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SOME RESULTS OF FRAMES IN KREIN SPACES

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ABSTRACT. In recently years, frames in Krein spaces had been considered. The paper presents a family of generators for a Krein space by their frames. These generators are dual frames and operator dual frames corresponding to a given frame in a Krein space. We characterize all generalized dual frames of a primary frame. Also, approximately dual frames in a Krein space are introduced and, we study the relation between approximately dual frames and operator duals in a Krein space. Finally, perturbation of frames in this space is considered.

Keywords: Approximately dual frames, Frames, Krein Spaces, Operator dual frames.
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1. Introduction

Frames have been first introduced in 1952 by Duffin and Schaeffer [7], have been reintroduced in 1986 by Daubechies, Grossman, and Meyer [5], and have been popularized from then on. One of the most important application of frames is generating a space (Banach space, Hilbert space, Hilbert C^*-module). A frame generates elements of a space but the coefficients in reconstruction process are not unique and it is a useful and important property of frames in signal processing, filter banks, and etc. [13], [15]. A family of generators corresponding to a frame is the family of dual frames. In [1], the authors have considered sufficient conditions for existence of dual frames.

The indefinite inner product space has been introduced by Pontrjagin [16], and afterward Ginzberg has presented the notion of Krein spaces that is an extension of Hilbert spaces for studying in quantum fields [9]. From 2012, some researchers have studied frames on Krein spaces [8], [10], [12]. Now, we investigate some properties of frames in Krein spaces and their duals and the related subjects.

This paper is organized as follows. Section 2 contains a brief account of basic definitions of Krein spaces and frames. In Section 3, duals and operator duals for frames in Krein spaces are introduced, and some operator duals are constructed. Also, all of operator duals for a given frame in a Krein space are characterized. Finally, approximately duals for frames in this space are considered in Section 4.

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2. **Preliminaries**

In this section, we review some basic concepts of frames and Krein spaces. For more detail, we refer interesting readers to [2] and [3] and the references therein.

An indefinite inner product space $(\mathcal{K}, [.,.])$ is called a Krein space, if there exist two subspace $\mathcal{K}^+$ and $\mathcal{K}^-$ of $\mathcal{K}$ such that $(\mathcal{K}^+, [.,.])$ and $(\mathcal{K}^-, -[.,.])$ are Hilbert spaces and $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$.

In this manuscript, the notation $(\mathcal{K}, J)$ or $\mathcal{K}$ denotes a Krein space with the fundamental symmetry J and $\{e_k\}_{k\in}$ is a J-orthonormal basis for $\mathcal{K}$. Also, $\{\delta_k\}_{k\in\mathbb{N}}$ is the canonical J-orthonormal basis for $\ell^2(\mathbb{N})$ where J induces standard Hilbert space $\ell^2(\mathbb{N})$ and $\mathcal{B}(\mathcal{K})$ shows the set of all of bounded operators on $\mathcal{K}$.

Frames have been extended in various spaces. Frames for Krein spaces have been introduced in three forms that will be mentioned in follow. In 2012, J-frames for Krein spaces have been introduced [10]. A Bessel family $\{f_k\}_{k\in}$with the synthesis operator T is known as J-frame for a krein space $\mathcal{K} =_1 \oplus_2$, if $R(T_1)$ is a maximal uniformly J-positive subspace of $\mathcal{K}$ and $R(T_2)$ is a maximal uniformly J-negative subspace of $\mathcal{K}$, where P_i is the orthogonal projection onto $_i$ and $T_i = TP_i$, for $i = 1, 2$. But it had some defects. Afterward, another definition for frames in a Krien space has been presented in [8]; a sequence $\{f_k\}_{k\in}$ is a frame for $\mathcal{K}$, if there exist constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{K}$,

$$A\|f\|_J^2 \leq \sum_{k\in} |[f, f_k]|^2 \leq B\|f\|_J^2.$$

But the above definitions are dependent to fundamental symmetry of the Krein space $\mathcal{K}$ while it is not unique. Recently, Karamkar and Monowar Hossein presented a definition for frames in a Krein space $\mathcal{K}$ such that is independent of fundamental symmetry and also has not the defects of the last definitions. We study some properties of frames according to this definition. The sequence $\{f_k\}_{k\in} \subseteq \mathcal{K}$ is a frame for $\mathcal{K}$, if $\{f_k^+\}_{k\in\mathbb{N}}$ and $\{f_k^-\}_{k\in\mathbb{N}}$ are frames for $^+$ and $^-$, respectively, these mean that the following inequalities hold.

$$A_1[f^+, f^+] \leq \sum_{k\in} |[f^+, f_k^+]|^2 \leq B_1[f^+, f^+],\ \forall f^+ \in \mathcal{K}^+,$$

$$A_2[f^-, f^-] \leq \sum_{k\in} |[f^-, f_k^-]|^2 \leq B_2[f^-, f^-],\ \forall f^- \in \mathcal{K}^-,$$

where $-\infty < B_2 \leq A_2 < 0 < A_1 \leq B_1 < +\infty$ are real numbers. The pairs (A_1, A_2) and (B_1, B_2) are called a pair of lower frame bounds and a pair of upper frame bounds, respectively.

The corresponding operators to a frame for are defined similar to Hilbert case. Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for . The bounded linear operators T_1 and T_2 are defined by

$$T_1 : \ell^2() \to \mathcal{K}^+,\ T_1\{c_k\}_{k\in} = \sum_{k\in} c_k f_k^+,$$
$$T_2 : \ell^2() \to \mathcal{K}^-,\ T_2\{c_k\}_{k\in} = \sum_{k\in} c_k f_k^-.$$

The pair (T_1, T_2) is called to be the pair of pre-frame operators of $\{f_k\}_{k\in}$. The adjoint of T_1 and T_2 are given by

$$T_1^* : \mathcal{K}^+ \to \ell^2(),\ T_1^*(f^+) = \{[f^+, f_k^+]\}_{k\in},$$
$$T_2^* : \mathcal{K}^- \to \ell^2(),\ T_2^*(f^-) = \{[f^-, f_k^-]\}_{k\in}.$$

The pair of frame operators (S_1, S_2) is defined by $S_1 := T_1T_1^*$ and $S_2 := T_2T_2^*$. The reconstruction formula holds as follows for $f \in \mathcal{K}$,

$$f = \sum_{k\in}[f^+, S_1^{-1}f_k^+]f_k^+ - \sum_{k\in}[f^-, S_2^{-1}f_k^-]f_k^-.$$

For the rest of this study, the notation '$\mp$' shows the positive and negative parts of an elements of a Krein space and we use this notation when there is no confusion.

A frame $\{g_k\}_{k\in}$ in $\mathcal{K}$ is called to be a dual frame for frame $\{f_k\}_{k\in}$, if it is satisfying in the following equalities.

$$f^\mp = \sum_{k\in}[f^\mp, g_k^\mp]f_k^\mp = \sum_{k\in}[f^\mp, f_k^\mp]g_k^\mp, \qquad \forall f^\mp \in \mathcal{K}^\mp.$$

The frame $\{S^{-1}f_k\}_{k\in}$ is said the canonical dual of $\{f_k\}_{k\in}$, where for $f \in \mathcal{K}$, $Sf = S_1f^+ - S_2f^-$.

3. Operator dual frames

One of the most important applications of frames is generating elements of a space. Dual frames corresponding to a given frame play main role in these applications. Studying dual frames has been interested to many researchers, for example Dehghan and Hasankhani have extended dual frames to a larger family which they have called operator dual frames, [6]. Also, we consider this concept and its properties in Krein spaces according to the special class of invertible operators on a Krein space. It is well known that two invertible operators on $^+$ and $^-$ give an invertible operator on , but we may not have two invertible operators on $^+$ and $^-$ of an invertible operator on ; it means that the restriction of an invertible operator on to $^+$ and $^-$ necessarily is not invertible. To see this, let $\{e_i\}_{i\in}$ be a J-orthonormal basis for $\mathcal{K}$ such that for $i \in$,

$$[e_i, e_i] = \begin{cases} 1 & , i = 2k \\ -1 & , i = 2k+1 \end{cases}$$

Define the operator γ by

$$\gamma(e_{2k}) = e_{2k+1}, \qquad \gamma(e_{2k+1}) = e_{2k}, \qquad \forall k \in \mathbb{N}.$$

γ is a bounded and invertible operator on , but the operator $P_1\gamma\mid_{\mathcal{K}^+}$ is not invertible because

$$P_1(\gamma f) = (\gamma f)^+ : \mathcal{K} \longrightarrow \mathcal{K}^+,\ P_1\gamma\mid_{\mathcal{K}^+}: \mathcal{K}^+ \longrightarrow \mathcal{K}^+,$$

$$P_1(\gamma e_{2k}) = P_1(e_{2k+1}) = 0.$$

Let $\{f_k\}_{k\in}$ be a frame for $\mathcal{K}$. A frame $\{g_k\}_{k\in}$ is called an operator dual frame for $\{f_k\}_{k\in}$, if there exist bounded invertible operators $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$ such that for each $f^\mp \in \mathcal{K}^\mp$,

$$f^\mp = \sum_{k\in}[\gamma_i f^\mp, g_k^\mp] f_k^\mp \quad , i = 1, 2.$$

The bounded invertible operators in the definition of operator dual are unique. Let $\mathcal{G} = \{g_k\}_{k\in}$ be an operator dual frame for $\mathcal{F} = \{f_k\}_{k\in}$ in $\mathcal{K}$ and (T_1, T_2) and (U_1, U_2) are their pairs of pre-frame operators, respectively. Then there exist bounded invertible operators $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$ such that for each $f^\mp \in \mathcal{K}^\mp$,

$$f^\mp = \sum_{k\in}[\gamma_i f^\mp, g_k^\mp] f_k^\mp = T_i\{[\gamma_i f^\mp, g_k^\mp]\}_{k\in} = T_i U_i^* \gamma_i f^\mp,$$

so $T_i U_i^* \gamma_i = I$. Now, we can say that the frame $\{g_k\}_{k\in}$ is an operator dual frame of $\{f_k\}_{k\in}$ with corresponding bounded invertible operators γ_1 and γ_2, and we can shortly represent an operator dual frame with triple $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$.

The family of operator duals corresponding to a given frame is larger than the family of its ordinary dual frames. Every ordinary dual frame is an operator dual frame; means that if $\{g_k\}_{k\in}$ is a dual frame of $\{f_k\}_{k\in}$, then $(\{g_k\}_{k\in}, I^+, I^-)$ is an operator dual frame of frame $\{f_k\}_{k\in}$. Operator dual frames have some more properties with respect to dual frames. The following remark expresses one of these properties. A frame $\{f_k\}_{k\in}$ with the pair of frame operators (S_1, S_2) is an operator dual for itself; it means that $(\{f_k\}_{k\in}, S_1^{-1}, S_2^{-1})$ is an operator dual frame for $\{f_k\}_{k\in}$. Let m be a natural number, and let $\{g_k^i\}_{k\in}$ be a dual frame of frame $\{f_k\}_{k\in}$ in $\mathcal{K}$, for $i = 1, 2, \cdots, m$. If $\{c_i\}_{i=1}^m$ and $\{d_i\}_{i=1}^m$ are finite sequences of complex numbers such that $\sum\limits_{i=1}^m c_i \neq 0$, $\sum\limits_{i=1}^m d_i \neq 0$, assume that $h_k^+ = \sum\limits_{i=1}^m c_i(g_k^i)^+$, and $h_k^- = \sum\limits_{i=1}^m d_i(g_k^i)^-$, for $k \in \mathbb{N}$. Now, set $\theta_1 = (\sum\limits_{i=1}^m c_i)^{-1} I^+$, and $\theta_2 = (\sum\limits_{i=1}^m d_i)^{-1} I^-$. The operators θ_1 and θ_2 are bounded invertible operators on $\mathcal{K}^+$ and $\mathcal{K}^-$, respectively. Then we have for $f^+ \in \mathcal{K}^+$ and $f^- \in \mathcal{K}^-$,

$$\begin{aligned} f^+ = \sum_{k\in}[f^+, (g_k^i)^+] f_k^+ &= \sum_{i=1}^m c_i \frac{1}{\sum\limits_{i=1}^m c_i} \sum_{k\in}[f^+, (g_k^i)^+] f_k^+ \\ &= \sum_{k\in}[\frac{1}{\sum\limits_{i=1}^m c_i} f^+, \sum_{i=1}^m c_i (g_k^i)^+] f_k^+ = \sum_{k\in}[\theta_1 f^+, h_k^+] f_k^+. \end{aligned}$$

Similarly, $f^- = \sum\limits_{k\in}[\frac{1}{\sum\limits_{i=1}^{m} d_i} f^-, \sum\limits_{i=1}^{m} d_i(g_k^i)^-]f_k^i = \sum\limits_{k\in}[\theta_2 f^-, h_k^-]f_k^-$.

These show that $(\{h_k\}_{k\in}, \theta_1, \theta_2)$ is an operator dual frame of $\{f_k\}_{k\in}$. The following lemma shows that the operator duality property is a symmetric relation. The proof is similar to Hilbert case [6]. Let $\mathcal{F} = \{f_k\}_{k\in}$ and $\mathcal{G} = \{g_k\}_{k\in}$ be Bessel sequences in $\mathcal{K}$. Then for invertible operators $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$, the following statements are equivalent.

a. $f^\mp = \sum\limits_{k\in\mathbb{N}} [\gamma_i f^\mp, f_k^\mp]g_k^\mp$, for $f^\mp \in \mathcal{K}^\mp$, and $i = 1, 2$.

b. $f^\mp = \sum\limits_{k\in\mathbb{N}} [\gamma_i^* f^\mp, g_k^\mp]f_k^\mp$, for $f^\mp \in \mathcal{K}^\mp$, and $i = 1, 2$.

c. $[f^\mp, g^\mp] = \sum\limits_{k\in\mathbb{N}} [\gamma_i f^\mp, f_k^\mp][g_k^\mp, g^\mp]$, for $f^\mp, g^\mp \in \mathcal{K}^\mp$, and $i = 1, 2$.

If one of the conditions in Lemma 3 is satisfying, then $\{g_k\}_{k\in}$ and $\{f_k\}_{k\in}$ are frames for $\mathcal{K}$.

The operator duality relation is not transitive. The following example shows this fact. Let $\{e_k\}_{k\in}$ be an J-orthonormal basis for $\mathcal{K}$. Set

$$\begin{aligned}\{f_k\}_{k\in} &= \{e_1, 0, e_2, 0, e_3, 0, \cdots\},\\ \{g_k\}_{k\in} &= \{e_1, e_1, e_2, e_2, e_3, \cdots\},\\ \{h_k\}_{k\in} &= \{0, e_1, 0, e_2, 0, e_3, \cdots\}.\end{aligned}$$

Then for each $f^\mp \in \mathcal{K}^\mp$, we have

$$\begin{aligned}\sum_{k\in}[f^\mp, g_k^\mp]f_k^\mp &= [f^\mp, e_1^\mp]e_1^\mp + [f^\mp, e_1^\mp]0 + [f^\mp, e_2^\mp]e_2^\mp + [f^\mp, e_2^\mp]0 + \cdots\\ &= \sum_{k=2n+1, n\in}[f^\mp, g_k^\mp]f_k^\mp = \sum_{k\in}[f^\mp, e_k^\mp]e_k^\mp = f^\mp.\end{aligned}$$

So $\{g_k\}_{k\in}$ is a dual frame for $\{f_k\}_{k\in\mathbb{N}}$ and an operator dual frame of $\{f_k\}_{k\in}$ by Remark 3. Also, $\{h_k\}_{k\in}$ is a dual frame for $\{g_k\}_{k\in}$ and it also is an operator dual frame by Remark 3. But $\{h_k\}_{k\in}$ is not an operator dual of $\{f_k\}_{k\in}$ since

$$\sum_{k\in}[f^\mp, f_k^\mp]h_k^\mp = [f^\mp, e_1^\mp]0 + [f^\mp, 0]e_1^\mp + [f^\mp, e_2^\mp]0 + \cdots = 0, \ \forall f^\mp \in \mathcal{K}^\mp.$$

Therefore, for all invertible operators $\gamma_i \in \mathcal{B}(\mathcal{K}^\mp)$,

$$\sum_{k\in}[\gamma_i f^\mp, f_k^\mp]h_k^\mp = 0.$$

The authors presented the definitions for Riesz bases as follows [2]; if $\{e_k\}_{k\in}$ is a J-orthonormal basis and γ is a bounded invertible operator on , then $\{\gamma e_k\}_{k\in}$ is called a Riesz basis.

Now, according to the relation between invertible operators on $\mathcal{K}$ and invertible operators on $\mathcal{K}^+$ and $\mathcal{K}^-$, and the definition of frames in this manuscript, we study a specific category of Riesz bases in Krien spaces. Let $\{e_k\}_{k\in}$ be a J-orthonormal basis for $\mathcal{K}$ and assume that there exist bounded invertible operators $\gamma_1 : \mathcal{K}^+ \longrightarrow \mathcal{K}^+$ and $\gamma_2 : \mathcal{K}^- \longrightarrow \mathcal{K}^-$. Then the triple $(\{e_k\}_{k\in}, \gamma_1, \gamma_2)$ or $\{\gamma e_k\}_{k\in}$ is a Riesz basis for $\mathcal{K}$, where γ is the bounded invertible operator corresponding to the pair operators (γ_1, γ_2).

The orthonormal bases can be dual frames for each other. The following proposition shows that not only this result is true for orthonormal bases, but also it holds for Riesz bases. Every two Riesz bases are operator dual frames in Krein spaces.

Proof. Let $\{e_k\}_{k\in}$ be a $J-$orthonormal basis for $\mathcal{K}$ and let $(\{e_k\}_{k\in}, \gamma_1, \gamma_2)$ and $(\{e_k\}_{k\in}, \theta_1, \theta_2)$ be Riesz bases.

Assume that ξ_i is the inverse of $\theta_i\gamma_i^*$. Set $f_k^\mp = \gamma_i e_k^\mp$ and $g_k^\mp = \theta_i e_k^\mp$, $k \in$, for $i = 1, 2$, we obtain

$$f^\mp = \theta_i\gamma_i^*\xi_i f^\mp = \theta_i(\sum_{k\in\mathbb{N}}[\gamma_i^*\xi_i f^\mp, e_k^\mp]e_k^\mp)$$
$$= \sum_{k\in\mathbb{N}}[\xi_i f^\mp, \gamma_i e_k^\mp]\theta_i e_k^\mp = \sum_{k\in\mathbb{N}}[\xi_i f^\mp, f_k^\mp]g_k^\mp, \ \forall f^\mp \in \mathcal{K}^\mp.$$

These relations show that the triple $(\{f_k\}_{k\in}, \xi_1, \xi_2)$ is an operator dual frame for $\{g_k\}_{k\in}$. □

In the following proposition, some operator duals for a frame are constructed. By an operator dual of a given frame, another operator dual can be constructed for a primary frame in a Krein space. Also, an operator dual is constructed by an operator dual on a subspace. Moreover, one of the differences between ordinary dual frames and operator dual frames is presented; an ordinary dual frame for a subspace could not be extended to a ordinary dual frames for the space but it is possible for operator dual frames. Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for $\mathcal{K}$ with the pair of frame operators (S_1, S_2).

a. If the triple $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame for $\{f_k\}_{k\in}$ for $\mathcal{K}$ and α, β are complex numbers, then the triple $(\{h_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame of $\{f_k\}_{k\in}$, where for $k \in$, $h_k = h_k^+ + h_k^-$, is defined by
$$h_k^\mp = \alpha g_k^\mp + (1-\alpha)(\gamma_i^{-1})^* S_i^{-1} f_k^\mp, i = 1, 2.$$

b. If the triple $(\{g_k\}_{k\in}, \theta_1, \theta_2)$ is an operator dual frame for $\{f_k\}_{k\in}$ for $V_1 := \overline{span}\{g_k^+\}_{k\in}$ and $V_2 := \overline{span}\{g_k^-\}_{k\in}$, respectively, then there exist invertible operators $\gamma_1 \in^+$ and $\gamma_2 \in^-$ such that the triple $(\{h_k\}_{k\in}, \gamma_1, \gamma_2)$ defined by
$$h_k^\mp = \theta_i^* g_k^\mp + S_i^{-1} f_k^\mp \ , \ \forall k \in,$$
is an operator dual frame of $\{f_k\}_{k\in}$ for $\mathcal{K}$.

c. Let $\{g_k\}_{k\in}$ be a dual frame of $\{f_k\}_{k\in}$ for $V := \overline{span}\{g_k\}_{k\in}$. Then $(\{h_k\}_{k\in}, \gamma_1, \gamma_2)$ defined by
$$h_k^\mp = g_k^\mp + S_i^{-1} f_k^\mp,$$
is an operator dual frame of $\{f_k\}_{k\in}$ for $\mathcal{K}$.

Proof. The parts a and c are similar to Hilbert case [6]. To see b, suppose that the subspaces $V_1^{\perp}$ and $V_2^{\perp}$ of $\mathcal{K}^+$ and $\mathcal{K}^-$ are orthogonal complements of V_1 and V_2, respectively. Therefore, the operators θ_1 and θ_2 can be extended to spaces $\mathcal{K}^+$ and $\mathcal{K}^-$ by using orthogonal projections P_i and Q_i of subspaces V_i and $V_i^{\perp}$ $(i = 1, 2)$, respectively,

$$\theta_i^{'} := \theta_i P_i + Q_i.$$

Then for each $f^+ \in V_1$ and $g^+ \in \mathcal{K}^+$, we have

$$\begin{aligned}
[(\theta_1^{'})^* f^+, g^+] &= [f^+, \theta_1 P_1 g^+ + Q_1 g^+] \\
&= [f^+, \theta_1 P_1 g^+] + [f^+, Q_1 g^+] \\
&= [\theta_1^* f^+, P_1 g^+] \\
&= [\theta_1^* f^+, P_1 g^+ + Q_1 g^+] \\
&= [\theta_1^* f^+, (P_1 + Q_1) g^+] = [\theta_1^* f^+, g^+]
\end{aligned}$$

and so $(\theta_1^{'})^* = \theta_1^*$. Similarly $(\theta_2^{'})^* = \theta_2^*$.

Now, suppose that $\gamma_1 := I^+ - \frac{1}{2}P_1$, $\gamma_2 := I^- - \frac{1}{2}P_2$, where I^+ and I^- denote the identity operators on $\mathcal{K}^+$ and $\mathcal{K}^-$, respectively. One obtains that

$$\|I^+ - \gamma_1\| = \|I^+ - I^+ + \frac{1}{2}P_1\| = \|\frac{1}{2}P_1\|. \tag{1}$$

On the other hand $P_1 + Q_1 = I^+$, then $\|P_1 + Q_1\| = 1$ and for $f^+ \in \mathcal{K}^+$,

$$\|P_1 f^+ + Q_1 f^+\|^2 = \|P_1 f^+\|^2 + \|Q_1 f^+\|^2 = \|f^+\|^2,$$

therefore $\|P_1 f^+\| \leq \|f^+\|$ and $\|\frac{1}{2}P_1\| < 1$. By the equality (1), γ_1 is invertible. Similarly, γ_2 is invertible. Now, for each $f^+ \in \mathcal{K}^+$,

$$\begin{aligned}
&\sum_{k\in}[\gamma_1 f^+, h_k^+] f_k^+ \\
=&\sum_{k\in}[P_1 f^+ + Q_1 f^+ - \frac{1}{2}P_1 P_1 f^+ - \frac{1}{2}P_1 Q_1 f^+, \theta_1^* g_k^+ + S_1^{-1} f_k^+] f_k^+ \\
=&\sum_{k\in}[\frac{1}{2}P_1 f^+ + Q_1 f^+, \theta_1^* g_k^+ + S_1^{-1} f_k^+] f_k^+ \\
=&\frac{1}{2}\sum_{k\in}[\theta_1 P_1 f^+, g_k^+] f_k^+ + \frac{1}{2}\sum_{k\in}[S_1^{-1} P_1 f^+, f_k^+] f_k^+ + \sum_{k\in}[S_1^{-1} Q_1 f^+, f_k^+] f_k^+ \\
=&\frac{1}{2}P_1 f^+ + \frac{1}{2}P_1 f^+ + Q_1 f^+ = f^+.
\end{aligned}$$

Also, $\sum_{k\in}[\gamma_2 f^-, h_k^-] f_k^- = f^-$, for $f^- \in \mathcal{K}^-$. Thus the triple $(\{h_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame for $\{f_k\}_{k\in}$. □

The following proposition shows that the sum of finite operator duals is also an operator dual frame under some conditions. If $\{(\{g_k^i\}_{k\in}, \theta_1^i, \theta_2^i)\}_{i=1}^{j}$, $j \in \mathbb{N}$, is a finite set of operator dual frames for $\{f_k\}_{k\in}$, such that for $i = 1, 2$, the operator

$\sum_{i=1}^{j}(\theta_r^i)^{-1}$ is invertible, then the triple $(\{\sum_{i=1}^{j} g_k^i\}_{k\in\mathbb{N}}, (\sum_{i=1}^{j}(\theta_r^i)^{-1})^{-1}, (\sum_{i=1}^{j}(\theta_2^i)^{-1})^{-1})$ is an operator dual frame for $\{f_k\}_{k\in}$.

Proof. Assume that ρ_r is the inverse of operator $\sum_{i=1}^{j}(\theta_r^i)^{-1}$, for $r = 1, 2$. Then for $f^{\mp} \in \mathcal{K}^{\mp}$,

$$\sum_{k\in\mathbb{N}}[\rho_r f^{\mp}, \sum_{i=1}^{j}(g_k^i)^{\mp}]f_k^{\mp} = \sum_{i=1}^{j}\sum_{k\in\mathbb{N}}[\rho_r f^{\mp}, (g_k^i)^{\mp}]f_k^{\mp} = (\sum_{i=1}^{j}(\theta_r^i)^{-1})\rho_r f^{\mp}.$$

□

The invertible operators preserve the operator dual property; it means that each invertible operator transforms an operator dual frame into another operator dual frame. In the following proposition, it will be reviewed. Its proof is similar to Proposition 2.5 [4] and is omitted. Let $U_1, V_1 \in \mathcal{B}(\mathcal{K}^+)$ and $U_2, V_2 \in \mathcal{B}(\mathcal{K}^-)$ be invertible operators and $U = U_1 + U_2$ and $V = V_1 + V_2$. Then $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames if and only if $\{Uf_k\}_{k\in}$ and $\{Vg_k\}_{k\in}$ are operator dual frames.

4. characterization of operator dual frames

Every frame has at least an operator dual frame, it is given in Remark 3. In this section, we will characterize all operator dual frames of a given frame in a Krein space.

Let $\{f_k\}_{k\in}$ be a frame for $\mathcal{K}$ with the pair of pre-frame operators (T_1, T_2). The set of all operator dual frames for $\{f_k\}_{k\in}$ is the set of all triple $(\{V\delta_k\}_{k\in}, \gamma_1, \gamma_2)$, where $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$ are invertible and $V = V_1 + V_2$, such that $V_1 : \ell^2() \longrightarrow \mathcal{K}^+$, and $V_2 : \ell^2() \longrightarrow \mathcal{K}^-$ are bounded left inverses of $T_1^*\gamma_1$ and $T_2^*\gamma_2$, respectively.

Proof. Let γ_i, $i = 1, 2$, be invertible operators on $^{\mp}$, respectively. Suppose that V_i is a bounded left inverse of $T_i^*\gamma_i$ and for $f^{\mp} \in \mathcal{K}^{\mp}$, it is well known that $(T_i^*\gamma_i)f^{\mp} \in \ell^2()$ and $V_i(T_i^*\gamma_i)f^{\mp} = f^{\mp}$, $i = 1, 2$. So V_i is surjective and $\{V_i\delta_k\}_{k\in}$ is a frame for $\mathcal{K}^{\mp}$. Moreover, for $f^{\mp} \in \mathcal{K}^{\mp}$,

$$f^{\mp} = V_i T_i^*\gamma_i f^{\mp} = V_i(\sum_{k\in}[\gamma_i f^{\mp}, f_k^{\mp}]\delta_k^{\mp}) = \sum_{k\in}[\gamma_i f^{\mp}, f_k^{\mp}]V_i\delta_k, \quad i = 1, 2.$$

Then the triple $(\{V\delta_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame of $\{f_k\}_{k\in}$.

Now, if $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame of $\{f_k\}_{k\in}$ with the pair of pre-frame operators (V_1, V_2), then $g_k = V\delta_k$, for $k \in \mathbb{N}$, when $V = V_1 + V_2$, and for $f^{\mp} \in \mathcal{K}^{\mp}$, and $i = 1, 2$,

$$f^{\mp} = \sum_{k\in}[\gamma_i f^{\mp}, f_k^{\mp}]g_k^{\mp} = \sum_{k\in}[\gamma_i f^{\mp}, f_k^{\mp}]V_i\delta_k = V_i T_i^*\gamma_i f^{\mp}.$$

Then $I^{\mp} = V_i T_i^*\gamma_i$ and so V_i is a bounded left inverse of $T_i^*\gamma_i$, $i = 1, 2$. □

The following lemma and theorem are similar to the Hilbert case and their proofs are omitted [3]. If $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$ are invertible operators and $\{f_k\}_{k\in}$ is a frame for $\mathcal{K}$ with the pair of pre-frame operators (T_1, T_2) and the pair of frame operators (S_1, S_2), then all of bounded left inverses of $T_i^*\gamma_i$, are the operators with the following form

$$\gamma_i^{-1}Si^{-1}T_i + W_i(I - T_i^*S_i^{-1}T_i);\ i = 1, 2,$$

where $W_1 : \ell^2() \longrightarrow \mathcal{K}^+$ and $W_2 : \ell^2() \longrightarrow \mathcal{K}^-$ are bounded operators and I is the identity operator on $\ell^2()$.

Now, we characterize all operator dual frames of a given frame in a Krein space. The following theorem illustrates it. If $\{f_k\}_{k\in}$ is a frame for $\mathcal{K}$, with the pair of frame operators (S_1, S_2) and the pair of pre-frame opeators (T_1, T_2), then operator dual frames of $\{f_k\}_{k\in}$ are the triples $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$, where for $k \in \mathbb{N}$, and $i = 1, 2$,

$$g_k^{\mp} = \gamma_i^{-1}S_i^{-1}f_k^{\mp} + h_k^{\mp} - \sum_{j\in}[S_i^{-1}f_k^{\mp}, f_j^{\mp}]h_j^{\mp},$$

and $\{h_k\}_{k\in}$ is a Bessel sequence in $\mathcal{K}$, and $\gamma_1 \in \mathcal{B}(\mathcal{K}^+)$, and $\gamma_2 \in \mathcal{B}(\mathcal{K}^-)$ are invertible operators.

5. **approximately dual frames**

Approximately dual frames as an effective and attractive duality principle in frame theory had been introduced by Christensen and Laugesen in [4]. Recently, its extension is presented by Dehghan and Hasankhani Fard [6]. In this section, approximately dual frames for frames in a Krein space are reviewed.

The Bessel sequences $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ with the pair of pre-frame operators (T_1, T_2) and (U_1, U_2), respectively, are called approximately dual frames whenever at least one of the following relations is true for $i = 1$ and $i = 2$.

$$\|I - T_iU_i^*\| < 1 \ or \ \|I - U_iT_i^*\| < 1.$$

In the following, we investigate the relation between approximately dual frames and operator dual frames. Each approximately dual frame of a frame in a Krein space is an operator dual frame.

Proof. Assume that two Bessel sequences $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are approximately dual frames in $\mathcal{K}$ with the pair of pre-frame operators (T_1, T_2) and (U_1, U_2), respectively. Then

$$\|I - U_iT_i^*\| < 1 \ or \ \|I - T_iU_i^*\| < 1,\ (i = 1, 2),$$

and $T_iU_i^*$ or $U_iT_i^*$ is invertible.

If $U_iT_i^*$ is invertible, then so is $T_iU_i^*$, and for $f^{\mp} \in \mathcal{K}^{\mp}$ and $i = 1, 2$

$$f^{\mp} = (U_iT_i^*)(U_iT_i^*)^{-1}f^{\mp} = \sum_{k\in}[(U_iT_i^*)^{-1}f^{\mp}, f_k^{\mp}]g_k^{\mp},$$

and

$$f^{\mp} = \sum_{k\in}[(T_iU_i^*)^{-1}f^{\mp}, g_k^{\mp}]f_k^{\mp}.$$

These equalities show that the triple $(\{f_k\}_{k\in}, (U_1T_1^*)^{-1}, (U_2T_2^*)^{-1})$ is an operator dual frame of $\{g_k\}_{k\in}$ and the triple $(\{g_k\}_{k\in}, (T_1U_1^*)^{-1}, (T_2U_2^*)^{-1})$ is an operator dual frame of $\{f_k\}_{k\in}$. Therefore, $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames for each other. □

In Proposition 5, the inverse of relationship is not true. It means that approximately dual frames are not necessarily operator dual frames and the set of all approximately dual frames of a given frame is a proper subset of the set of all operator dual frames. Let $\{f_k\}_{k\in}$ be a frame with the canonical dual $\{\tilde{f}_k\}_{k\in}$ and let ξ be a nonzero complex number such that $|\xi - 1| \geq 1$. Then $(\{\xi f_k\}_{k\in}, \xi^{-1}I^+, \xi^{-1}I^-)$ is an operator dual frame of $\{\tilde{f}_k\}_{k\in}$ for $\mathcal{K}$, but $\{\xi f_k\}_{k\in}$ is not an approximately dual frame of $\{\tilde{f}_k\}_{k\in}$.

Firstly, the sequence $\{\xi f_k\}_{k\in}$ is an operator dual frame of $\{\tilde{f}_k\}_{k\in}$ for $\mathcal{K}$ because

$$f^{\mp} = \sum_{k\in}[f^{\mp}, f_k^{\mp}]\tilde{f}_k^{\ \mp} = \sum_{k\in}[\xi^{-1}I^{\mp}f^{\mp}, \xi f_k^{\mp}]\tilde{f}_k^{\ \mp},\ \forall f^{\mp} \in \mathcal{K}^{\mp}.$$

Now, we consider the approximately dual property for $\{\xi f_k\}_{k\in\mathbb{N}}$. Assume (T_1, T_2) and (U_1, U_2) are the pairs of pre-frame operators of $\{\xi f_k\}_{k\in}$ and $\{\tilde{f}_k\}_{k\in}$, respectively. Then for each $f^{\mp} \in \mathcal{K}^{\mp}$ and $i = 1, 2$, we have

$$\|(I^{\mp} - T_iU_i^*)f^{\mp}\| = \|f^{\mp} - \sum_{k\in}[f^{\mp}, \tilde{f}_k^{\ \mp}]\xi f_k^{\mp}\| = \|f^{\mp} - \xi f^{\mp}\|.$$

Then

$$\|I^{\mp} - T_iU_i^*\| = |\xi - 1| \geq 1.$$

So $\{\xi f_k\}_{k\in}$ and $\{\tilde{f}_k\}_{k\in}$ are not approximately dual frames. We saw that every operator dual is not necessarily an approximately dual. But some operator duals are approximately duals under the conditions. The following proposition illustrates these conditions. If $(\{f_k\}_{k\in}, \gamma_1, \gamma_2)$ and $(\{g_k\}_{k\in}, \gamma_1^*, \gamma_2^*)$ are operator dual frames, such that $\|I^+ - \gamma_1^{-1}\| < 1$ and $\|I^- - \gamma_2^{-1}\| < 1$, then $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are approximately dual frames.

Proof. Let (T_1, T_2) and (U_1, U_2) be the pairs of pre-frame operators of $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$, respectively. Then by Remark 3, for $i = 1, 2$, we have $\gamma_i^{-1} = T_iU_i^*$, and the assumption concludes $\|I^{\mp} - T_iU_i^*\| < 1$. So $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are approximately dual frames. □

The following theorem offers a way to construct the operator dual frames by the Bessel sequences. Suppose that $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are Bessel sequences in $\mathcal{K}$ with the pairs of pre-frame operators (T_1, T_2) and (U_1, U_2), respectively. If there exist constants $\alpha, \beta, \lambda, \mu \in [0, 1)$ such that

$$\|f^{\mp} - U_iT_i^*f^{\mp}\| \leq \alpha\|U_iT_i^*f^{\mp}\| + \beta\|f^{\mp}\|,\ \forall f^{\mp} \in \mathcal{K}^{\mp}.$$

then $(\{f_k\}_{k\in}, (U_1T_1^*)^{-1}, (U_2T_2^*)^{-1})$ is an operator dual frame of $\{g_k\}_{k\in}$.

Proof. By [17], the operator $U_iT_i^*$ is invertible and so the result is clear. □

6. perturbation of operator dual frames

The concept of perturbation in frame theory has been introduced since 1990, [11] and [14]. Recently, this notion has been considered for g-dual frames in [6].

In this section, the perturbation for operator dual frames in a Krein space is investigated. We consider some conditions that preserve the operator duality relation. Let $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$ be an operator dual frame of $\{h_k\}_{k\in}$ in $\mathcal{K}$ and let $\{f_k\}_{k\in}$ be a sequence in $\mathcal{K}$. Assume that there exist constants $\alpha, \beta, \lambda, \mu \geq 0$, such that for all finite sequences $\{c_k\}_{k\in}$ and $\{d_k\}_{k\in}$

$$\|\sum_{k\in} c_k(f_k^+ - h_k^+)\| \leq \alpha\|\sum_{k\in} c_k h_k^+\| + \beta(\sum_{k\in} |c_k|^2)^{\frac{1}{2}}, \tag{2}$$

$$\|\sum_{k\in} d_k(f_k^- - h_k^-)\| \leq \lambda\|\sum_{k\in} d_k h_k^-\| + \mu(\sum_{k\in} |d_k|^2)^{\frac{1}{2}}, \tag{3}$$

a. If (B_1, B_2) is the pair of upper frame bounds of $\{g_k\}_{k\in}$ such that $\alpha+\beta\sqrt{B_1}\|\gamma_1\| < 1$ and $\lambda + \mu\sqrt{B_2}\gamma_2\| < 1$, then $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

b. If (S_1, S_2) and (D_1, D_2) are the pair of frame operators and the pair of upper frame bounds of $\{h_k\}_{k\in}$, respectively, and $\alpha + \beta\sqrt{D_1}\|S_1^{-1}\| < 1$ and $\lambda + \mu\sqrt{D_2}\|S_2^{-1}\| < 1$, then $\{f_k\}_{k\in}$ and $\{h_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

Proof. Assume that $(T_1, T_2), (U_1, U_2)$ and (V_1, V_2) are the pairs of frame operators of $\{f_k\}_{k\in}$, $\{g_k\}_{k\in}$ and $\{h_k\}_{k\in}$, respectively. Since $\{h_k^+\}_{k\in}$ and $\{h_k^-\}_{k\in}$ are frames with the pair of upper bounds (C_1, C_2), the operators V_1 and V_2 are bounded and $\|V_1\| \leq \sqrt{C_1}$, $\|V_2\| \leq \sqrt{C_2}$.

The inequality (2) implies that for every finite sequence $\{c_k\}_{k\in}$, we have

$$\begin{aligned}\|\sum_{k\in} c_k f_k^+\| &\leq \|\sum_{k\in} c_k(h_k^+ - f_k^+)\| + \|\sum_{k\in} c_k h_k^+\| \\ &\leq \alpha\|\sum_{k\in} c_k h_k^+\| + \beta(\sum_{k\in} |c_k|^2)^{\frac{1}{2}} + \|\sum_{k\in} c_k h_k^+\| \\ &= (1+\alpha)\|\sum_{k\in} c_k h_k^+\| + \beta(\sum_{k\in} |c_k|^2)^{\frac{1}{2}},\end{aligned}$$

Similarly, by the inequality (3) for every finite sequence $\{d_k\}_{k\in}$,

$$\|\sum_{k\in} d_k f_k^-\| \leq (1+\lambda)\|\sum_{k\in} d_k h_k^-\| + \mu(\sum_{k\in} |d_k|^2)^{\frac{1}{2}}.$$

This calculation is established even for each $\{c_k\}_{k\in}, \{d_k\}_{k\in} \in \ell^2()$.

As a result $\sum_{k\in} c_k(f_k^+ - h_k^+)$ and $\sum_{k\in} d_k(f_k^- - h_k^-)$ converge and the inequalities (2) and (3) hold for all sequences $\{c_k\}_{k\in}, \{d_k\}_{k\in} \in \ell^2()$. Therefore, the operators $W_1 : \ell^2() \longrightarrow \mathcal{K}^+$ and $W_2 : \ell^2() \longrightarrow \mathcal{K}^-$ are defined by $W_1(\{c_k\}_{k\in}) = \sum_{k\in} c_k(f_k^+ - h_k^+)$ and $W_2(\{d_k\}_{k\in}) = \sum_{k\in} d_k(f_k^- - h_k^-)$ and they are well-defined and bounded. So the sequence $\{f_k - h_k\}_{k\in}$ is a Bessel sequence by [3]. Similarly, the pair of pre-frame operators (T_1, T_2) are well-defined and bounded and hence $\{f_k\}_{k\in}$ is also a Bessel sequence.

To prove a, let $\alpha + \beta\sqrt{B_1}\|\gamma_1\| < 1$ and $\lambda + \mu\sqrt{B_2}\|\gamma_2\| < 1$. Since $\{g_k\}_{k\in}$ is a frame, for $f^+ \in \mathcal{K}^+$, the series $\sum_{k\in} |[\gamma_1 f^+, g_k^+]|^2$ is convergent and the sequence $\{[\gamma_1 f^+, g_k^+]\}_{k\in} \in \ell^2()$. On the other hand, $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$ is an operator dual frame of $\{h_k\}_{k\in}$ in $\mathcal{K}$, so by (2),

$$\begin{aligned}\|f^+ - \sum_{k\in}[\gamma_1 f^+, g_k^+]f_k^+\| &= \|\sum_{k\in}[\gamma_1 f^+, g_k^+]h_k^+ - \sum_{k\in}[\gamma_1 f^+, g_k^+]f_k^+\| \\ &= \|\sum_{k\in}[\gamma_1 f^+, g_k^+](h_k^+ - f_k^+)\| \\ &\leq \alpha\|\sum_{k\in}[\gamma_1 f^+, g_k^+]h_k^+\| + \beta(\sum_{k\in}|[\gamma_1 f^+, g_k^+]|^2)^{\frac{1}{2}} \\ &\leq \alpha\|f^+\| + \beta\sqrt{B_1}\|\gamma_1 f^+\| = (\alpha + \beta\sqrt{B_1}\|\gamma_1\|)\|f^+\|,\end{aligned}$$

then $\|I^+ - T_1U_1^*\gamma_1\| < 1$. A similar argument shows that $\|I^- - T_2U_2^*\gamma_2\| < 1$. So $T_iU_i^*$ is invertible, for $i = 1, 2$. Suppose that θ_i is the inverse of operator $T_iU_i^*$, $i = 1, 2$. Then we have

$$f^\mp = T_iU_i^*\theta_i f^\mp = \sum_{k\in}[\theta_i f^\mp, g_k^\mp]f_k^\mp, \ \forall f^\mp \in \mathcal{K}^\mp.$$

Hence $(\{g_k\}_{k\in}, \theta_1, \theta_2)$ is an operator dual frame of $\{f_k\}_{k\in}$.

Proof b. Suppose that $\alpha + \beta\sqrt{D_1}\|S_1^{-1}\| < 1$ and $\lambda + \mu\sqrt{D_2}\|S_2^{-1}\| < 1$. By Remark 3 and the properties of $\{f_k\}_{k\in}$ and $\{h_k\}_{k\in}$, for each $f^+ \in \mathcal{K}^+$, we obtain

$$\begin{aligned}&\|f^+ - \sum_{k\in}[S_1^{-1}f^+, h_k^+]f_k^+\| \\ =&\|\sum_{k\in}[S_1^{-1}f^+, h_k^+](h_k^+ - f_k^+)\| \\ \leq&\alpha\|f^+\| + \beta\sqrt{D_1}\|S_1^{-1}f^+\| = (\alpha + \beta\sqrt{D_1}\|S_1^{-1}\|)\|f^+\|,\end{aligned}$$

so $\|I^+ - T_1V_1^*S_1^{-1}\| < 1$, and similarly $\|I^- - T_2V_2^*S_2^{-1}\| < 1$. Therefore, $T_iV_i^*$ is invertible, for $i = 1, 2$. If ρ_i is the inverse of $T_iV_i^*$, $i = 1, 2$, then

$$f^\mp = T_iV_i^*\rho_i f^\mp = \sum_{k\in}[\rho_i f^\mp, h_k^\mp]f_k^\mp, \ \forall f^\mp \in \mathcal{K}^\mp,$$

this equality concludes that triple $(\{h_k\}_{k\in}, \rho_1, \rho_2)$ is an operator dual frame for $\{f_k\}_{k\in}$. □

Assume $\{f_k\}_{k\in}$ is a sequence in $\mathcal{K}$.

a. Assume that $\{g_k\}_{k\in}$ and $\{h_k\}_{k\in}$ are operator dual frames. If there exist constants $\alpha, \lambda \in [0, 1)$ such that

$$\|\sum_{k\in} c_k(f_k^+ - h_k^+)\| \leq \alpha\|\sum_{k\in} c_k h_k^+\|\ ,\ \|\sum_{k\in} d_k(f_k^- - h_k^-)\| \leq \lambda\|\sum_{k\in} d_k h_k^-\|,$$

for all finite sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$, then the pair of frames $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ and the pair of frames $\{f_k\}_{k\in}$ and $\{h_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

b. Assume that $\{g_k\}_{k\in}$ is a frame for $\mathcal{K}$ with the pair of frame operators (S_1, S_2) and the pair of upper frame bounds (B_1, B_2), and $\{f_k\}_{k\in}$ is a sequence in $\mathcal{K}$, and there exist constants $\alpha, \beta, \lambda, \mu \geq 0$, such that

$$\|\sum_{k\in} c_k(f_k^+ - g_k^+)\| \leq \alpha\|\sum_{k\in} c_k g_k^+\| + \beta(\sum_{k\in}|c_k|^2)^{\frac{1}{2}},$$
$$\|\sum_{k\in} d_k(f_k^- - g_k^-)\| \leq \lambda\|\sum_{k\in} d_k g_k^-\| + \mu(\sum_{k\in}|d_k|^2)^{\frac{1}{2}},$$

for all finite sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$. If $\alpha + \beta\sqrt{B_1}\|S_1^{-1}\| < 1$ and $\lambda + \mu\sqrt{B_2}\|S_2^{-1}\| < 1$, then $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

Proof. For see the part a, it is enough that in Theorem 6, set $\beta = \mu = 0$. To proof b, by Remark 3, the conditions in Theorem 6 satisfy so $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$ by part *b*. □

Let $(\{g_k\}_{k\in}, \gamma_1, \gamma_2)$ be an operator dual frame of $\{h_k\}_{k\in}$ and let $\{f_k\}_{k\in}$ be a sequence in $\mathcal{K}$. Assume that $\theta_1 \in \mathcal{B}(\mathcal{K}^+)$ and $\theta_2 \in \mathcal{B}(\mathcal{K}^-)$ are invertible operators such that for some $\eta, \xi > 0$,

$$\sum_{k\in} |[\theta_1 f^+, f_k^+ - h_k^+]|^2 \leq \eta\|f^+\|^2,\ \forall f^+ \in \mathcal{K}^+, \tag{4}$$

$$\sum_{k\in} |[\theta_2 f^-, f_k^- - h_k^-]|^2 \leq \xi\|f^-\|^2,\ \forall f^- \in \mathcal{K}^-, \tag{5}$$

a. If (B_1, B_2) is the pair of upper frame bounds of $\{g_k\}_{k\in}$ and $\eta < B_1^{-1}(\|\gamma_1\|\ \|\theta_1^{-1}\|)^{-2}$ and $\xi < B_2^{-1}(\|\gamma_2\|\|\theta_2^{-1}\|)^{-2}$, then $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

b. If (S_1, S_2) and (D_1, D_2) are the pair of frame operators and the pair of upper frame bounds of $\{h_k\}_{k\in}$, resp., and $\eta < D_1^{-1}(\|S_1^{-1}\|\|\theta_1^{-1}\|)^{-2}$, $\xi < D_2^{-1}(\|S_2^{-1}\|\|\theta_2^{-1}\|)^{-2}$, then $\{f_k\}_{k\in}$ and $\{h_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

Proof. For $f \in$, set $g^+ = \theta_1 f^+$ and $g^- = \theta_2 f^-$, the inequalities (4) and (5) conclude that

$$\sum_{k\in} |[g^+, f_k^+ - h_k^+]|^2 \leq \eta\|\theta_1^{-1}\|^2\|g^+\|^2\ ,\ \sum_{k\in} |[g^-, f_k^- - h_k^-]|^2 \leq \xi\|\theta_2^{-1}\|^2\|g^-\|^2.$$

So $\{f_k - h_k\}_{k\in}$ is a Bessel sequence with the pair bounds $(\eta\|\theta_1^{-1}\|^2, \xi\|\theta_2^{-1}\|^2)$, and

$$\|\sum_{k\in} c_k(f_k^+ - h_k^+)\| \leq \sqrt{\eta}\|\theta_1^{-1}\|(\sum_{k\in}|c_k|^2)^{\frac{1}{2}},\ \forall\{c_k\}_{k\in\mathbb{N}} \in \ell^2(),$$

$$\|\sum_{k\in} d_k(f_k^- - h_k^-)\| \leq \sqrt{\xi}\|\theta_2^{-1}\|(\sum_{k\in}|d_k|^2)^{\frac{1}{2}},\ \forall\{d_k\}_{k\in\mathbb{N}} \in \ell^2().$$

Set $\alpha = \lambda = 0, \beta = \sqrt{\eta}\|\theta_1^{-1}\|$, and $\mu = \sqrt{\xi}\|\theta_2^{-1}\|$ in Theorem 6. Therefore (2) and (3) hold.

Now, by the assumptions in part a, we have

$$\sqrt{\eta}\|\theta_1^{-1}\|\sqrt{B_1}\|\gamma_1\| < 1, \quad \sqrt{\xi}\|\theta_2^{-1}\|\sqrt{B_2}\|\gamma_2\| < 1,$$

and the part a in Theorem 6 concludes that the frames $\{f_k\}_{k\in}$ and $\{g_k\}_{k\in}$ are operator dual frames in $\mathcal{K}$.

The conditions in b obtain the inequalities in part b of Theorem 6 and the proof is complete. □

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