_2\ldots).$$

Let $f_{1,\infty} = \{\sigma, \sigma^{-2}, \sigma^2, \sigma, \sigma^{-2}, \sigma^2, \ldots\}$. Then the 3-periodic NDS $(\Sigma^2, f_{1,\infty})$, has the shadowing property. Indeed, $f_{1,\infty}^{[3]} = \{g_n\}_{n=1}^{\infty}$, where $g_n(\omega) = \sigma(\omega)$ for any $n \geq 1$ and $\omega \in \Sigma^2$. Since the NDS $(\Sigma^2, f_{1,\infty})$ is equicontinuous and the mapping σ on Σ^2 has the shadowing property, using [14, Theorem 3.5] implies that $(\Sigma^2, f_{1,\infty})$ has the shadowing property. On the other hand, since σ is topologically mixing, there exists an integer $N \in \mathbb{N}$ such that, for any $n \geq N$,

$$\sigma^n(U) \cap V = f_1^{3n}(U) \cap V \neq \emptyset.$$

This implies that $(\Sigma^2, f_{1,\infty})$ is topologically transitive, but since $N_{f_{1,\infty}}(U, V) = \{3N, 3N+3, 3N+6, \ldots\}$, so $(\Sigma^2, f_{1,\infty})$ cannot be topologically mixing. Using Theorem 1.1 yields that $(\Sigma^2, f_{1,\infty})$ does not have the ergodic shadowing property.

Example 4.7. Let $f_i : S^1 \to S^1$ be defined by $f_i(x) = \frac{anx}{n+1}(\text{mod } 1)$ with $a > 1$. Then the NDS $(S^1, f_{1,\infty})$ is uniformly expanding; see [10]. In [10], the authors proved that any uniformly expanding NDS has the shadowing and topologically mixing properties. Since the NDS $(S^1, f_{1,\infty})$ is equicontinuous, Theorem 1.1 implies that it is chain mixing and has pseudo-orbital specification and ergodic shadowing properties.

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