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A NOTE ON SOME DISTANCE FORMULAE IN 3-DIMENSIONAL MAXIMUM SPACE

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Abstract. In this paper, we consider 3-dimensional analytical space furnishing with maximum metric and we give some distance formulas about relations of distances between a point and a line, a point and a plane and between two lines in terms of maximum metric.

Keywords: Maximum metric, Maximum 3-space, Distance of a point to a plane, Distance of a point to a line, Distance between two lines.
2020 MSC: Primary 51F99, 51K05, 51K99, 51M15, 51M20.

1. INTRODUCTION

If we want to measure the distance between two points in a space, then we use commonly Euclidean distance. Despite, it is so popular and ancient, it is not practical in the real world. For example, we imagine that we were on a mountain climb. When climbing, we saw another climbing team on the opposite hillside. In this point, there is a logical question such that ”What is the distance between two climbing teams?” in our mind. For the answer to the question, the following comment can be made: If we have wings to fly freely, we can use the Euclidean distance to get a correct result. However, when we consider the impossibility of a human being to fly using a vehicle, the distance we want to measure is much more than we think. Of course, it is possible to increase such examples. So, as is clear from this example, the Euclidean metric is not useful in such situations. Therefore, in some situations some other distance functions like as taxicab, chinese checker and maximum distances would be sufficient instead of Euclidean distance. Especially the maximum distance is a very useful model in the real world and its applications are so much. For example, urban planning at the macro dimensions can be used for nearly calculations.

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Let $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_m)$ be two points in $\mathbb{R}^m$. Therefore, Euclidean distance is the most commonly used distance- it calculates the root of square differences between coordinates of a pair of objects;

$$D_{XY} = \sqrt{\sum_{k=1}^{m} (x_k - y_k)^2}$$

Manhattan distance, city block distance or mostly known as Taxicab distance indicates distance between points in a city road grid. It calculates the absolute differences between coordinates of a pair of objects;

$$D_{XY} = \sum_{k=1}^{m} |x_k - y_k|$$

Chebyshev distance is also called as Maximum value distance. It calculates maximum absolute difference between the corresponding coordinates of a pair of objects;

$$D_{XY} = \max_{k \in \{1,\ldots,m\}} |x_k - y_k|$$

Minkowski distance is the generalized metric distance which is defined as follows:

$$D_{XY} = \left(\sum_{k=1}^{m} |x_k - y_k|^p \right)^{1/p}$$

Note that when $p = 2$, the Minkowski distance is the Euclidean distance. When $p = 1$, the Minkowski distance is the Taxicab distance. Chebyshev distance is a special case of Minkowski distance with $p \to \infty$ (taking a limit). So Chebyshev distance is also called as L_∞ distance.

To find the distance between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, maximum metric $d_M : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ is defined by

$$d_M(P_1, P_2) = \max \{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$$

instead of well known Euclidean metric

$$d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

$\mathbb{R}^3_M$ and $\mathbb{R}^3_E$ are denote analytic 3-space which is furnished by Maximum metric and Euclidean metric, respectively. Linear structure of the $\mathbb{R}^3_M$ is almost the same of $\mathbb{R}^3_E$, that is, points, lines and planes are the same with Euclidean case and angles are measured by the same way, there is only one difference, this difference is that using different distance function in 3-dimensional analytical space. Therefore it is so important to work on notions concerned to distance in geometric studies, because change of metric can give rise to some interesting results.

According to d_M-metric, the shortest way between points P_1 and P_2 is a line segments which is parallel to a coordinate axis, as shown in Figure 1. Thus the shortest distance between P_1 and P_2 is Euclidean length of this segment.

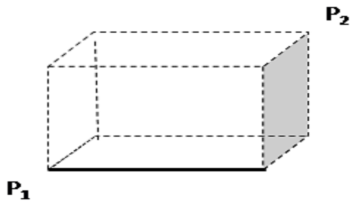

Figure 1

In this work, we study Maximum geometry versions of some topics of the Euclidean space that include the notion of distance. These topics are distances of a point to a plane, a point to a line and distance between two lines. Versions of taxicab and chinese checkers of these topics have been given in [1] and [12], respectively. Also the similar topics for different space are examined in [4, 6] These works have attracted our attention to this subject.

2. MAIN RESULTS

In this section, we give maximum analogues of distance of a point to a plane and a line, and distance between two lines in $\mathbb{R}^3_M$. If metric is changed in a metric space, then the notations about metric are changed. For example, the sphere in $\mathbb{R}^3_M$ is the cube which faces are parallel to coordinate planes. The cube which is a sphere of $\mathbb{R}^3_M$ is one of the Platonic solids. The taxicab sphere is an octahedron which is a Platonic solid. In [7] the metric spaces whose spheres are dodecahedron and icosahedron were given. Furthermore, in the recent years considerable amount of articles can be found in the literature about some metric spaces whose spheres are some convex polyhedra. For some examples of these articles can be seen to [2, 3, 5, 9–11, 13]. Thus in this sense studies in maximum metric space are valueable.

The following property and theorem without proof are given from [8]. These statements are useful tools to give distance formulas. Therefore following property states that the ratio of Euclidean and maximum distance between two points. In the other words, this property expresses how the transition exists between Euclidean and maximum metric.

Let l be a line through the points P_1 and P_2. If l has direction vector (p, q, r) then it is easy to see that

$$\frac{d_E(P_1, P_2)}{\sqrt{p^2 + q^2 + r^2}} = \frac{d_M(P_1, P_2)}{\max\{|p|, |q|, |r|\}}.$$

The following theorem show that all translations in $\mathbb{R}^3$ are isometries in $\mathbb{R}^3_M$.

Theorem 2.1. *The Maximum distance is invariant under all translations in analytic* $3-$ *space. That is,*

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \ni T(x, y, z) = (x + a, y + b, z + c), \qquad a, b, c \in \mathbb{R}$$

does not change the distance between any two points in $\mathbb{R}^3_M$.

The following proposition gives the formula of distance between a point and a plane in the $M-$space, $\mathbb{R}^3_M$.

Proposition 2.2. *$M-$ distance of a point $P = (x_o, y_o, z_o)$ to a plane $\mathcal{P} : Ax + By + Cz + D = 0$ is*

$$d_M(P, \mathcal{P}) = \frac{|Ax + By + Cz + D|}{\max\{|A + B + C|, |-A + B + C|, |A - B + C|, |A + B - C|\}}.$$

Proof. In the M-space, $\mathbb{R}^3_M$, distance from a point P to a plane $\mathcal{P}$ is defined as

$$d_M(P, \mathcal{P}) = \min\{d_M(P, X) \mid X \in \mathcal{P}\}.$$

To find this distance the sphere (cube) with the center P would be considered. While the radius of the sphere (cube) is gradually increased, if the point such that the cube and the plane intersect firstly, then the required distance would be found easily. So if the cube is inflated, then $d_M(P, \mathcal{P}) = d_M(P, Q)$ where Q is the intersection point of the plane and the cube, and also Q must be a corner of the cube. Thus if we consider lines l_i, $i = 1, 2, 3, 4$ passing through P and a corner of the cube, each of l_i has a direction vector which is an element of $\Delta = \{(1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$. Let $P_i = l_i \cap \mathcal{P}$ for $i = 1, 2, 3, 4$. So, $d_M(P, \mathcal{P}) = \min\{d_M(P, P_i) : i = 1, 2, 3, 4\}$. For example, let $(p, q, r) = (1, 1, 1)$ be the direction vector of the line l_1. Thus, the equation of the line l_1 which is passing through P is

$$\frac{x - x_o}{1} = \frac{y - y_o}{1} = \frac{z - z_o}{1} = t_1.$$

According to this coordinates of the any point $P_1 = (x, y, z)$ on the line l_1 are $x = x_0 + t_1$,
$y = y_o + t_1, z = z_0 + t_1$. Also this point provides plane equation, since it is on the plane. Thus, one can find the following equation:

$$A(x_0 + t_1) + B(y_0 + t_1) + C(z_0 + t_1) + D = 0.$$

If the equation is solved for t_1, then one can find the following result:

$$t_1 = \frac{Ax_0 + By_0 + Cz_0 + D}{-(A + B + C)}.$$

So in this case

$$\begin{aligned} d_M(P, \mathcal{P}) &= d_M(P, P_1) \\ &= \max\{|x_0 - (x_0 + t_1)|, |y_0 - (y_0 + t_1)|, |z_0 - (z_0 + t_1)|\} \\ &= t_1 = \left|\tfrac{Ax_0 + By_0 + Cz_0 + D}{-A + B + C}\right|. \end{aligned}$$

Similarly, if one take the lines which direction vector

$$(p, q, r) = (1, -1, 1), (-1, 1, 1) \text{ or } (1, 1, -1)$$

then the results are obtained as follows:

$$d_M(P, P_i) = \left|\frac{Ax_0 + By_0 + cz_0 + D}{\alpha_i}\right|$$

where $\alpha_2 = -A + B + C$, $\alpha_3 = A - B + C$, $\alpha_4 = A + B - C$. Note that obviously all of $|A + B + C|, |-A + B + C|, |A - B + C|$ and $|A + B - C|$ can not be zero. Therefore the required formulae giving maximum distance from the point to the plane is obtained. □

The next proposition gives the formula to find the distance between a point to a line in $\mathbb{R}^3_M$.

Proposition 2.3. *M-distance of a point $P = (x_o, y_o, z_o)$ to a line l given by*

$$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

is

$$d_M(P, l) = \frac{\max\{\alpha, \beta, \gamma\}}{\Delta} .$$

where $\alpha = |p(y_0 - b) - q(x_0 - a)|$, $\beta = |p(z_0 - c) - r(x_0 - a)|$, $\gamma = |q(z_0 - c) - r(y_0 - b)|$ and

$$\Delta = \begin{cases} \max\{|p+q|, |p-q|\} & \text{if } \alpha \geq \beta \text{ and } \alpha \geq \gamma \\ \max\{|p+r|, |p-r|\} & \text{if } \beta \geq \alpha \text{ and } \beta \geq \gamma \\ \max\{|q+r|, |q-r|\} & \text{if } \gamma \geq \alpha \text{ and } \gamma \geq \beta. \end{cases}$$

Proof. We consider a cube with the center P to find distance of a point $P = (x_o, y_o, z_o)$ to a line l given by $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$. If we enlarge the radius of the cube, then the maximum distance between P and l is equal to the maximum distance between P and the first intersection point of the cube and the line l.

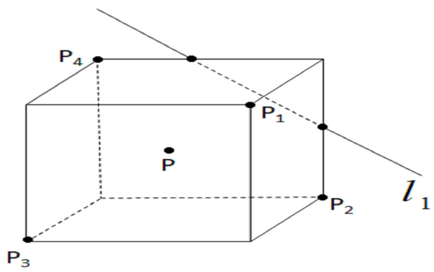

Figure 2

Consider the cube with the center $P = (x_o, y_o, z_o)$ and let the length of an edge of this cube be k. If we enlarge sufficiently the radius of the cube, then the cube and the line l intersects. So four of vertices of the cube are $P_1 = (x_o + k, y_o + k, z_o + k), P_2 = (x_o - k, y_o + k, z_o - k), P_3 = (x_o + k, y_o - k, z_o - k)$ and $P_4 = (x_o - k, y_o - k, z_o + k)$. Line l and the cube would intersect at an edge of the cube. The edges of the cube lies on a line which direction vector is $(1, 0, 0), (0, 1, 0)$ or $(0, 0, 1)$, and passing through P_1, P_2, P_3 or P_4. For example, let's consider the edge on the line with direction vector $(1, 0, 0)$ and passing through P_1. Hence this edge is on the line $x = x_0 + k + \lambda, y = y_0 + k, z = z_0 + k$, where $\lambda \in \mathbb{R}$ and line l is $x = p\mu + a, y = q\mu + b, z = r\mu + c$, where $\mu \in \mathbb{R}$.

Now we can find the intersection point of the line l and the cube. So we obtain $k = \frac{r(y_0-b)-q(z_o-c)}{q-r}$ as the maximum distance between P and l.
Similarly the other cases can be easily found. Therefore we obtain the required formulae. □

The following proposition gives the formula to compute the distance between two lines in $\mathbb{R}^3_M$.

Proposition 3: M-distance between any two lines given by

$$l \ldots \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} = \lambda$$

$$l' \ldots \frac{x-a'}{p} = \frac{y-b'}{q} = \frac{z-c'}{r} = \mu$$

in $\mathbb{R}^3_M$ can be expressed as follows:
If l is parallel to l', then $d_M(l,l') = d_M(A,l')$, where $A = (a,b,c)$ is on the line l. If l is not parallel to l', then

$$d_M(l,l') = \frac{|(rq' - r'q)(a-a') + (pr' - rp')(b-b') + (p'q - pq')(c-c')|}{\max\{|\alpha+\beta+\gamma|, |-\alpha+\beta+\gamma|, |\alpha-\beta+\gamma|, |\alpha+\beta-\gamma|\}}$$

where $\alpha = p'q - pq'$, $\beta = pr' - p'r$ and $\gamma = rq' - r'q$.

Proof. The distance between two lines can be stated as $d_M(l,l') = \min\{d_M(X,X') \,|X \in l, X' \in l'\}$. If l is parallel to l', then it can be taken as $(p,q,r) = (p',q',r')$ and $P = (a' + \mu p, b' + \mu q, c' + \mu r)$ for any point P on l', without loss of generality. Then it can be easily computed using the formula given by proposition 2 that $d_M(l,P) = d_M(A,l)$, where $A = (a,b,c)$.

If l is not parallel to l', then at least one of $pq' - qp', qr' - rq', rp' - pr'$ is not zero. Otherwise these lines would be parallel to each other. Let's consider the points $P = (p\lambda + a, q\lambda + b, r\lambda + c)$ and $P' = (p'\mu + a', q'\mu + b', r'\mu + c')$ which are on l and l', respectively. Thus if $d_M(P,P')$ is minimum, then direction vector of the line l' passing through P and P' is an element of

$$\{(1,1,1), (-1,1,1), (1,-1,1), (1,1,-1)\}.$$

For example, one can take $(p',q',r') = (1,1,1)$ as direction vector of l'. Since

$$\frac{p\lambda + a - p'\mu - a'}{1} = \frac{q\lambda + b - q'\mu - b'}{1} = \frac{r\lambda + c - r'\mu - c'}{1},$$

then

$$\mu = \frac{(q-r)(a-a') + (-p+r)(b-b') + (p-q)(c-c')}{(p'q - pq') + (pr' - rp') + (rq' - r'q)}$$

and

$$\lambda = \frac{(q'-r')(a-a') + (-p'+r')(b-b') + (p'-q')(c-c')}{(p'q - pq') + (pr' - rp') + (rq' - r'q)}.$$

So we can obtain that

$$
\begin{aligned}
d_M(P,P') &= \max\{|p\lambda + a - p'\mu - a'|, |q\lambda + b - q'\mu - b'|, |r\lambda + c - r'\mu - c'|\} \\
&= \left| \tfrac{(rq'-r'q)(a-a')+(pr'-rp')(b-b')+(p'q-pq')(c-c')}{(p'q-pq')+(pr'-p'r)+(rq'-r'q)} \right|.
\end{aligned}
$$

Other cases can be obtained by the similiar way. And as a subcase if (p', q', r') is an element of $\{(1,0,0), (0,1,0), (0,0,1)\}$, then

$$
d_M(l,l') = \frac{|(rq' - r'q)(a - a') + (pr' - rp')(b - b') + (p'q - pq')(c - c')|}{\max\{|p'q - pq'|, |pr' - rp'|, |rq' - r'q|\}}
$$

similarly if (p', q', r') is an element of
$\{(1,1,0), (1,0,1), (1,-1,0), (1,0,-1), (0,1,-1), (0,1,1)\}$, then
$d_M(l,l') = \frac{|(rq'-r'q)(a-a')+(pr'-rp')(b-b')+(p'q-pq')(c-c')|}{\max\{|(p'q-pq')\pm(pr'-rp')|, |(p'q-pq')\pm(rq'-r'q)|, |(pr'-rp')\pm(rq'-r'q)|\}}$. □

3. CONCLUSION

We give some fundamental formulae to compute the distance between a point to a line, between a point to a plane, and between two lines in maximum $3-$dimensional space by a synthetic approach. The (unit) sphere of $\mathbb{R}^3_M$ is the cube which faces are parallel to coordinate planes. The cube is one of the Platonic solids. The similar article about taxicab $3-$dimensional space [1] gives the taxicab analogues of these distance by a different process. Taxicab sphere is an octahedron which is a Platonic solid. In the recent years a lot of articles or studies can be found about the metric spaces in each the sphere is a convex polyhedron. Therefore the similar study can be interesting in dodecahedron, icosahedron space or in any other convex polyhedron spaces.

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