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PROPER L_K-BIHARMONIC HYPERSURFACES IN THE EUCLIDEAN SPHERE WITH TWO PRINCIPAL CURVATURES

M. AMINIAN* AND M. NAMJOO

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ABSTRACT. In this paper we classify proper L_k-biharmonic hypersurfaces M, in the unit Euclidean sphere should have two principal curvatures and we show that they are open pieces of standard products of spheres. Also we study proper L_k-biharmonic compact hypersurfaces M with respect to $tr(S^2 \circ P_k)$ and H_k where S is the shape operator, P_k is the Newton transformation and H_k is the k-th mean curvature of M, and by definiteness assumption of P_k, we show that H_{k+1} is constant.

Keywords: L_k operator, biharmonic hypersurfaces, Chen conjecture.
2020 MSC: Primary 53C40, 53C42.

1. Introduction and Statement of Results

Harmonic and biharmonic maps are critical points of energy and bienergy functionals, equivalently these maps are solutions of PDE systems when tension and bitension fields are zero, respectively, [20, 22]. In [5, 8, 9], We have generalized these functionals and the notions of tension and bitension fields to introduce L_k-harmonic and L_k-biharmonic maps. We recall that the natural generalization of the Laplace operator is the L_k operator, [26, 27], which is the linearized operator of the $(k+1)$th mean curvature of a hypersurface for $k = 0, \ldots, n-1$, when $k = 0$, $L_0 = \Delta$.

Let $\varphi : M^n \to \mathbb{R}^m$ be an isometric immersion from a Riemannian manifold M^n into the Euclidean space $\mathbb{R}^m$, by the Beltrami formula $\Delta\varphi = n\vec{H}$, so φ is harmonic if and only if M is minimal, i.e., $\vec{H} = 0$, where Δ is the Laplace operator on M, and $\vec{H}$ is the mean curvature vector field of M. Inspired by this result, B.Y. Chen in [14] made the conjecture: Any biharmonic submanifold of a Euclidean space is minimal. Several authors have proved it in some cases (cf. [1, 16, 18, 19, 21]). Chen conjecture has also been generalized as follows [13]: Any biharmonic submanifold of a Riemannian manifold of nonpositive sectional curvature is minimal. This conjecture has been proved in numerous cases as in [2, 10, 13, 23, 24]. Both conjectures are still open in their full generality for ambient spaces with constant non-positive sectional curvature. On the other hand, Y-L. Ou and L. Tang in [25] have shown that the Generalized Chen conjecture is false, by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space of negative sectional curvature. By way of

*Corresponding author
E-mail: mehran.aminian@vru.ac.ir
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contrast, there are several families of examples of proper biharmonic submanifolds in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$ (cf. [11]).

Let $\varphi : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion from a connected oriented Riemannian manifold into the Euclidean space $\mathbb{R}^{n+1}$ with N as the unit normal direction. We have, [3],

$$L_k\varphi = (k+1)\binom{n}{k+1}H_{k+1}N, \tag{1}$$

where $k = 0, \cdots, n-1$ and H_{k+1} is $(k+1)$th mean curvature of M. When $k = 0$, above equation reduces to $\Delta\varphi = nH_1N = n\vec{H}$ which is the Beltrami equation. Inspired by Chen conjecture, we proposed the L_k-conjecture: Every Euclidean hypersurface $\varphi : M^n \to \mathbb{R}^{n+1}$ satisfying the condition $L_k^2\varphi = 0$ for some k, $0 \leq k \leq n-1$, has zero $(k+1)$th mean curvature, namely it is k-minimal. We have proved the L_k-conjecture in case of Euclidean hypersurfaces with at most two principal curvatures, [7], and also in case of space forms with three principal curvatures we consider it in [6]. Hereafter we have generalized the notions of tension and bitension fields to introduce L_k-harmonic and L_k-biharmonic maps (see below Definition 2.1 and Definition 2.2). By splitting of the Amin-bitension field with respect to its normal and tangent components we get the following characterization:

Theorem 1.1 ([8]). *Let M be a connected, oriented isometrically immersed Riemannian hypersurface in a simply connected space form $R^{n+1}(c), c = 0, \pm 1$. Then M is L_k-biharmonic hypersurface if and only if the following equations are satisfied:*

$$\begin{aligned} &\binom{n}{k+1}H_{k+1}\nabla H_{k+1} + 2(S \circ P_k)(\nabla H_{k+1}) = 0\,, \\ &L_kH_{k+1} - \binom{n}{k+1}H_{k+1}\left(nH_1H_{k+1} - (n-k-1)H_{k+2} - c(k+1)H_k\right) = 0. \end{aligned} \tag{2}$$

Hereby we generalized the L_k-conjecture for hypersurfaces of simply connected space forms as follows:

L_k*-*conjecture 1.2 ([8]). Let $\varphi : M^n \to R^{n+1}(c), c = 0, \pm 1$, be a connected oriented hypersurface immersed into a simply connected space form $R^{n+1}(c)$. If M is an L_k-biharmonic hypersurface, then H_{k+1} is zero.

For $c = 0, -1$, the L_k-conjecture is proved in some cases as hypersurface M has two principal curvatures, or M is weakly convex, or M is complete with some constraint on it and on L_k, and it is shown that there is not any L_k-biharmonic hypersurface M^n in $\mathbb{H}^{n+1}$ with two principal curvatures of multiplicities greater than one, [8].

For the case $c = +1$, the L_k-conjecture is false by considering hypersurface $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$, so $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ is proper ($H_{k+1} \neq 0$) L_k-biharmonic hypersurface. It leads us to the following characterization of proper totally umbilic hypersurfaces in the unit Euclidean sphere.

Remark 1.3 ([8]). Let M^n be a connected totally umbilic isometrically immersed hypersurface in $\mathbb{S}^{n+1}$. Then M is proper L_k-biharmonic hypersurface if and only if it is an open piece of $\mathbb{S}^n(\frac{\sqrt{2}}{2})$.

We extend this result to hypersurfaces having two distinct principal curvatures and we show that they are open pieces of the standard products of spheres (Theorem 1.4 and Theorem 1.5). Because of differentiation of proofs, in Theorem 1.4, we consider hypersurfaces having two distinct principal curvatures with both multiplicities greater than one, and in Theorem 1.5, we consider hypersurfaces having two distinct principal curvatures with both multiplicities 1 and $n-1$, in the unit Euclidean sphere $\mathbb{S}^n$.

Theorem 1.4. *Let M^n be an isometrically immersed Riemannian hypersurface in $\mathbb{S}^{n+1}$ having two distinct principal curvatures both with multiplicities greater than one. Then M is proper L_k-biharmonic hypersurface if and only if it is an open piece of $\mathbb{S}^m(\frac{1}{\sqrt{\alpha^2+1}}) \times \mathbb{S}^{n-m}(\frac{\alpha}{\sqrt{\alpha^2+1}})$ where $m \geq 2, n-m \geq 2, \alpha > 0$, and α satisfy the following equations:*

$$\sum_i \binom{m}{i}\binom{n-m}{k+1-i}(-\alpha^2)^i \neq 0\,, \tag{3}$$

$$\begin{aligned} &(k+2)\binom{m}{k+2}(-\alpha^2)^{k+1} - (n-k)\binom{m}{n-k}(-\alpha^2)^{m+k-n} \\ &+ \sum_{i\neq k+2,\, m+k-n}\Big[-m + \frac{n-m}{\alpha^2} - \frac{(k+2)(n+i-m-k-1)}{(k+2-i)\alpha^2} \\ &\qquad - \frac{(n-k)(k+1-i)}{(n+i-m-k)}\Big]\binom{m}{i}\binom{n-m}{k+1-i}\left(-\alpha^2\right)^i = 0\,. \end{aligned} \tag{4}$$

Theorem 1.5. *Let M^n be an isometrically immersed Riemannian hypersurface in $\mathbb{S}^{n+1}$ having two distinct principal curvatures with multiplicities 1 and $n-1$. Then M is proper L_k-biharmonic hypersurface if and only if it is an open piece of $\mathbb{S}^1(\frac{\sqrt{2}}{2}) \times \mathbb{S}^{n-1}(\frac{\sqrt{2}}{2})$ where $n \neq 2(k+1)$.*

Remark 1.6. In [10], it is shown that the only proper biharmonic hypersurfaces having at most two distinct principal curvatures in Euclidean sphere $\mathbb{S}^{n+1}$ are open pieces of $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ or $\mathbb{S}^m(\frac{\sqrt{2}}{2}) \times \mathbb{S}^{n-m}(\frac{\sqrt{2}}{2})$ where $n \neq 2m$ that we see it by Remark 1.3, Theorem 1.4 and replacing $k=0$ in equations (3) and (4).

Easily by considering $k=1$ in Remark 1.3, Theorem 1.4 and Theorem 1.5, we get the following result for proper L_1-biharmonic hypersurfaces which has at most two principal curvatures.

Corollary 1.7. *The only proper L_1-biharmonic hypersurfaces in $\mathbb{S}^{n+1}$ which has at most two distinct principal curvatures are open pieces of $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ or $\mathbb{S}^m(\frac{\sqrt{2}}{2}) \times \mathbb{S}^{n-m}(\frac{\sqrt{2}}{2})$ where $n \neq \frac{4m+1\pm\sqrt{8m+1}}{2}$. Especially the only proper L_1-biharmonic surfaces in $\mathbb{S}^3$ are open pieces of $\mathbb{S}^2(\frac{\sqrt{2}}{2})$ or $\mathbb{S}^1(\frac{\sqrt{2}}{2}) \times \mathbb{S}^1(\frac{\sqrt{2}}{2})$.*

In the following we shall study the proper L_k-biharmonic compact hypersurfaces with respect to $tr(S^2 \circ P_k)$ and H_k and by assuming definiteness of transformation P_k, we show that H_{k+1} is constant. The following results is an extension of Propositions 3.12 and 3.13 in [11] for proper L_k-biharmonic hypersurfaces.

Proposition 1.8. *Let M be a compact and proper L_k-biharmonic hypersurface in $\mathbb{S}^{n+1}$. If H_{k+1} is nowhere zero on M and $tr(S^2 \circ P_k) \geq (n-k)\binom{n}{k}H_k$ or $tr(S^2 \circ P_k) \leq (n-k)\binom{n}{k}H_k$, then $tr(S^2 \circ P_k) = (n-k)\binom{n}{k}H_k$. In addition, if P_k is definite then H_{k+1} is constant.*

Proposition 1.9. *Let M be a compact and proper L_k-biharmonic hypersurface in $\mathbb{S}^{n+1}$. If P_k is positive definite and $tr(S^2 \circ P_k) \geq (n-k)\binom{n}{k}H_k$, or that P_k is negative definite and*
$tr(S^2 \circ P_k) \leq (n-k)\binom{n}{k}H_k$, then $tr(S^2 \circ P_k) = (n-k)\binom{n}{k}H_k$ and H_{k+1} is constant.

2. Preliminaries

We recall the prerequisites from [3, 8, 15, 26]. Let $\varphi : M^n \to R^{n+1}(c)$ be an isometric immersion from a connected oriented Riemannian manifold M^n into the simply connected Riemannian space form $R^{n+1}(c)$ of constant sectional curvature c which is the Euclidean space $\mathbb{R}^{n+1}$ for $c = 0$ and the Hyperbolic space $\mathbb{H}^{n+1}$ for $c = -1$ and the Euclidean Sphere $\mathbb{S}^{n+1}$ for $c = +1$, $<,>_{g_M}$ the induced Riemannian metric on M by φ, N the unit normal vector field, ∇ and $\overline{\nabla}$ the Levi-Civita connections on M and $R^{n+1}(c)$, respectively. For simplicity we also denote the induced connection on the pullback bundle $\varphi^* TR^{n+1}(c)$ by $\overline{\nabla}$. Let X, Y be vector fields on M. We have the following formula for the shape operator of M,

$$d\varphi(SX) = -\overline{\nabla}_X N \,.$$

The shape operator $S : \mathcal{X}(M) \to \mathcal{X}(M)$ is a self-adjoint linear operator. Let $k_1, \ldots, k_n$ be its eigenvalues which are called principal curvatures of M . Define $s_0 = 1$ and

$$s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} k_{i_1} \cdots k_{i_k} \,.$$

The k-th mean curvature of M is defined by

$$\binom{n}{k} H_k = s_k \,.$$

For $k = 1$, $H_1 = \frac{1}{n} tr(S)$ is the mean curvature of M. If M has two principal curvatures, we denote them by

$$k_1 = \cdots = k_m = f,\, k_{m+1} = \cdots = k_n = g \,,$$

By assumption that $\binom{l}{r} = 0$ if $r > l$ or $r < 0$, we can write s_k as

$$s_k = \sum_i \binom{m}{i}\binom{n-m}{k-i} f^i g^{k-i}\,. \tag{5}$$

The Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - S \circ P_{k-1}\,,\ 1 \le k \le n.$$

From the Cayley-Hamilton theorem, one gets that $P_n = 0$. Each P_k is a self adjoint linear operator which commutes with S.

We recall that the natural generalization of the Laplace operator is the L_k operator, [17, 26, 27], which is the linearized operator of the $(k+1)$th mean curvature of a hypersurface for $k = 0, \ldots, n-1$, and it is defined by

$$L_k f = tr(P_k \circ \nabla^2 f)\,,$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of f and is defined by $\left\langle (\nabla^2 f)X,\, Y\right\rangle_{g_M} = \langle \nabla_X(\nabla f),\, Y\rangle_{g_M}$ for all vector fields $X, Y \in \mathcal{X}(M)$, and ∇f is the gradient vector field of f.

Here we recall following useful properties of the shape operator and the Newton transformation P_k to be used later. Let X, Y be tangent vector fields on M, then we have

$$(\nabla_X S)Y = (\nabla_Y S)X \text{ (Codazzi equation)},$$

$$divP_k = 0, \tag{6}$$

$$tr(S^2 \circ P_k) = s_1 s_{k+1} - (k+2)s_{k+2}\,. \tag{7}$$

Here we mention definition of Amin-tension field and Amin-L_k operator and then L_k-biharmonic map introduced in [8].

Definition 2.1. The Amin-tension field of φ is defined by

$$A_k(\varphi) = \sum_{i,j} P_{k\ ij}\left(\overline{\nabla}_{e_i} d\varphi(e_j) - d\varphi(\nabla_{e_i} e_j)\right) \tag{8}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M and $P_{k\ ij} = \langle P_k(e_i), e_j\rangle$. For a vector field $V \in \mathcal{X}(\varphi)$, the Amin-$L_k$ operator is defined by:

$$\bar{L}_k V = \sum_{i,j} P_{k\ ij}\left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} V - \overline{\nabla}_{\nabla_{e_i} e_j} V\right)\,. \tag{9}$$

One can see that in local coordinates $\{x^i\}$ for M, $\{y^\alpha\}$ for $R^{n+1}(c)$, $g_M = (g_{M\,ij})$ and $\varphi = (\varphi^\alpha)$, the Amin-tension field has the following expression:

$$A_k(\varphi) = \left(L_k\varphi^\gamma + g_M^{ii'} g_M^{jj'} \left\langle P_k(\frac{\partial}{\partial x^i}), \frac{\partial}{\partial x^j}\right\rangle_{g_M} \frac{\partial \varphi^\alpha}{\partial x^{i'}}\frac{\partial \varphi^\beta}{\partial x^{j'}}\overline{\Gamma}^\gamma_{\alpha\beta} \circ \varphi\right)\frac{\partial}{\partial y^\gamma} \circ \varphi\,,$$

where $\overline{\Gamma}^\gamma_{\alpha\beta}$'s are Christoffel symbols of the Levi-Civita connection $\overline{\nabla}$.

Definition 2.2. The map φ is an L_k-biharmonic if it satisfies the following equation:

$$\bar{L}_k(A_k(\varphi)) + P_{k\,ij}\bar{R}(A_k(\varphi), d\varphi(e_i))d\varphi(e_j) = 0\,,$$

where $\bar{R}$ is curvature tensor of $R^{n+1}(c)$ and $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M. The L.H.S of above equation is called Amin-bitension field $A_{2\,k}(\varphi)$. An L_k-biharmonic map is proper if $H_{k+1} \neq 0$.

3. Proof of Main Results

In this section, we prove our main results mentioned in Introduction. Here we mention the following auxiliary theorem which easily it can be obtained by using proofs of theorems 4.5 and 4.6 of [8]. Below we use Theorem 3.1 to prove Theorem 1.4 and Theorem 1.5.

Theorem 3.1. *Let M^n be a connected, oriented isometrically immersed L_k-biharmonic hypersurface in $\mathbb{S}^{n+1}$ having at most two principal curvatures. Then H_{k+1} is constant.*

Proof of Theorem 1.4. Suppose f and g denote principal curvatures of M with multiplicities m and $n-m$, respectively. By Theorem 3.1, s_{k+1} is constant. So by formulae (5), for example g is a smooth function of f. Take $g = F(f)$ for some smooth function $F : \mathbb{R} \to \mathbb{R}$. Let $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M which are the eigenvectors of the shape operator S of M w.r.t. the globally chosen unit normal vector field N. Since the multiplicities are greater than one, equations $Se_i = fe_i$ $i \leq m$, $Se_i = ge_i$ $i > m$ and the Codazzi equation, $(\nabla_{e_i}S)e_j = (\nabla_{e_j}S)e_i$, imply that

$$\nabla_{e_i}f = 0 \qquad i \leq m, \tag{10}$$

$$\nabla_{e_i}g = 0 \qquad i > m\,. \tag{11}$$

We have $\nabla_{e_i}g = F'(f)\nabla_{e_i}f$. So equations (10) and (11) imply that $\nabla_{e_i}g = 0$ for each i. Thus g is constant and since s_{k+1} is constant by our assumption, f is also constant. So M is an isoparametric hypersurface in $\mathbb{S}^{n+1}$. Then, by the classical results on isoparametric hypersurfaces in the Euclidean sphere, we get that $fg = -1$. Let $f > 0$ and $f = \alpha$ and so $g = \frac{-1}{\alpha}$. By Example 3.4 of [4], M is an open piece of $\mathbb{S}^m(\frac{1}{\sqrt{\alpha^2+1}}) \times \mathbb{S}^{n-m}(\frac{\alpha}{\sqrt{\alpha^2+1}})$. Since s_{k+1} is non-zero constant, straightforward calculations using equations (5), (2) and properties of combinations, we get equations (3) and (4). □

Proof of Theorem 1.5. Suppose f and g denote principal curvatures of M with multiplicities 1 and $n-1$, respectively. By Theorem 3.1, s_{k+1} is non-zero constant. Now by formulae (5), if $g = 0$ then f is constant. Therefore M is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ but by the classical results on isoparametric hypersurfaces in the Euclidean sphere, we know that $fg = -1$ which is a contradiction. So we get $g \neq 0$ and again by formulae (5), we have

$$f = \frac{s_{k+1} - \binom{n-1}{k+1}g^{k+1}}{\binom{n-1}{k}g^k}\,. \tag{12}$$

Using equations (2) and (12), we get a non zero polynomial of variable g. So g and then f is constant. Therefore M is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ and by the classical results on isoparametric hypersurfaces in the Euclidean sphere as Theorem 1.4, it is an open piece of $\mathbb{S}^1(\frac{1}{\sqrt{\alpha^2+1}}) \times \mathbb{S}^{n-1}(\frac{\alpha}{\sqrt{\alpha^2+1}})$ where $\alpha > 0$ and α satisfy equations (3) and (4) by replacing $m = 1$ which yields that $\alpha = 1$ and $n \neq 2(k+1)$. □

To prove Proposition 1.8 and Proposition 1.9, we shall need the following lemmas for properties of L_k operator. In Lemma 3.2 and Lemma 3.4, we introduce extra properties of L_k operator by use of Amin-L_k operator.

Lemma 3.2. *Let f_1 and f_2 be smooth functions on M and, X and Y be smooth vector fields on M. Then*

$$i)\ L_k(f_1 f_2) = f_2 L_k f_1 + f_1 L_k f_2 + 2\left\langle P_k(\nabla f_1), \nabla f_2 \right\rangle_{\varphi^* g}$$

$$ii)\ L_k \left\langle X, Y \right\rangle_{\varphi^* g} = \left\langle \bar{L}_k d\varphi(X), d\varphi(Y) \right\rangle_g + \left\langle d\varphi(X), \bar{L}_k d\varphi(Y) \right\rangle_g + 2\sum_i \left\langle \overline{\nabla}_{P_k(e_i)} d\varphi(X), \overline{\nabla}_{e_i} d\varphi(Y) \right\rangle_g$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame on M.

Remark 3.3. For part (i) of Lemma 3.2, the reader can see (cf. [4]). Here we give a proof for part (ii) of it.

Proof of part (ii) of Lemma 3.2. Assume a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every $i,\ j$. From (6), we get that $\sum_i \nabla_{e_i} P_{k\,ij}(p) = 0$ for every j. So at p, we have

$$\begin{aligned}
L_k \left\langle X, Y \right\rangle_{\varphi^* g} &= \sum_{i,j} P_{k\,ij} \left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} \left\langle d\varphi(X), d\varphi(Y) \right\rangle_g \right) \\
&= \sum_{i,j} P_{k\,ij} \left(\overline{\nabla}_{e_i} \left(\left\langle \overline{\nabla}_{e_j} d\varphi(X), d\varphi(Y) \right\rangle_g + \left\langle d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y) \right\rangle_g \right) \right) \\
&= \sum_{i,j} P_{k\,ij} \Big(\left\langle \overline{\nabla}_{e_j} d\varphi(X), \overline{\nabla}_{e_i} d\varphi(Y) \right\rangle_g + \left\langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} d\varphi(X), d\varphi(Y) \right\rangle_g \\
&\quad + \left\langle \overline{\nabla}_{e_i} d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y) \right\rangle_g + \left\langle d\varphi(X), \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} d\varphi(Y) \right\rangle_g \Big) \\
&\underset{(9)}{=} \left\langle \bar{L}_k d\varphi(X), d\varphi(Y) \right\rangle_g + \left\langle d\varphi(X), \bar{L}_k d\varphi(Y) \right\rangle_g \\
&\quad + 2\sum_{i,j} P_{k\,ij} \left\langle \overline{\nabla}_{e_j} d\varphi(X), \overline{\nabla}_{e_i} d\varphi(Y) \right\rangle_g \\
&= \left\langle \bar{L}_k d\varphi(X), d\varphi(Y) \right\rangle_g + \left\langle d\varphi(X), \bar{L}_k d\varphi(Y) \right\rangle_g \\
&\quad + 2\sum_{i} \left\langle \overline{\nabla}_{P_k(e_i)} d\varphi(X), \overline{\nabla}_{e_i} d\varphi(Y) \right\rangle_g .
\end{aligned}$$

□

Lemma 3.4. *Let f_1 and f_2 be smooth functions on M and, X and Y be smooth vector fields on M. Suppose that support of f_1 and X is in a compact domain. Then*

$$i) \quad \int_M f_1\, L_k f_2 \,\mathrm{d}M = \int_M f_2\, L_k f_1 \,\mathrm{d}M\,,$$

$$ii) \quad \int_M \left\langle d\varphi(X), \bar{L}_k d\varphi(Y)\right\rangle_g \mathrm{d}M = \int_M \left\langle \bar{L}_k d\varphi(X), d\varphi(Y)\right\rangle_g \mathrm{d}M\,.$$

Remark 3.5. For part (i) of Lemma 3.4, the reader can see (cf. [12]). Here we give a proof for part (ii) of it.

Proof of part (ii) of Lemma 3.4. Assume a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every $i,\ j$. From (6), we get that $\sum_i \nabla_{e_i} P_{k\ ij}(p) = 0$ for every j. Let's define a well-defined vector field Z on M as

$$Z := P_{k\ ij} \left\langle d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y)\right\rangle_g e_i\ .$$

So at p, we have

$$\begin{aligned} divZ =& P_{k\ ij} \left\langle \overline{\nabla}_{e_i} d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y)\right\rangle_g + P_{k\ ij} \left\langle d\varphi(X), \overline{\nabla}_{e_i}\overline{\nabla}_{e_j} d\varphi(Y)\right\rangle_g \\ =& P_{k\ ij} \left\langle \overline{\nabla}_{e_i} d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y)\right\rangle_g + \left\langle d\varphi(X), \bar{L}_k d\varphi(Y)\right\rangle_g\ . \end{aligned}$$

Therefore by Divergence Theorem we get

$$\begin{aligned} \int_M \left\langle d\varphi(X), \bar{L}_k d\varphi(Y)\right\rangle_g \mathrm{d}M &= -\int_M P_{k\ ij} \left\langle \overline{\nabla}_{e_i} d\varphi(X), \overline{\nabla}_{e_j} d\varphi(Y)\right\rangle_g \mathrm{d}M \\ &= \int_M \left\langle \bar{L}_k d\varphi(X), d\varphi(Y)\right\rangle_g \mathrm{d}M\,. \end{aligned}$$

□

The following proposition is a known result of Maximum Principle for operators. Here for convenience, we give a proof for Proposition 3.6 to be used later.

Proposition 3.6. *Let f be a smooth function on M and its support be in a compact domain. If P_k is definite and $L_k f = 0$ then f is constant.*

Proof. By Lemma 3.2, we have

$$L_k f^2 = 2\left\langle P_k(\nabla f), \nabla f\right\rangle_{\varphi^* g}\,. \tag{13}$$

Now using Lemma 3.4, we get

$$\int_M L_k f^2 \mathrm{d}M = 0\,. \tag{14}$$

So equations (13) and (14) result in

$$\int_M \left\langle P_k(\nabla f), \nabla f\right\rangle_{\varphi^* g} \mathrm{d}M = 0\,.$$

Since P_k is definite, we get $\left\langle P_k(\nabla f), \nabla f\right\rangle_{\varphi^* g} = 0$ and so $\nabla f = 0$. Therefore f is constant. □

Proof of Proposition 1.8. We know by equations (7) and (2),

$$L_k s_{k+1} = s_{k+1}\left(tr(S^2 \circ P_k) - (n-k)s_k\right) . \tag{15}$$

By Lemma 3.4, we have $\int_M L_k s_{k+1} \mathrm{d}M = 0$. So equation (15) yields that

$$\int_M s_{k+1}\left(tr(S^2 \circ P_k) - (n-k)s_k\right) \mathrm{d}M = 0 .$$

By the hypothesis, above integrand does not change sign, thus $s_{k+1}\left(tr(S^2 \circ P_k) - (n-k)s_k\right) = 0$ and since $s_{k+1} \neq 0$, we get $tr(S^2 \circ P_k) = (n-k)s_k$. By equation (15), $L_k s_{k+1} = 0$ and if P_k is definite then Proposition 3.6 gives that s_{k+1} is constant. □

Proof of Proposition 1.9. We get by equations (7) and (2), and Lemma 3.2,

$$\begin{aligned} L_k s_{k+1}^2 &= 2s_{k+1} L_k s_{k+1} + 2\left\langle P_k \nabla s_{k+1}, \nabla s_{k+1}\right\rangle_{\varphi^* g} \\ &= 2s_{k+1}^2\left(tr(S^2 \circ P_k) - (n-k)s_k\right) + 2\left\langle P_k \nabla s_{k+1}, \nabla s_{k+1}\right\rangle_{\varphi^* g} . \end{aligned} \tag{16}$$

By Lemma 3.4, we have $\int_M L_k s_{k+1}^2 \mathrm{d}M = 0$. Assume that P_k is positive definite and $tr(S^2 \circ P_k) \geq (n-k)s_k$, in other case the proof is similar. So equation (16) yields that $\int_M \left\langle P_k \nabla s_{k+1}, \nabla s_{k+1}\right\rangle_{\varphi^* g} \mathrm{d}M \leq 0$ and because of P_k is positive definite we get $\left\langle P_k \nabla s_{k+1}, \nabla s_{k+1}\right\rangle_{\varphi^* g} \leq 0$ and so $\nabla s_{k+1} = 0$. Therefore s_{k+1} is constant and by equation (2), $tr(S^2 \circ P_k) = (n-k)s_k$. □

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MEHRAN AMINIAN
DEPARTMENT OF MATHEMATICS
VALI-E-ASR UNIVERSITY OF RAFSANJAN
RAFSANJAN, IRAN.
E-mail address: `mehran.aminian@vru.ac.ir`

MEHRAN NAMJOO
DEPARTMENT OF MATHEMATICS
VALI-E-ASR UNIVERSITY OF RAFSANJAN
RAFSANJAN, IRAN.
E-mail address: `namjoo@vru.ac.ir`