Journal of Mahani Mathematical Research Center
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ON A CLASS OF BANACH FUNCTION SPACES AND ITS RELATION TO SOME CLASSICAL SPACES

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Article type: Research Article
(Received: 25 August 2019, Revised: 14 February 2021; Accepted: 06 April 2021)
(Communicated by A. Armandnejad)

Abstract. Assume that $\mathbb{F}$ denotes a specific space in the class $\mathbb{F}_{\alpha,p}$ constructed by H. Khodabakhshian [2] as a class of separable Banach function spaces similar to the well-known James function spaces. In this note, we prove that $l_p(\alpha)$ is isomorphic to a complemented subspace of $\mathbb{F}_{\alpha,p}$, and that $\mathbb{F}_{\alpha,2}$ is a closed subspace of the Waterman–Shiba space αBV^2.

Keywords: Banach space, Complemented subspace, Generalized bounded variation.
2020 MSC: Primary 46B04, 46B20.

1. Introduction

In [2], H. Khodabakhshian introduced a class of separable Banach function spaces $\mathbb{F}_{\alpha,p}$ similar to the well-known James function spaces, with the following properties.

(1) The James function space JF belongs to this class.
(2) Each member of this class is a separable Banach space with non-separable dual.
(3) The sequence space $X_{\alpha,1}$, constructed by Azimi and Hagler [3], embeds in $\mathbb{F}_{\alpha,1}$.

In this paper, we show that $l_p(\alpha)$ is isomorphic to a complemented subspace of $\mathbb{F}_{\alpha,p}$, and that $\mathbb{F}_{\alpha,p}$ is a closed subspace of the Waterman–Shiba space $\alpha BV^{(p)}$ for $p = 2$.

We use the standard Banach space notation as can be found in [4] and [5]. Let X be a (real) normed space, and let (x_n) be a non-zero sequence in X. We say that (x_n) is a (*Schauder*) *basis* for X if for each $x \in X$, there exists a unique sequence (a_n) of scalars such that $x = \sum_{i=1}^{\infty} a_n x_n$, where the series converges to x in norm.

Let X be a linear space. Then, a *linear projection* on X (or just a *projection* on X) is a linear map $P : X \to X$ which is an idempotent, that is, $P(P(x)) \equiv P^2(x) = P(x)$ for every $x \in X$.

A sequence $(x_n) \subseteq X$ is said to be *normalized* if $\|x_n\| = 1$ for every n; it is called *monotone* if $x_n \neq 0$ for every $n \in \mathbb{N}$, and the following inequality holds

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DOI: 10.22103/jmmrc.2021.14614.1100

How to cite: J. Fathi Mourjani, *On a class of Banach function spaces and its relation to some classical spaces*, J. Mahani Math. Res. Cent. 2021; 10(1): 103-110.

for $K \in \mathbb{N}$ and $(a_n)_{n=1}^{K+1} \subseteq \mathbb{R}$.

$$\left\| \sum_{n=1}^{K} a_n x_n \right\| \leq \left\| \sum_{n=1}^{K+1} a_n x_n \right\| .$$

A subspace Y of a Banach space X is said to be *complemented* in X if there exists a bounded projection $P : X \to Y$ such that $P(X) = Y$.

Let (α_i) be a decreasing sequence such that $\alpha_1 = 1$, $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$. By $l_p(\alpha)$ we denote the space of all sequences $x = (x_i)$ such that $\sum_{i=1}^{\infty} \alpha_i |x_i|^p$ converges with respect to the norm $||x||_{l_p(\alpha)} = (\sum_{i=1}^{\infty} \alpha_i |x_i|^p)^{\frac{1}{p}}$.

For any i, let $e_i = \left(\underbrace{0, \dots, 0}_{i-1}, \left(\frac{1}{\alpha_i}\right)^{\frac{1}{p}}, 0, \dots \right)$. We know that $\{e_i : i \in \mathbb{N}\}$ is a normalized basis for $l_p(\alpha)$.

Now, we recall the construction of the spaces $\mathbb{F}_{\alpha,p}$. For $1 \leq p < \infty$, the function space $\mathbb{F}_{\alpha,p}$ is defined as the completion of the space of all equivalence classes of the linear spans of characteristic functions of partitions of $[0, 1]$, equipped with the following norm.

$$\|f\| = \sup_{0=x_0<x_1<\cdots<x_n=1} \left[\sum_{i=1}^{n} \alpha_i | \int_{x_{i-1}}^{x_i} f(x)dx|^p \right]^{\frac{1}{p}} .$$

Clearly, we identify those functions which are equal almost everywhere.

A function f defined on $[0, 1]$ is said to be of bounded $p - \alpha-$variation $(1 \leq p < \infty)$ if

$$\|f\| = \sup_{0=t_0<t_1<\cdots<t_n=1} \left[\sum_{i=1}^{n} \alpha_i |f(t_{i-1}) - f(t_i)|^p \right]^{\frac{1}{p}} < \infty.$$

The function $\|.\|$ is a norm on the set αBV^p of all functions f such that $f(0) = 0$ and $\|f\| < \infty$. The normed space $(\alpha BV^p, ||.||)$ is a Banach space.

The Waterman–Shiba space αBV^p has been introduced by M. Shiba in 1980 [6]. When $p = 1$, αBV^p is the well-known Waterman space αBV. See [7] and [8] for example. Throughout this paper, μ denotes the Lebesgue measure in $\mathbb{R}$. Every interval I has a strictly positive measure and defines a bounded linear functional I^* on $\mathbb{F}_{\alpha,p}$:

$$I^*(f) := \int_I f d\mu.$$

2. The main results

This section contains the proofs of our main results, namely, that $l_p(\alpha)$ is isomorphic to a complemented subspace of $\mathbb{F}_{\alpha,p}$, and that $\mathbb{F}_{\alpha,2}$ is a closed subspace of the Waterman–Shiba space αBV^2.

Lemma 2.1. *Let I_n be a sequence of successive intervals (that is,* $\sup(I_n) = \inf(I_{n+1})$*) such that $\mu(I_{2n-1}) = \mu(I_{2n})$ for every $n \in \mathbb{N}$. We set $\phi_n = \chi_{I_{2n-1}} - \chi_{I_{2n}}$ for every $n \in \mathbb{N}$. Then, the following inequalities hold for $(a_n) \in \mathbb{R}$.*

$$\Big\|\sum_n a_n e_n\Big\|_{\ell_p(\alpha)} \leq \Big\|\sum_n a_n \frac{\phi_n}{\mu(I_{2n-1})}\Big\|_{\mathbb{F}_{\alpha,p}} \leq 2\Big\|\sum_n a_n e_n\Big\|_{\ell_p(\alpha)}.$$

Proof. Let the partition $(I_n)_n$ of $I = [0,1]$ satisfy the asserted conditions, and consider some $(a_n) \in \mathbb{R}$. For the first inequality, we consider the partition (I_{2n-1}). Then,

$$\begin{aligned}\Big\|\sum_n a_n \tfrac{\phi_n}{\mu(I_{2n-1})}\Big\|_{\mathbb{F}_{\alpha,p}} &\geq \Big\|\sum_k \left(\int_{I_{2k-1}} \sum_n a_n \tfrac{\phi_n}{\mu(I_{2n-1})}\right) e_k\Big\|_{\ell_p(\alpha)} \\ &\geq \Big\|\sum_n a_n e_n\Big\|_{\ell_p(\alpha)}.\end{aligned}$$

For the second inequality, let $(A_j)_j$ be any partition of $[0,1]$. Since

$$\sum_n a_n \frac{\phi_n}{\mu(I_{2n-1})}\Big|_{I_k} = \begin{cases} a_k & \text{if } k \text{ is odd,} \\ -a_k & \text{if } k \text{ is even,} \end{cases}$$

we may assume that each of the intervals A_j is a finite union of successive intervals I_n. The following estimates can be easily obtained for each interval A_j.

$$\left|\int_{A_j} \sum_n a_n \frac{\phi_n}{\mu(I_{2n-1})}\right| \leq \begin{cases} 0 & \text{if } A_j = [I_{2k-1}, I_{2m}],\ k \leq m, \\ |a_m| & \text{if } A_j = [I_{2k-1}, I_{2m-1}],\ k \leq m, \\ |a_k| & \text{if } A_j = [I_{2k}, I_{2m}],\ k \leq m, \\ |-a_k + a_m| & \text{if } A_j = [I_{2k}, I_{2m-1}],\ k < m. \end{cases}$$

Since the intervals are successive, each a_n appears at most two times. Therefore,

$$\Big\|\sum_n a_n \frac{\phi_n}{\mu(I_{2n-1})}\Big\|_{\mathbb{F}_{\alpha,p}} \leq 2\Big\|\sum_n a_n e_n\Big\|_{\ell_p(\alpha)}.$$

□

In the sequel, we denote by $\langle A\rangle$ the linear subspace generated by a subset A of a normed space Y.

Theorem 2.2. *The space $\ell_p(\alpha)$ is isomorphic to a complemented subspace of $\mathbb{F}_{\alpha,p}$.*

Proof. Let (I_n) and (φ_n) be as in Lemma 2.1. We prove that the space generated by (φ_n) is a complemented subspace of $\mathbb{F}_{\alpha,p}$. Consider the map

$$P : \mathbb{F}_{\alpha,p} \longrightarrow \overline{\left\langle \left\{ \frac{\phi_n}{\mu(I_{2n-1})} : n \in \mathbb{N} \right\} \right\rangle}$$

defined by

$$g \longmapsto \sum_n I^*_{2n-1}(g) \frac{\phi_n}{\mu(I_{2n-1})}.$$

It follows from the definition of the map $I^*_{2n_1}$ that

$$I^*_{2n-1}\left(\frac{\phi_n}{\mu(I_{2k-1})}\right) = \delta_{n,k},$$

and by Lemma 1,

$$\begin{aligned}\Big\| \sum_n I^*_{2n-1}(g)\tfrac{\phi_n}{\mu(I_{2n-1})}\Big\|_{\mathbb{F}_{\alpha,p}} &\leq 2\Big\| \sum_n I^*_{2n-1}(g)e_n\Big\|_{\ell_p(\alpha)} \\ &= 2\Big\| \sum_n \left(\int_{I_{2n-1}} g\right) e_n\Big\|_{\ell_p(\alpha)} \leq 2\|g\|_{\mathbb{F}_{\alpha,p}}.\end{aligned}$$

Therefore, P is a projection with $||P|| \leq 2$. □

A finite and ordered sequence $G = \{t_i : i = 1, ..., n \text{ and } t_1 < \cdots < t_n\}$ in $[0, 1]$, not necessarily containing 0 and 1, determines the following seminorm on αBV^2.

$$| f |^G = \left[\sum_{i=1}^{n} \alpha_i |f(t_{i-1}) - f(t_i)|^2\right]^{\frac{1}{2}}.$$

A finite and ordered sequence $G = \{(r_i, t_i) : r_1 < t_1 \leq r_2 < t_2 \leq \cdots \leq r_n < t_n\}$ of ordered pairs in $[0, 1]$, not necessarily containing 0 and 1, determines the following seminorm on αBV^2.

$$| f |_G = \left[\sum_{i=1}^{n} \alpha_i |f(t_i) - f(r_i)|^2\right]^{\frac{1}{2}}.$$

Note that $| f |^G \leq \| f \|$ and $| f |_G \leq \| f \|$.

Definition 2.3. For $\delta > 0$, let $S(\delta)$ denote the collection of all seminorms $|.|_G$ determined by finite sequences $G = \{r_1 < t_1 \leq r_2 < t_2 \leq \cdots \leq r_n < t_n\}$ of ordered pairs in $[0, 1]$ with the property that $t_i - r_i \leq \delta$ for $1 \leq i \leq n$. Define $\sigma_2^\delta : \alpha BV^2 \to \mathbb{R}$ by

$$\sigma_2^\delta(f) = \sup_{|.|_G \in S(\delta)} |f|_G.$$

Now, define

$$\alpha BV_0^2 = \left\{ f \in \alpha BV^2 : \lim_{\delta\to 0} \sigma_2^\delta(f) = 0 \right\}.$$

Definition 2.4. By the following proposition, αBV_0^2 is a Banach space.

Proposition 2.5. *The space αBV_0^2 is a closed subspace of αBV^2.*

Proof. It is easy to see that αBV_0^2 is a linear subspace of αBV^2. To show that αBV_0^2 is closed, we consider the sequence $(h_j) \in \alpha BV_0^2$, where $h_j \to h$ for some $h \in \alpha BV^2$. Given $\epsilon > 0$, choose J and $\delta > 0$ such that $\| h - h_J \| < \frac{\epsilon}{2}$ and $\sigma_2^\delta(h_J) < \frac{\epsilon}{2}$. Let $|.|_G$ be an arbitrary seminorm on $S(\delta)$. Then,

$$|h|_G = |h - h_J + h_J|_G \leq \|h - h_J\| + \sigma_2^\delta(h_J) < \epsilon.$$

This shows that $h \in \alpha BV_0^2$. □

Definition 2.6. For $n \in N^* := \mathbb{N} \cup \{0\}$ and an integer $1 \leq k \leq 2^n$, let

$$D_n^k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right].$$

The sequence $\left(\left(D_n^k\right)_{k=1}^{2^n}\right)_{n=0}^{\infty}$ is called a collection of dyadic intervals in $[0,1]$. An element of $[0,1]$ is a dyadic rational if it is of the form $\frac{k}{2^n}$ for some $n \in N^*$ and $k \in \{0, \ldots, 2^n\}$.

Definition 2.7. Let $f : [0,1] \to \mathbb{R}$ be a continuous, piecewise linear function. We say that f changes monotonicity at $t \in (0,1)$ if there exists $\epsilon > 0$ for which one of the following holds.

1. $f|_{(t-\epsilon,t)}$ is constant, and $f|_{(t+\epsilon,t)}$ is strictly increasing or decreasing.
2. $f|_{(t-\epsilon,t)}$ is strictly increasing or decreasing, and $f|_{(t+\epsilon,t)}$ is constant.
3. $f|_{(t-\epsilon,t)}$ is strictly increasing (respectively, decreasing), and $f|_{(t+\epsilon,t)}$ is strictly decreasing (respectively, increasing).

We will adopt the convention that every $f : [0,1] \to \mathbb{R}$ changes monotonicity at 0 and 1.
The following proposition is a fairly straightforward generalization of Proposition 1.7 in [1].

Proposition 2.8. *Let $f : [0,1] \to \mathbb{R}$ be a continuous, piecewise linear function with $f(0) = 0$, and let (α_i) be a decreasing sequence such that $\alpha_1 = 1$, $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$. Then, there exists a finite, ordered sequence $(d_i)_{i=0}^m$ in $[0,1]$ such that f changes monotonicity at $d_i \in [0,1]$ for $i = 0, 1, \ldots, m$, and*

$$\|f\| = \left[\sum_{i=1}^{n} \alpha_i |f(d_i) - f(d_{i-1})|^2\right]^{\frac{1}{2}}.$$

Proof. We assume, without loss of generality, that f is not constant on any subinterval I of $[0,1]$. In this case, the points where f changes monotonicity, say

$$0 = d_0 < d_1 < \cdots < d_l = 1,$$

satisfy item 3 of Definition 4. It is sufficient to show that for a given finite, ordered sequence $G = (s_i)_{i=0}^n$ in $[0,1]$, there exists a finite, ordered subsequence $(t_i)_{i=0}^m$ of $(d_i)_{i=0}^l$ such that

$$\mid f \mid^G = \left[\sum_{i=1}^{n} \alpha_i |f(s_i) - f(s_{i-1})|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{m} \alpha_i |f(t_i) - f(t_{i-1})|^2\right]^{\frac{1}{2}}. \quad (1)$$

For $1 \leq i \leq l$, let $G \cap [d_{i-1}, d_i] = \{s_j, \ldots, s_k\}$. If we remove s_i from G, for some $j < i < k$, then we can increase $\mid f \mid^G$. Thus we may assume that for each $1 \leq i \leq l$, $G \cap [d_{i-1}, d_i]$ has at most two elements. Beginning with $i = 1$, we inductively "adjust" $G \cap [d_{i-1}, d_i]$ to be a subset of $\{d_{i-1}, d_i\}$ in such a way that we increase $\mid f \mid^G$; the ith adjustment, for $1 \leq i \leq l$, is done as follows. (We deal with the extreme cases $i = 1$ and l similarly.) For notational convenience, we assume that the adjusted set G is still labeled as $\{s_0, \ldots, s_n\}$. Assume that f is increasing on $[d_{i-1}, d_i]$. We consider two cases.

Case 1: $G \cap [d_{i-1}, d_i] = \{s_{j-1}, s_j\}$.

- If $f(s_{j-2}) \geq f(s_{j-1})$, then we replace s_{j-1} with d_{j-1}. If $f(s_j) \geq f(s_{j+1})$, then s_j is replaced by d_i; otherwise, we remove s_j from G.
- If $f(s_{j-2}) < f(s_{j-1})$, then we remove s_{j-1} from G. Also, if $f(s_j) \geq f(s_{j+1})$, s_j is replaced by d_i; otherwise, we remove s_j from G.

Case 2: $G \cap [d_{i-1}, d_i] = \{s_j\}$.

- If $f(s_{j-1}) \geq f(s_j) \geq f(s_{j+1})$, then we add d_{i-1} and d_i to G, and remove s_j from it. If $f(s_j) < f(s_{j+1})$, then s_j is replaced by d_{i-1}.
- If $f(s_{j-1}) < f(s_j)$ and $f(s_j) > f(s_{j+1})$, then s_j is replaced by d_i, and if $f(s_j) < f(s_{j+1})$, then we remove s_j from G.

The case where f is decreasing can be handled similarly. □

Lemma 2.9. *Let $f_1(t) = t$ for $t \in [0, 1]$,*

$$f_{2^n+1} = \left(\frac{\alpha_1 + \alpha_2}{2}\right)^{\frac{1}{2}} \begin{cases} 2^{n+\frac{1}{2}} t & t \in [0, \frac{1}{2^{n+1}}] \\ -2^{n+\frac{1}{2}} t + \sqrt{2} & t \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \\ 0 & t \in [0,1] \setminus D_n^1 \end{cases},$$

for $n \geq 0$, and

$$f_{2^n+k} = \left(\frac{\alpha_2 + \alpha_3}{2}\right)^{\frac{1}{2}} f_{2^n+1}\left(t - \frac{k-1}{2^n}\right)$$

for $1 < k \leq 2^n$. Then (f_n) is a normalized, monotone Schauder basis for the space $\mathbb{F}_{\alpha,2}$.

Proof. Note that $supp(f_{2^n+k}) = D_n^k$. It is clear that $[(f_n)] \subseteq \mathbb{F}_{\alpha,2}$. So, we only need to prove the reverse inclusion.

Let f be a continuous, piecewise linear function which is linear on, say, m intervals $([t_{i-1}, t_i])_{i=1}^m$. Given $\epsilon > 0$, we can find a piecewise linear function g which is linear on m intervals $([c_{i-1}, c_i])_{i=1}^m$ with dyadic rational endpoints, and hence an element of $< (f_n) >$, with the property that $\|f - g\|_\infty < \frac{\varepsilon}{2\sqrt{2m-1}}$. Note that if $s, s' \in [0, 1]$, then this shows that

$$|(f-g)(s) - (f-g)(s')| < \frac{\varepsilon}{2\sqrt{2m-1}}. \qquad (1)$$

The function $f - g$ is continuous and piecewise linear, and the points where it changes monotonicity form a subset of $\cup_{i=0}^m (c_i, t_i)$.

By Proposition 4, there exists a finite, ordered sequence $G = \{s_0, < s_1 < \cdots < s_n\}$ in $[0, 1]$ such that $G \subseteq \cup_{i=0}^m (c_i, t_i)$ and $|f - g|^G = \|f - g\|$. Note that $n \leq 2m - 1$. As a result of this discussion and inequality (1), we conclude that $\|f - g\| \leq \epsilon$.

To show that (f_n) is a monotone basis for $\mathbb{F}_{\alpha,2}$, it is enough to prove that if $K \in \mathbb{N}$ and $(a_n)_{n=1}^{K+1} \subseteq \mathbb{R}$, then

$$\left\| \sum_{n=1}^{K} a_n f_n \right\| \leq \left\| \sum_{n=1}^{K+1} a_n f_n \right\|. \qquad (2)$$

Note that $\sum_{n=1}^{K} a_n f_n$ and $\sum_{n=1}^{K+1} a_n f_n$ are piecewise linear functions. Furthermore, if $\sum_{n=1}^{K} a_n f_n$ changes monotonicity at some $t \in [0,1]$, then $\sum_{n=1}^{K+1} a_n f_n$ changes monotonicity at that t as well. Thus by Proposition 4, (2) follows. □

Theorem 2.10. $\mathbb{F}_{\alpha,2} = \alpha BV_0^2$.

Proof. The proof that $\mathbb{F}_{\alpha,2} \subseteq \alpha BV_0^2$ is rather straightforward. Indeed, since αBV_0^2 is a Banach space, it is sufficient to show that the normalized monotone Schauder basis (f_n) of $\mathbb{F}_{\alpha,2}$ belongs to αBV_0^2. This is an elementary observation and we omit the details.

We will show that $\alpha BV_0^2 \subseteq \mathbb{F}_{\alpha,2}$. Let $\epsilon > 0$ and $f \in \alpha BV_0^2$ be given, and let $\epsilon_0 > 0$ be such that $14\epsilon_0^2 < \epsilon^2$. We choose $\delta > 0$ and $N \in \mathbb{N}$ such that $\sigma_2^\delta(f) < \epsilon_0$ and $\frac{3}{\delta N^2} < 1$. Since $\alpha BV_0^2 \subseteq C[0,1]$, $f \in C[0,1]$. Thus, we may choose $n \in \mathbb{N}$ in such a way that if $g \in \mathbb{F}_{\alpha,2}$ is linear on each D_n^k and $g(\frac{k}{2^n}) = f(\frac{k}{2^n})$ for $0 \leq k \leq 2^n$, then $\|f - g\|_\infty < \frac{\epsilon_0}{2N}$. We also require that $2^{-n} < \frac{\delta}{3}$.

Let $G = \{r_1 < s_1 \leq \cdots \leq r_m < s_m\}$ be a finite, ordered sequence of ordered pairs in $[0,1]$, determining the seminorm $|.|_G$. We partition G into three finite, ordered sequences G_1, G_2 and G_3 (so that $|.|_G^2 = |.|_{G_1}^2 + |.|_{G_2}^2 + |.|_{G_3}^2$) as follows. Let i be in $\{1, ..., m\}$.

1. If $s_i - r_i > \frac{\delta}{3}$, then we consider r_i and s_i as elements of G_1. Thus, $|f - g|_{G_1}^2 < \frac{3}{\delta}\left(\frac{\varepsilon_0}{N}\right)^2 < \epsilon_0^2$.
2. If $(r_i - s_i) \subseteq D_n^k$ for some k, then r_i and s_i are contained in G_2. It follows that
$$|g|_{G_2}^2 \leq \epsilon_0^2.$$
Also, $|f|_{G_2}^2 \leq \sigma_2^\delta(f)^2 < \epsilon_0^2$.
3. The remaining s_is and r_is are considered as the elements of G_3. Let $G_3 = \{0 \leq r_1' < s_1' \leq \cdots \leq r_t' < s_t' \leq 1\}$. Then, $s_i' - r_i' < \frac{\delta}{3}$ for $i = 1, ..., t$, and both s_i' and r_i' cannot be in the same dyadic interval D_n^K. Split G_3 into $G_{3,e}$ and $G_{3,o}$ by putting s_i' and r_i' in $G_{3,e}$ if i is even, and putting them in $G_{3,o}$ otherwise. Thus, for example, if $s_i', r_i' \in G_{3,e}$ and $s_i' \in D_n^k$, then $D_n^k \cap G_{3,e} = \{s_i'\}$. Applying the Intermediate Value Theorem to f and using the fact that $2^{-n} < \frac{\delta}{3}$, we can find (say, if t is even)
$$G_{3,e}'' = \{0 \leq r_1'' < s_1'' \leq \cdots \leq r_{\frac{t}{2}}'' < s_{\frac{t}{2}}'' \leq 1\}$$
with $|.|_{G_{3,e}''} \in S(\delta)$ and $f(r_i'') = g(r_{2i}')$, $f(s_i'') = f(s_{2i}')$ for $i = 0, ..., \frac{t}{2}$. Thus $|g|_{G_{3,e}}^2 = |f|_{G_{3,e}''}^2 < \varepsilon_0^2$. Similarly, $|g|_{G_{3,o}}^2 < \varepsilon_0^2$.

Hence,

$$\begin{aligned} |f-g|_G^2 &= |f-g|_{G_1}^2 + |f-g|_{G_2}^2 + |f-g|_{G_3}^2 \\ &< \epsilon_0^2 + \big(|f|_{G_2} + |g|_{G_2}\big)^2 + \big(|f|_{G_3} + |g|_{G_3}\big)^2 \\ &< \epsilon_0^2 + (2\epsilon_0)^2 + \Big(\epsilon_0 + \big(|g|_{G_{3,e}}^2 + |g|_{G_{3,o}}^2\big)^{\frac{1}{2}}\Big)^2 \\ &< \epsilon_0^2 + (2\epsilon_0)^2 + (\epsilon_0 + \sqrt{2}\epsilon_0)^2 \\ &< 14\epsilon_0 \\ &< \epsilon^2. \end{aligned}$$

This completes the proof. □

3. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

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