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EXISTENCE AND STABILITY OF SOLUTIONS FOR A NONLINEAR FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

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Abstract. This paper investigates the existence and interval of existence, uniqueness and Ulam stability of solutions on initial value type problem of a nonlinear Caputo fractional Volterra-Fredholm integro-differential equation in Banach spaces.

Keywords: Fractional Volterra-Fredholm integro-differential equation, Caputo sense, Fixed point technique.
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1. Introduction

Over the past few decades, fractional calculus, as a promising field of research in mathematics, has demonstrated an outstanding advancement in terms of its growing applications to numerous real-world problems in areas such as viscoelasticity, rheology, continuum mechanics, electromagnetic theory, disaster control and so on [16, 21]. Due to the nonlinearity of problems related to science and technology, fractional calculus has become an indispensable mathematical tool in analyzing and determining the different possibilities of defining real/complex ordered derivatives and integrals that can be of great use while representing such problems [12, 21, 22]. Applying the concepts of fractional calculus to integro-differential equations further advanced its scope of research in mathematical modeling and control. The main difference between integro-differential equations and fractional-order integro-differential equations lies in the fact that while the former involves derivation and integration of integer order, the latter involves the same in a non-integer/an arbitrary order [15, 16]. This has given rise to a stiff increase in the use of these equations to model real life problems of science and engineering over integer modeling, as it supersedes integer modeling in terms of its efficiency to translate realistic situations into mathematical formulations more precisely [16, 19]. In recent years, many authors focus on the development of techniques for discussing the solutions of

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fractional differential and integro-differential equations. For instance, we can remember the following works:

Ibrahim and Momani [15] studied the existence and uniqueness of solutions of a class of fractional order differential equations, Karthikeyan and Trujillo [19] proved existence and uniqueness of solutions for fractional integro-differential equations with boundary value conditions, Bahuguna and Dabas [2] applied the method of lines to establish the existence and uniqueness of a strong solution for the partial integrodifferential equations, Matar [23] deliberated the existence of solutions for nonlocal fractional semilinear integro-differential equations in Banach spaces via Banach fixed point theorem.

Momani [18] considered the following fractional differential equations:

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f(t, u(t)),\ 0 < \alpha \leq 1,\\ &u(t_0) = u_0. \end{aligned}$$

Devi and Sreedhar [4] considered the following Caputo fractional integro-differential equation of the type:

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f(t, u(t), I^{\alpha}u(t)),\ 0 < \alpha \leq 1,\\ &u(0) = u_0. \end{aligned}$$

Momani et al. [17] proved the Local and global uniqueness results by using Bihari's inequality for the fractional integro-differential equation:

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f(t, u(t)) + \int_{t_0}^{t} Z(t, s, u(s))ds,\ 0 < \alpha \leq 1\\ &u(0) = u_0. \end{aligned}$$

Ahmad and Sivasundaram [1] studied some existence and uniqueness results in a Banach space for the fractional integro-differential equation:

$${}^cD^{\alpha}u(t) = f(t, u(t)) + \int_{t_0}^{t} Z(t, s, u(s))ds,\ 0 < \alpha < 1$$

$$u(0) = u_0 - g(u),\ g \in C([0, T], X).$$

Wu and Liu [28] discussed the existence and uniqueness of solutions for fractional integro-differential equations of the type:

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f\Big(t, u(t), \int_{0}^{t} Z(t, s, u(s)ds\Big),\ t \in [0, 1],\\ &u(0) = u_0. \end{aligned}$$

Recently, in [3, 7, 13, 14, 17, 27] the author's obtained the result on existence and uniqueness of solutions for fractional integro-differential equations by using the fixed point theorem of Banach space with contraction mapping principle.

In this position, by utilizing and mixing interesting ideas of the above mentioned manuscripts, we intend to check some specific aims about the existence and interval of existence, uniqueness, Ulam stability of solutions for the following proposed initial value type problem of a Caputo fractional Volterra-Fredholm integro-differential equation

$$(1)\qquad {}^cD^{\alpha}u(t) = \theta u(t) + f\Big(t, u(t), \int_0^t Z_1(t,s)u(s)ds, \int_0^T Z_2(t,s)u(s)ds\Big),$$

$$(2)\qquad u(0) = u_0,\ t \in J := [0,T], \theta > 0,$$

where ${}^cD^{\alpha}$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$, $f : J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $Z_1, Z_2 : J \times J \longrightarrow \mathbb{R}$ are continuous functions satisfying some conditions which will be stated later, and $\zeta_1 := \sup\{|Z_1(t,s)| : 0 \leq s \leq t \leq T\}$, $\zeta_2 := \sup\{|Z_2(t,s)| : 0 \leq s \leq t \leq T\}$.

The main objective of the present paper is to study the new existence, uniqueness and Ulam stability results by means of the Banach contraction principle, Schaefer's fixed point theorem and Pachpatte's integral inequality for Caputo fractional Volterra-Fredholm integro-differential equations with initial value condition.

The rest of this paper is organized as follows. In Section 2, some essential notations, definitions and lemmas related to fractional calculus are recalled. In Section 3, the new existence and uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential equation have been proved. In Section 4, we study the Ulam stability of the problem (1)-(2). In Section 5, an illustrative example is presented. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. **Preliminaries**

In this section, we outline some basic concepts of fractional calculus and modern tools of functional analysis, and state some fixed-point theorems related to our work. For more details, see [5, 6, 8–11, 22–24, 26].

Let $C(J,\mathbb{R})$ denotes the Banach space of all continuous functions on J. For any function $h \in C(J,\mathbb{R})$, $\|h\|_{\infty} = \sup\{|h(t)| : t \in J\}$. $L^1(J)$ denotes the space of all real functions defined on J which are Lebesgue integrable with the norm $\|h\|_{L^1} = \int_0^T |h(t)|dt$.

Definition 2.1. [21] (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function h is defined as

$$\begin{aligned} J^{\alpha}h(t) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}h(s)ds, \qquad \alpha \in \mathbb{R}^+, \\ J^0h(t) &= h(t), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. [21] **(Caputo fractional derivative).** The fractional derivative of $h(t)$ in the Caputo sense is defined by

$$
\begin{aligned}
{}^{c}D^{\alpha}h(t) &= J^{m-\alpha}D^{m}h(t) \\
&= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}\frac{\partial^m h(s)}{\partial s^m}ds, & m-1<\alpha<m, \\ \\ \frac{\partial^m h(t)}{\partial t^m}, & \alpha=m, \quad m\in N, \end{cases}
\end{aligned} \tag{3}
$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^{\alpha}J^{v}h = J^{\alpha+v}h, \quad \alpha, v>0.$
(2) $J^{\alpha}h^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}h^{\beta+\alpha},$
(3) $D^{\alpha}h^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}h^{\beta-\alpha}, \quad \alpha>0, \quad \beta>-1.$
(4) $J^{\alpha}D^{\alpha}h(t) = h(t)-h(a),\ 0<\alpha<1.$
(5) $J^{\alpha}D^{\alpha}h(t) = h(t)-\sum_{k=0}^{m-1}h^{(k)}(0^+)\frac{(t-a)^k}{k!}, \quad t>0.$

Definition 2.3. [21] **(Riemann-Liouville fractional derivative).** The Riemann Liouville fractional derivative of order $\alpha>0$ is normally defined as

$$D^{\alpha}h(t) = D^{m}J^{m-\alpha}h(t), \qquad m-1<\alpha\le m, \quad m\in\mathbb{N}. \tag{4}$$

Definition 2.4. [26] Let $T: X \longrightarrow X$ be a mapping on a normed space $(X, \|.\|)$. A point $x\in X$ for which $Tx=x$ is called a fixed point of T.

Definition 2.5. [21] The mapping T on a normed space $(X, \|.\|)$ is called contractive if there is a non-negative real number $c\in(0,1)$, such that $\|Tx-Ty\|\le c\|x-y\|$ for all $x, y\in X$.

Theorem 2.6. *[22] (Banach fixed point theorem) Let $(X, \|.\|)$ be a complete normed space, and let the mapping $T: X\longrightarrow X$ be a contraction mapping. Then T has exactly one fixed point.*

Theorem 2.7. *[21] (Schaefer's fixed point theorem) Let X be a normed space, T a continuous mapping of X into X which is compact on each bounded subset E of X. Then either*
(i) The equation $x=\lambda T(x)$ has a solution for $\lambda=1$, or
(ii) The set of all such solutions x, for $0<\lambda<1$, is unbounded.

Lemma 2.8. *[25] (Pachpatte's inequality) Let $u(t), f(t), q(t)\in C(J,\mathbb{R}_+)$ and let $n(t)\in C(J,\mathbb{R}_+)$ be and nondecreasing for which the inequality*

$$u(t)\le n(t)+\int_0^t f(s)u(s)ds+\int_0^t f(s)\int_0^s q(r)u(r)drds,$$

holds for any $t\in\mathbb{R}_+$. Then

$$u(t)\le n(t)\Big[1+\int_0^t f(s)\exp\Big(\int_0^s (f(r)+q(r))dr\Big)ds\Big].$$

3. Existence and Uniqueness Results

In this section, we shall give an existence and uniqueness results of Eq.(1), with the initial condition (2). Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exists a constant $L_f > 0$, for each $t \in J$ and $u_i, v_i, y_i \in \mathbb{R},\ i = 1, 2,$ such that

$$\|f(t, u_1, v_1, y_1) - f(t, u_1, v_1, y_1)\| \leq L_f(\|u_1 - u_2\| + \|v_1 - v_2\| + \|y_1 - y_2\|).$$

(A2) There exists a constant $a_f > 0$, for each $t \in J$ and $u, v, y \in \mathbb{R}$, such that

$$\|f(t, u, v, y)\| \leq a_f(1 + \|u\| + \|v\| + \|y\|).$$

First, we will state the following axiom lemma.

Lemma 3.1. *Let $0 < \alpha \leq 1$. Assume that f, Z_1 and Z_2 are continuous functions. If $u \in C(J, \mathbb{R})$ then u satisfies the problem (1)-(2) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) &= u_0 + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)ds,\ t \in J. \end{aligned} \tag{5}$$

Theorem 3.2. *If the hypotheses (A1) and (A2) hold, and*

$$M := \Big[\frac{(\theta + L_f)T^\alpha + L_f(\zeta_1 + \zeta_2)T^{\alpha+1}}{\Gamma(\alpha+1)}\Big] < 1. \tag{6}$$

Then the problem (1)-(2) has a unique solution on J.

Proof. We transform the Cauchy problem (1)-(2) to be applicable to fixed point problem and define the operator $\Upsilon : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ by

$$\begin{aligned} \Upsilon(u)(t) &= u_0 + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)ds. \end{aligned} \tag{7}$$

Let $u, v \in C(J, \mathbb{R})$. Then for each $t \in J$, we have

$$
\begin{aligned}
&\|\Upsilon(u)(t)-\Upsilon(v)(t)\| \\
\leq\ & \frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\Big(s,u(s),\int_0^s Z_1(t,\sigma)u(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big) \\
&-f\Big(s,v(s),\int_0^s Z_1(t,\sigma)v(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)v(\sigma)d\sigma\Big)\Big\|ds \\
\leq\ & \frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds \\
&+\frac{L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\|u(s)-v(s)\|+\int_0^s |Z_1(t,\sigma)|\|u(\sigma)-v(\sigma)\|d\sigma \\
&+\int_0^T |Z_2(t,\sigma)|\|u(\sigma)-v(\sigma)\|d\sigma\Big)ds \\
\leq\ & \frac{\theta+L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds \\
&+\frac{L_f(\zeta_1+\zeta_2)T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds \\
\leq\ & \Big[\frac{(\theta+L_f)T^\alpha+L_f(\zeta_1+\zeta_2)T^{\alpha+1}}{\Gamma(\alpha+1)}\Big]\|u-v\|_\infty .
\end{aligned}
$$

Thus,

$$\|\Upsilon(u)-\Upsilon(v)\|_\infty \leq M\|u-v\|_\infty .$$

Since $M<1$, we conclude that Υ is a contraction map. By Banach contraction principle, we deduce that Υ has a unique fixed point which is a solution of the problem (1)-(2). □

Next, we will prove the existence of solution for the problem (1)-(2) in the space $C(J,X)$ by means of Schaefer's fixed point theorem.

Theorem 3.3. *If the hypotheses (A1) and (A2) hold. Then the problem* (1)-(2) *has at least one solution on* J.

Proof. The proof will be given in some steps.

Step 1. The operator Υ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \longrightarrow u$ in $C(J,\mathbb{R})$. Then for each $t\in J$

$$
\begin{aligned}
&\|\Upsilon(u_n)(t)-\Upsilon(u)(t)\| \\
\leq\ & \frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_n(s)-u(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\sup_{t\in J}\Big\|f\Big(s,u_n(s),\int_0^s Z_1(t,\sigma)u_n(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)u_n(\sigma)d\sigma\Big) \\
&-f\Big(s,u(s),\int_0^s Z_1(t,\sigma)u(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)\Big\|ds,
\end{aligned}
$$

since f, Z_1 and Z_2 are continuous functions and $u_n \longrightarrow u$, then we have

$$\|\Upsilon(u_n)-\Upsilon(u)\|_\infty \longrightarrow 0, \text{ as } n\longrightarrow\infty.$$

Step 2. The operator Υ maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

We need to show that for any $\rho>0$, there exists a positive constant r such that for each $u\in B_\rho=\{u\in C(J,\mathbb{R}) : \|u\|_\infty\leq\rho\}$, we have $\|\Upsilon(u)\|_\infty\leq r$. Thus, for each $t\in J$, we have

$$
\begin{aligned}
&\|\Upsilon(u)(t)\| \\
\leq\ & \|u_0\|+\frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\Big\|f\Big(s,u(s),\int_0^s Z_1(t,\sigma)u(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)\Big\|ds \\
\leq\ & \|u_0\|+\frac{\theta\rho T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a_f\Big(1+\|u(s)\|+ \\
&\int_0^s |Z_1(t,\sigma)|\|u(\sigma)\|d\sigma+\int_0^T |Z_2(t,\sigma)|\|u(\sigma)\|d\sigma\Big)ds \\
\leq\ & \|u_0\|+\frac{\theta\rho T^\alpha}{\Gamma(\alpha+1)}+\frac{a_f(1+\rho+\rho(\zeta_1+\zeta_2)T)T^\alpha}{\Gamma(\alpha+1)} \\
:=\ & r.
\end{aligned}
$$

Thus

$$\|\Upsilon(u)\|_\infty\leq r.$$

Step 3. The operator Υ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in (0,T],\ t_1 < t_2,\ B_\rho$ be a bounded set of $C(J,\mathbb{R})$ as in step 2, and let $u \in B_\rho$. Then

$$\begin{aligned}
&\|\Upsilon(u)(t_1) - \Upsilon(u)(t_2)\| \\
\leq\ & \frac{\theta}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] \|u(s)\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] \\
&\Big\| f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)\Big\| ds \\
&+ \frac{\theta}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|u(s)\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \\
&\Big\| f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big)\Big\| ds \\
\leq\ & \frac{\theta\rho}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] ds \\
&+ \frac{a_f(1+\rho+(\zeta_1+\zeta_2)\rho T)}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] ds \\
&+ \frac{\theta\rho}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds + \frac{a_f(1+\rho+(\zeta_1+\zeta_2)\rho T)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\
\leq\ & \frac{\theta\rho + a_f(1+\rho+(\zeta_1+\zeta_2)\rho T)}{\Gamma(\alpha+1)} [2(t_2-t_1)^\alpha + (t_1^\alpha - t_2^\alpha)] \\
&\longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.
\end{aligned}$$

As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $\Upsilon : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\Omega = \{u \in C(J,\mathbb{R}) : u = \sigma\Upsilon(u),\ \text{for some } \sigma \in (0,1)\},$$

is bounded. Let $u \in \Omega$, then $u = \sigma\Upsilon(u)$, for some $\sigma \in (0,1)$. Thus, for each $t \in J$ we have

$$\begin{aligned}
u(t) &= \sigma\Big[u_0 + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big) ds\Big].
\end{aligned}$$

This implies by (A1) and (A2) that for each $t \in J$ we have

$$\begin{aligned} \|\Upsilon(u)(t)\| &\leq \|u_0\| + \frac{\theta\rho T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1+\rho+\rho(\zeta_1+\zeta_2)T)T^\alpha}{\Gamma(\alpha+1)} \\ &:= L. \end{aligned}$$

Thus

$$\|\Upsilon(u)\|_\infty \leq L.$$

This shows that the set Ω is bounded. Now applying Schaefer's fixed point theorem, we deduce that Υ has a fixed point which is a solution of the problem (1)-(2).

□

4. **Stability Results**

In this part, we study the Ulam stability of the problem (1)-(2). Now we consider the Ulam stability for the following problem

$$(8) \quad \begin{aligned} {}^cD^\alpha v(t) = \; & \theta u(t) + \\ & f\Big(t, u(t), \int_0^t Z_1(t,s)u(s)ds, \int_0^T Z_2(t,s)u(s)ds\Big), \; t \in J, \end{aligned}$$

and the following inequalities:

$$(9) \quad \begin{aligned} &\Big|{}^cD^\alpha v(t) - \theta v(t) - f\Big(t, v(t), \int_0^t Z_1(t,s)v(s)ds, \int_0^T Z_2(t,s)v(s)ds\Big)\Big| \\ &\leq \epsilon, \; t \in J, \end{aligned}$$

$$(10) \quad \begin{aligned} &\Big|{}^cD^\alpha v(t) - \theta v(t) - f\Big(t, v(t), \int_0^t Z_1(t,s)v(s)ds, \int_0^T Z_2(t,s)v(s)ds\Big)\Big| \\ &\leq \epsilon\varphi(t), \; t \in J, \end{aligned}$$

$$(11) \quad \begin{aligned} &\Big|{}^cD^\alpha v(t) - \theta v(t) - f\Big(t, v(t), \int_0^t Z_1(t,s)v(s)ds, \int_0^T Z_2(t,s)v(s)ds\Big)\Big| \\ &\leq \varphi(t), \; t \in J. \end{aligned}$$

Definition 4.1. [20] The Eq. (8) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C(J, \mathbb{R})$ of inequality (9) there exists a solution $u \in C(J, \mathbb{R})$ of equation (8) with

$$|v(t) - u(t)| \leq \epsilon C_f, t \in J.$$

Definition 4.2. [20] The Eq. (8) is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0,\infty),[0,\infty))$, $\psi_f(0) = 0$ such that for each solution $v \in C(J, \mathbb{R})$ of inequality (9) there exists a solution $u \in C(J, \mathbb{R})$ of equation (8) with

$$|v(t) - u(t)| \leq \epsilon\psi_f, \; t \in J.$$

Definition 4.3. [20] The Eq. (8) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C(J, \mathbb{R})$ of inequality (10) there exists a solution $u \in C(J, \mathbb{R})$ of equation (8) with

$$|v(t) - u(t)| \leq \epsilon C_f \varphi(t), \ t \in J.$$

Theorem 4.4. *Assumes that (A1), (A2) and* (6) *are fulfilled. Then the problem* (1)-(2) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $v \in C(J, \mathbb{R})$ be a function which satisfies inequality (9) and let $u \in C(J, \mathbb{R})$ be the unique solution of the following problem

$$(12) \qquad {}^cD^\alpha u(t) = \theta u(t) + f\Big(t, u(t), \int_0^t Z_1(t,s)u(s)ds, \int_0^T Z_2(t,s)u(s)ds\Big),$$

$$(13) \qquad v(0) = u_0. \ t \in J, \ 0 < \alpha \leq 1.$$

Using Lemma 3.1, we obtain

$$\begin{aligned} u(t) &= u_0 + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\Big(s, u(s), \int_0^s Z_1(t,\sigma)u(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)u(\sigma)d\sigma\Big) ds. \end{aligned}$$

By integrating (9), we obtain:

$$(14) \qquad \begin{aligned} &\Big\| v(t) - u_0 - \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &f\Big(s, v(s), \int_0^s Z_1(t,\sigma)v(\sigma)d\sigma, \int_0^T Z_2(t,\sigma)v(\sigma)d\sigma\Big) ds \Big\| \\ &\leq \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Using (A1), (A2) and the inequality (14), for every $t \in J$, we have:

$$\begin{aligned}
\|v(t)-u(t)\| &\leq \Big\|v(t)-u_0-\frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
& f\Big(s,v(s),\int_0^s Z_1(t,\sigma)v(\sigma)d\sigma,\int_0^T Z_2(t,\sigma)v(\sigma)d\sigma\Big)ds\Big\| \\
& +\frac{\theta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|v(s)-u(s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\Big(s,v(s),\int_0^s Z_1(t,\sigma)v(\sigma)d\sigma, \\
& \int_0^T Z_2(t,\sigma)v(\sigma)d\sigma\Big)\Big\|ds \\
&\leq \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}+\frac{\theta+L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|v(s)-u(s)\|ds \\
& +\frac{L_f(\zeta_1+\zeta_2)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \|v(\sigma)-u(\sigma)\|d\sigma+ \\
& \int_0^T \|v(\sigma)-u(\sigma)\|d\sigma\Big)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\|v(t)-u(t)\| &\leq \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}+\frac{\theta+L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|v(s)-u(s)\|ds \\
& +\frac{L_f(\zeta_1+\zeta_2)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \|v(\sigma)-u(\sigma)\|d\sigma+ \\
& \int_0^T \|v(\sigma)-u(\sigma)\|d\sigma\Big)ds \\
&\leq \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}+\int_0^t \frac{\theta+L_f}{\Gamma(\alpha)}(T-s)^{\alpha-1}\Big[\|v(s)-u(s)\| \\
& +\frac{L_f(\zeta_1+\zeta_2)}{(\theta+L_f)}\|v(\sigma)-u(\sigma)\|d\sigma\Big]ds. \qquad (15)
\end{aligned}$$

By applying Pachpatte's inequality given in Theorem 3.2 to the inequality (15), we obtain

$$\begin{aligned}\|v(t)-u(t)\| &\leq \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)}\Big[1+\int_0^T \frac{\theta+L_f}{\Gamma(\alpha)}(T-s)^{\alpha-1} \\ &\quad \times \exp\Big(\int_0^s \Big[\frac{\theta+L_f}{\Gamma(\alpha)}(T-\sigma)^{\alpha-1}+\frac{L_f(\zeta_1+\zeta_2)}{\theta+L_f}\Big]d\sigma\Big)ds\Big] \\ &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}\Big[1+\int_0^T \frac{\theta+L_f}{\Gamma(\alpha)}(T-s)^{\alpha-1} \\ &\quad \times \exp\Big(\int_0^s \Big[\frac{\theta+L_f}{\Gamma(\alpha)}(T-\sigma)^{\alpha-1}+\frac{L_f(\zeta_1+\zeta_2)}{\theta+L_f}\Big]d\sigma\Big)ds\Big] \\ &:= \epsilon\, C_f.\end{aligned}$$

Thus the problem (1)-(2) is Ulam-Hyers stable. □

Corollary 4.5. *If f and g in the nonlocal problem* (1)-(2) *satisfy the conditions (A1), (A2) and the inequality* (6) *hold, then the nonlocal problem* (1)-(2) *is generalized Ulam-Hyers stable.*

Theorem 4.6. *Assumes that (A1), (A2) and* (6) *are fulfilled. Further suppose there exist an increasing function $\varphi \in C(J,\mathbb{R}_+)$ and $\Psi_\varphi > 0$ such that $I^{\alpha}\varphi(t) \leq \Psi_\varphi \varphi(t)$, for any $t \in J$. Then the nonlocal problem* (1)-(2) *is Ulam-Hyers-Rassias stable.*

Proof. Under the assumptions of Theorem 4.4, we consider problem (1)-(2) and inequality (11). One can repeat the same process to verify that problem (1)-(2) is Ulam-Hyers-Rassias stable. □

5. An Example

We consider the following nonlocal Cauchy problem of Caputo fractional Volterra-Fredholm integro-differential equation

$$(16)\quad {}^cD^{0.5}u(t) = \frac{1}{5}|u(t)| + \frac{e^{-t}}{4+e^t}\Big[\frac{|u(t)|}{1+|u(t)|}\Big] + \frac{1}{5}\int_0^t \frac{e^{-t}}{(2+t)^2}|u(s)|ds + \frac{1}{5}\int_0^1 \frac{e^{-t}}{(3+t)^2}|u(s)|ds,$$

$$(17)\quad u(0) = 1,\ t \in J := [0,1].$$

From equations (16)-(17) and inequality (6), we have

$$\begin{aligned}M &= \Big[\frac{(\theta+L_f)T^{\alpha}+L_f(\zeta_1+\zeta_2)T^{\alpha+1}}{\Gamma(\alpha+1)}\Big] \\ &= \Big[\frac{(\frac{1}{5}+\frac{1}{5})+\frac{1}{5}(\frac{1}{4}+\frac{1}{9})}{\Gamma(\frac{1}{2}+1)}\Big] \\ &= 0.53 \\ &< 1.\end{aligned}$$

Then the problem (16)-(17) has a unique solution on J, and by Theorem 4.4, the problem (16)-(17) is Ulam-Hyers stable.

6. Conclusion

The main purpose of this paper was to present new existence, uniqueness and Ulam-Hyers stability results of the solution for Caputo fractional Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as Banach contraction principle, Schaefer's fixed point theorem and Pachpatte's integral inequality. Moreover, the results of references [1, 18, 28] appear as a special case of our results.

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