Journal of Mahani Mathematical Research Center
Print ISSN: 2251-7952 Online ISSN: 2645-4505

AN APPROACH TO CHANGE THE TOPOLOGY OF A TOPOLOGICAL SPACE WITH THE HELP OF ITS CLOSED SETS IN THE PRESENCE OF GRILLS

A. Talabeigi*

Dedicated to sincere professor Mehdi Radjabalipour on turning 75

Article type: Research Article
(Received: 25 January 2021, Revised: 16 February 2021, Accepted: 11 April 2021)
(Communicated by A. Salemi)

Abstract. The aim of this paper is introduce an approach to convert the topology of a topological space to another topology(in fact, a coarser topology). For this purpose, considering a closed set P of subsets of a topological space (X, τ) and a grill $\mathcal{G}$ on the space, we use the closure operator cl associated with τ, to define a new Kuratowski closure operator cl^*_P on X. The operator cl^*_P induces the desired topology. We then characterize the form of this resulting topology and also determine its relationship to the initial topology of the space. Some examples are given. Also, using a suitable grill in the method, we convert each topological space to corresponding D-space.

Keywords: Kuratowski closure operator, Kuratowski closure axioms, grill, D-topology, D-space.
2020 MSC: Primary 54A05, 54A10, 54C25.

1. Introduction

When geographic (spatial) data is modeled for use in GIS systems, we find that some of the modeled data must have spatial relationships with other data in the model. For example, taxi stations are located in specific locations and taxis are allowed to move in a certain area. These defined relationships can be presented in the form of topological laws. In fact, topological structures- e.g. topology, generalized topology, supra topology, m-topology, proximity spaces, closure spaces and etc- are models that can describe the geometric sharing of data and also provides a mechanism for establishing and maintaining spatial relationships between the data in the model. Now, if the range of taxi traffic expands or decreases, it changes the model and the relationships. So, application of topological structures in spatial geography can be very valuable, because the change in the role of points and places can be interpreted as a kind of change in the formed topological structure. Therefore, the issue of change in topological

*Corresponding author, ORCID: 0000-0001-5232-5008
e-mail: talabeigi@pnu.ac.ir
DOI: 10.22103/jmmrc.2021.17094.1132

How to cite: A. Talabeigi, *An approach to change the topology of a topological space with the help of its closed sets in the presence of grills*, J. Mahani Math. Res. Cent. 2021; 10(2): 9-19.

structures appears. Inspired by this issue, in this paper we intend to present a way to change the topology of a topological space.

It is noteworthy that for the first time, I(the author) was confronted with the concept of change in topology in the master course. On page 29 of [10], it is stated; “If $A \subseteq X$ and τ is any topology for X, then $\{U \cup (V \cap A) : U, V \in \tau\}$ is a topology for X. It is called the simple extension for τ over A.”. This was very interesting to me and caught my attention. Later I came across with a topic called “Closed Extension Topology” in [1], but the method did not appeal to me. I was constantly struggling with the issue of changing in topology until I finally realized that the best way to change the topology is to use its twin, i.e., Kuratowski closure operator.

In summary, the purpose of this paper is to present a method based on which we can turn the topology of a desired topological space into another topology. Indeed in this method, using the initial Kuratowski closure operator cl corresponding to the desired topological space (X, τ), we try to define a new Kuratowski closure operator cl^* to create a new topology on the set X.

2. Preliminaries

In this section, we will introduce some requirements related to the paper. Denote $\mathcal{P}(X)$ as the power set of X. Then cl as an operator on $\mathcal{P}(X)$ is called a Kuratowski closure operator, if it satisfies the following Kuratowski closure axioms;

(1) : $cl(\emptyset) = \emptyset$ (It preserves the empty set),
(2) : $A \subseteq cl(A)$ for any $A \subseteq X$ (It is extensive),
(3) : $cl(cl(A)) = cl(A)$ for any $A \subseteq X$ (It is idempotent),
(4) : $cl(A \cup B) = cl(A) \cup cl(B)$ for any $A, B \subseteq X$(It is additive).

It is well-known that topological spaces are characterized by Kuratowski closure operators and vice versa, that is, in fact associated with any topology τ on a set X is a Kuratowski closure operator on the set X, denoted cl_τ (in short, cl), which gives for any subset $A \subseteq X$, the smallest closed set clA containing A. Also, on the other hand, corresponding to any Kuratowski closure operator cl on a set X, there exists a unique topology, say, τ on the set X in the form of $\tau = \{X - A : cl(A) = A\}$, see [10].

As stated in the introduction, the purpose of this paper is to present a method based on the use of the Kuratowski closure operator to change the associated topology. It is worth mentioning that in the desired method, we will use a mathematical tool called grill, which was first introduced by Choquet [5] in 1947 as follows;

A non-empty collection $\mathcal{G}$ of non-empty subsets of X is called a grill on X if

(1) : $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$ and
(2) : $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

It is worth noting that grill like ideals, nets and filters are very useful tool. It is also seen that in many situations, grills are more productive and more flexible than these concepts. For instance, we can see its role in proximity spaces([9]), closure spaces ([3,4]) and the theory of extension of compactification([2,7]).

For a space (X, τ), the following collections are examples of grills on X;

- the collection of all uncountable subsets of X,
- the collection of all nowhere dense subsets of X,
- for $A \subseteq X$ the collection $\{B \subseteq X : B \cap A \neq \emptyset\}$ and especially for every point p of X the collection $\{A \subseteq X : p \in A\}$,
- and also, $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$.

We denote the family of all grills on X with the symbol of $\mathcal{G}(X)$ and note that the maximum element of $\mathcal{G}(X)$ is $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$.

Remark 2.1. Here we remind the reader that, the condition (1) in definition of grills requires $X \in \mathcal{G}$ for any $\mathcal{G} \in \mathcal{G}(X)$ and from condition (2) it follows that $X \setminus A \in \mathcal{G}$ when $A \notin \mathcal{G}$ for any $A \subseteq X$ and $\mathcal{G} \in \mathcal{G}(X)$.

Now, let us dedicate the end of this section to a brief description of the method.

Let (X, τ) be an arbitrary topological space, $P \in \mathcal{P}(X)$ and $\mathcal{G}$ be a grill on (X, τ). We define the operator cl^*_P on X based on the grill $\mathcal{G}$ and the closure operator cl associated with τ, as follows;

$$cl^*_P(A) = cl^{\mathcal{G}}_P(A) = \begin{cases} clA & clA \notin \mathcal{G} \\ clA \cup P & clA \in \mathcal{G} \end{cases} \tag{1}$$

where $A \in \mathcal{P}(X)$(we prefer to use the symbol cl^*_P instead of the symbol $cl^{\mathcal{G}}_P$ if there is no ambiguity about the grill $\mathcal{G}$).

Considering a suitable condition about the set P in the next section, it will be shown that the operator cl^*_P satisfies Kuratowski closure axioms, that is, the operator can constitute a topology, say, τ^*_P on X.

3. Main Results

As proposed in the previous section, this section intends to provide a method to construct another topology on a topological space. To that end, first the general construction of a new Kuratowski closure operator from the old one, in any topological space is presented.

Theorem 3.1. *Let (X, τ) be a topological space and P be a closed subset of X. Also, let $\mathcal{G}$ be a grill on (X, τ). Then the operator $cl^*_P : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by*

$$cl^*_P(A) = \begin{cases} clA & clA \notin \mathcal{G} \\ clA \cup P & clA \in \mathcal{G} \end{cases} \tag{2}$$

where A be any subset of X, is a Kuratowski closure operator so induces a topology τ^*_P on X.

Proof. From $cl(\emptyset) = \emptyset \notin \mathcal{G}$, we have $cl^*_P(\emptyset) = \emptyset$. Also, it is clear that $A \subseteq cl^*_P(A)$ for any $A \subseteq X$.

We now verify that for any $A, B \subseteq X$, $cl^*_P(A \cup B) = cl^*_P(A) \cup cl^*_P(B)$.
Let A and B are two subsets of X and consider two cases;

- ∘ (i): $cl(A \cup B) \notin \mathcal{G}$
- ∘ (ii): $cl(A \cup B) \in \mathcal{G}$

In case (i); we have

$$cl^*_P(A \cup B) = cl(A \cup B) = clA \cup clB$$

and also, we have $clA = cl^*_P A$ and $clB = cl^*_P B$, because $cl(A \cup B) \notin \mathcal{G}$ implies that $clA, clB \notin \mathcal{G}$. So, in this case

$$cl^*_P(A \cup B) = clA \cup clB = cl^*_P(A) \cup cl^*_P(B).$$

In case (ii); we have

$$cl^*_P(A \cup B) = cl(A \cup B) \cup P = clA \cup clB \cup P$$

also, we have $clA \cup clB \cup P = cl^*_P A \cup cl^*_P B$, because $cl(A \cup B) \in \mathcal{G}$ implies that $clA \in \mathcal{G}$ or $clB \in \mathcal{G}$. So, in this case

$$cl^*_P(A \cup B) = clA \cup clB \cup P = cl^*_P(A) \cup cl^*_P(B).$$

Hence, cl^*_P has the property of additive.

We next show that $cl^*_P(cl^*_P(A)) = cl^*_P(A)$, for any $A \subseteq X$.

Here, let A be a subset of X and consider two cases;

- ∘ (i): $clA \notin \mathcal{G}$
- ∘ (ii): $clA \in \mathcal{G}$

In case (i); we have

$$cl^*_P(cl^*_P(A)) = cl^*_P(clA)\,(from\ \ cl(clA) = clA \notin \mathcal{G}) = clA = cl^*_P(A),$$

while in case (ii); we have

$cl^*_P(cl^*_P(A)) = cl^*_P((clA) \cup P)$(due to the additive property of cl^*_P shown above)$= cl^*_P(clA) \cup cl^*_P(P) = ((clA) \cup P) \cup cl(P)$(due to closedness of P in hypothesis)$= (clA) \cup P = cl^*_P A$. So, cl^*_P has the property of idempotency.

It follows that cl^*_P is a Kuratowski closure operator on X which gives rise to a topology(say) τ^*_P on X. □

Remark 3.2. In the above theorem, if the set P is selected equal to the empty set then, $cl^*_P = cl^*_\emptyset = cl$, so $\tau^*_P = \tau^*_\emptyset = \tau$. Hence from here onwards we assume that P is a nonempty set.

To determine the general form of τ^*_P, we first characterize the operator int^*_P as dual of the topological closure operator cl^*_P in the sense of $int^*_P(A) = X \setminus cl^*_P(X \setminus A)$ and $cl^*_P(A) = X \setminus int^*_P(X \setminus A)$ for any $A \subseteq X$.

Theorem 3.3. *Let (X, τ) be a topological space and cl^*_P be the operator constructed in Theorem 3.1. Then for any subset A of X the interior operator int^*_P as dual of the operator cl^*_P has the form of*

$$(3) \quad int^*_P(A) = \begin{cases} intA & P \cap A = \emptyset \text{ or } P \cap A \neq \emptyset \text{ with } cl(X \setminus A) \notin \mathcal{G} \\ (intA) \setminus P & P \cap A \neq \emptyset \text{ with } cl(X \setminus A) \in \mathcal{G} \end{cases}$$

Proof. Let A be a subset of X and consider two cases;
(i): $P \cap A \neq \emptyset$,
(ii): $P \cap A = \emptyset$.
In case (i), considering $cl(X \setminus A) \notin \mathcal{G}$, we have $cl^*_P(X \setminus A) = cl(X \setminus A)$, so

$$int^*_P(A) = X \setminus cl^*_P(X \setminus A) = X \setminus cl(X \setminus A) = intA,$$

while considering $cl(X \setminus A) \in \mathcal{G}$ leads to $cl^*_P(X \setminus A) = cl(X \setminus A) \cup P$, and so

$$\begin{aligned} int^*_P A &= X \setminus cl^*_P(X \setminus A) = X \setminus [cl(X \setminus A) \cup P] \\ &= (X \setminus cl(X \setminus A)) \cap (X \setminus P) = (intA) \setminus P. \end{aligned}$$

In case (ii) we have; $P \subseteq X \setminus A$. Here, considering $cl(X \setminus A) \notin \mathcal{G}$ leads to $cl^*_P(X \setminus A) = cl(X \setminus A)$, so

$$int^*_P(A) = X \setminus cl^*_P(X \setminus A) = X \setminus cl(X \setminus A) = intA,$$

while considering $cl(X \setminus A) \in \mathcal{G}$ requires that we have $cl^*_P(X \setminus A) = cl(X \setminus A) \cup P(\text{as } P \subseteq X \setminus A) = cl(X \setminus A)$ and so

$$int^*_P(A) = X \setminus cl^*_P(X \setminus A) = X \setminus cl(X \setminus A) = intA.$$

Therefore, according to the above we have the following formula

$$(4) \qquad int^*_P(A) = \begin{cases} intA & P \cap A \neq \emptyset \text{ with } cl(X \setminus A) \notin \mathcal{G} \\ (intA) \setminus P & P \cap A \neq \emptyset \text{ with } cl(X \setminus A) \in \mathcal{G} \\ intA & P \cap A = \emptyset \end{cases}$$

□

By determining the set $\{A \subseteq X : int^*_P A = A\}$ as the fixed points of the operator int^*_P, the topology τ^*_P is determined.

Corollary 3.4. *The topology τ^*_P as the set of fixed points of the operator int^*_P on the space (X, τ), has the following form;*

$$(5) \qquad \tau^*_P = \tau^{\mathcal{G}}_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{G}\}.$$

Proof. To determine τ^*_P, we note that $A \in \tau^*_P$ if and only if $int^*_P A = A$. Considering $cl(X \setminus A) \notin \mathcal{G}$ in case (i) in the proof of the Theorem 3.3, leads to $int^*_P A = intA$, so $A \in \tau^*_P$ if and only if $A \in \tau$, while considering $cl(X \setminus A) \in \mathcal{G}$ in case (i), leads to $int^*_P A = (intA) \setminus P$, so $A \in \tau^*_P$ if and only if $A = (intA) \setminus P$ and this is impossile. Therefore, no subset A of X, which intersects P and is valid under the condition $cl(X \setminus A) \in \mathcal{G}$, can belong to τ^*_P.

In case (ii) of the proof of the Therorem 3.3, because $int^*_P A = intA$, hence $A \in \tau^*_P$ if and only if $A \in \tau$. So according to the above we have

$$\tau^*_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A = cl(X \setminus A) \notin \mathcal{G}\}.$$

□

Below we describe different forms of τ^*_P by considering some special cases for the set P and grill $\mathcal{G}$.

Remark 3.5. If we choose P as a closed subset of (X, τ), then P is closed in (X, τ^*_P), hence if $P \neq X$ then P is not dense in (X, τ^*_P).

Proof. According to Formula (5) in Corollary 3.4, clearly closedness of P in (X, τ), implies closedness of P in (X, τ^*_P), so if $P \neq X$ be a closed set of (X, τ), we will have $cl^*_P P = P \neq X$ which completes the proof. □

Example 3.6. By choosing $P = X$ in Corollary 3.4, we will have;

$$\begin{aligned} \tau^*_X &= \{A \in \tau : A \cap X = \emptyset\} \cup \{A \in \tau : A \cap X \neq \emptyset,\ X \setminus A \notin \mathcal{G}\} \\ &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : X \setminus A \notin \mathcal{G}\}. \end{aligned}$$

Remark 3.7. If we put $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$ in Example 3.6, then the result will be the trivial topology.

Definition 3.8. A topology ς for a set X is called a D-topology (and (X, ς) is called D-space) whenever every non-empty open set is dense in X.

Remark 3.9. If we put $\mathcal{G}^* = \{A \subseteq X : intclA \neq \emptyset\}$ in Example 3.6, then there will be a way to extract D-topology $D(\tau)$, from the original topology of any topological space (X, τ). Because, we have

$$\begin{aligned} \tau^*_X &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : X \setminus A \notin \mathcal{G}\} \\ &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : intcl(X \setminus A) = \emptyset\} \\ &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : int(X \setminus A) = \emptyset\} \\ &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : clA = X\} = D(\tau). \end{aligned}$$

In the following we will try to calculate the rule of the Kuratowski closure operator $cl^{\mathcal{G}^*}_P$ corresponding to the D-topology $D(\tau)$.

From Remark 3.9, we have;

$$\begin{aligned} D(\tau) &= \{\emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : clA = X\} \\ &= \{A \in \tau : A \cap X = \emptyset\} \cup \{A \in \tau \setminus \{\emptyset\} : int(X \setminus A) = intcl(X \setminus A) = \emptyset\} \\ &= \{A \in \tau : A \cap X = \emptyset\} \cup \{A \in \tau : A \cap X \neq \emptyset,\ intcl(X \setminus A) = \emptyset\} \\ &= \{A \in \tau : A \cap X = \emptyset\} \cup \{A \subseteq X : A \cap X \neq \emptyset,\ cl(X \setminus A) \notin \mathcal{G}^*\}. \end{aligned}$$

Now, according to Corollary 3.4 and Theorem 3.3, we have;

$$int^*_{P=X}(A) = \begin{cases} \emptyset & A = \emptyset \\ (intA) \setminus P & intcl(X \setminus A) \neq \emptyset \\ intA & intcl(X \setminus A) = \emptyset \end{cases} = \begin{cases} intA & clintA = X \\ \emptyset & otherwise \end{cases} \tag{6}$$

and correspondingly;

$$X \setminus int^*_{P=X}(A) = \begin{cases} X \setminus intA & clintA = X \\ X & otherwise \end{cases} \tag{7}$$

Thus

$$cl^*_{P=X}(X \setminus A) = \begin{cases} cl(X \setminus A) & intcl(X \setminus A) = \emptyset \\ X & otherwise \end{cases} \tag{8}$$

and so;

$$cl^*_{P=X}(A) = \begin{cases} clA & A \text{ is a nowhere dense set} \\ X & otherwise \end{cases} \tag{9}$$

In the following remark, the relationship between τ and τ^*_P has been determined.

Remark 3.10. In general, τ^*_P is coarser than τ, because $\tau = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset\}$ and clearly, $\{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{G}\} \subseteq \{A \in \tau : P \cap A \neq \emptyset\}$.

In the following example, we show that by choosing the appropriate grills, τ^*_P and τ can be matched in some topological spaces.

Example 3.11. Let τ denotes the cofinite topology on a(an infinite) set X and $\mathcal{G}$ be the grill of all infinte subsets of X, then

$$\begin{aligned} \tau^*_P &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{G}\} \\ &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \ is\ finite(so\ is\ closed)\} \\ &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset\} = \tau. \end{aligned}$$

Here, by selecting X as an uncountable set equipped with cocountable topology τ and grill $\mathcal{G} = \{A \subseteq X : A \text{ is an uncountable subset of } X\}$, we will have the same result again.

As we have seen, considering $\mathcal{G}_* = \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{G}^* = \{A \subseteq X : intclA \neq \emptyset\}$, as grills on (X, τ) in τ^*_P, we have;

$$\tau^{\mathcal{G}_*}_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{X\}$$

and

$$\tau^{\mathcal{G}^*}_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : clA = X\}.$$

So, as we see $\tau^{\mathcal{G}_*}_P \subseteq \tau^{\mathcal{G}^*}_P$. In this regard, we have the following proposition.

Proposition 3.12. *Let $\mathcal{I}$ and $\mathcal{J}$ be grills on a space (X, τ) such that $\mathcal{I} \subseteq \mathcal{J}$, then $\tau^{\mathcal{J}}_P \subseteq \tau^{\mathcal{I}}_P$.*

Proof. If $\mathcal{I} \subseteq \mathcal{J}$, then for any $B \subseteq X$, $B \notin \mathcal{J}$ implies that $B \notin \mathcal{I}$. Putting $B = X \setminus A$, we have $\{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{J}\} \subseteq \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{I}\}$ and therefore $\tau^{\mathcal{J}}_P \subseteq \tau^{\mathcal{I}}_P$. □

Based on the above proposition, we have the following corollary.

Corollary 3.13. *For any grill $\mathcal{G}$ defined on a topological space (X,τ), we have*

$$\{A \in \tau : A \cap P = \emptyset\} \cup \{X\} \subseteq \tau_P^{\mathcal{G}} \subseteq \tau.$$

Proof. Clearly, $\mathcal{P}(X) \setminus \{\emptyset\} \supseteq \mathcal{G}$ for any grill $\mathcal{G} \in \mathcal{G}(X)$. Now, from Corollary 3.4, Proposition 3.12 and Remarks 3.7 and 3.10 we get the result. □

Example 3.14. Let $X = \{a,b,c\}$ and also, let $\tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}\}$ and $\mathcal{G} = \{\{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ are respectively a topology and a grill on X. If $P_0 = \emptyset$, $P_1 = \{b\}$, $P_2 = \{a,b\}$, $P_3 = \{b,c\}$ and $P_4 = X$, then we have;

$$\tau^*_{P_0} = \tau^*_{P_1} = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}\} = \tau,$$
$$\tau^*_{P_2} = \{\emptyset, \{c\}, X\},\ \tau^*_{P_3} = \{\emptyset, \{a\}, X\},\ \tau^*_{P_4} = \{\emptyset, X\}.$$

Next example shows that using two different closed subsets in our method may lead to the same topology.

Example 3.15. Put $X = \{a,b,c,d\}$ and let $\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $\mathcal{G} = \{\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}\}$ be respectively, a topology and a grill on X. Then for closed sets $\{c\}$ and $\{d\}$ of (X,τ), we have

$$\{A \in \tau : \{c\} \cap A = \emptyset\} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,d\}\}$$

and

$$\{A \in \tau : \{d\} \cap A = \emptyset\} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\},$$

also, we have

$$\{A \in \tau : \{c\} \cap A \neq \emptyset, X \setminus A \notin \mathcal{G}\} = \{\{a,b,c\}, \{a,b,c,d\}\}$$

and

$$\{A \in \tau : \{d\} \cap A \neq \emptyset, X \setminus A \notin \mathcal{G}\} = \{\{a,b,d\}, \{a,b,c,d\}\}.$$

So we will have,

$$\tau^*_{\{c\}} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\} = \tau^*_{\{d\}}.$$

Therefore, it is possible to obtain the same topology for different closed sets.

Proposition 3.16. *The set $\{A \in \tau : A \cap P = \emptyset\}$ as a part of τ^*_P, is a topology on $X \setminus P$, in fact it is the subspace topology from τ on $X \setminus P$.*

Proof. Clearly $\emptyset$ is in $\{A \in \tau : A \cap P = \emptyset\}$ and $X \setminus P$(since we choose P as a closed set of τ) is in $\{A \in \tau : A \cap P = \emptyset\}$. Now, let A_1 and A_2 are in $\{A \in \tau : A \cap P = \emptyset\}$, then $A_1 \cap A_2 \in \tau$ and also, $P \cap (A_1 \cap A_2) = \emptyset$. So $A_1 \cap A_2 \in \{A \in \tau : A \cap P = \emptyset\}$.

Also if for an arbitrary indexing set K, $\{A_k : k \in K\}$ be a subcollection of $\{A \in \tau : A \cap P = \emptyset\}$, then $\forall k \in K$ we have $A_k \in \tau$ and $P \cap A_k = \emptyset$. So $\cup_{k\in K} A_k \in \tau$ and $\cup_{k\in K} A_k \cap P = \emptyset$, hence $\cup_{k\in K} A_k \in \{A \in \tau : A \cap P = \emptyset\}$. Thus, we show that the set $\{A \in \tau : A \cap P = \emptyset\}$ forms a topology on $X \setminus P$, but as $X \setminus P$

is an open set in (X, τ), then we infer that the topology $\{A \in \tau : A \cap P = \emptyset\}$ is the subspace topology from τ on $X \setminus P$. $\square$

Proposition 3.16 brings to mind the following lemma.

Lemma 3.17. *Let $(Y, \tau^{'})$ be a topological space with a grill $\mathcal{G}$ on it and P be any set with $P \cap Y = \emptyset$. Then the collection $\tau^{'} \cup \{A \cup P : A \in \tau^{'}, \, cl(Y \setminus A) \notin \mathcal{G}\}$ is a topology on X, where $X = Y \cup P$.*

Proof. Assuming $Y = X \setminus P$ in Proposition 3.16 we conclude that the set $\{A \in \tau : A \cap P = \emptyset\}$ as a part of τ_P^* is the same as the subspace topology $(\tau_P^*)_{|_Y}$ induced by τ_P^* on Y, that is, $\{A \in \tau : A \cap P = \emptyset\} = (\tau_P^*)_{|_Y} = \tau^{'}$(therefore, we can consider Y as an original topological space from the beginning). So, from Corollary 3.4 we can write

$$\tau_P^* = (\tau_P^*)_{|_Y} \cup \{A \in \tau : P \cap A \neq \emptyset, \, X \setminus A \notin \mathcal{G}\}.$$

So from the beginning, we can assume $(Y, \tau^{'})$ as the initial topological space.

We now check that $\tau^{'} \cup \{A \cup P : A \in \tau^{'}, \, cl(Y \setminus A) \notin \mathcal{G}\}$ is a topology on X. Let $\Gamma = \{A \cup P : A \in \tau^{'}, \; cl(Y \setminus A) \notin \mathcal{G}\}$ and $\Gamma^* = \tau^{'} \cup \Gamma$. Clearly $\emptyset \in \Gamma^*$. Assuming $A = Y$, we will have $A \in \tau^{'}$ and $cl(Y \setminus A) = \emptyset \notin \mathcal{G}$, so $X = A \cup P \in \Gamma \subseteq \Gamma^*$.

Now, let B_1 and B_2 are in Γ^*, we show that $B_1 \cap B_2$ is in Γ^*. Here if both B_1 and B_2 are in $\tau^{'}$, then clearly $B_1 \cap B_2$ is in $\tau^{'}$ and so $B_1 \cap B_2$ is in Γ^*. But if both B_1 and B_2 are in Γ, taking $B_1 = A_1 \cup P$ and $B_2 = A_2 \cup P$ for $A_1, A_2 \in \tau^{'}$ and $cl(Y \setminus A_1), cl(Y \setminus A_2) \notin \mathcal{G}$, we will have $B_1 \cap B_2 = (A_1 \cap A_2) \cup P \in \Gamma$, because $A_1 \cap A_2 \in \tau^{'}$ and $cl(Y \setminus (A_1 \cap A_2)) = cl(Y \setminus A_1) \cup cl(Y \setminus A_2) \notin \mathcal{G}$. So, here $B_1 \cap B_2 = (A_1 \cap A_2) \cup P \in \Gamma^*$.

Also for $B_1 \in \tau$ and $B_2 \in \Gamma$, by placing $B_1 = A_1$ and $B_2 = A_2 \cup P$ for $A_1, A_2 \in \tau$ we will have $B_1 \cap B_2 = A_1 \cap (A_2 \cup P) = A_1 \cap A_2 \in \tau \subseteq \Gamma^*$.

Also, for any $\{B_\alpha\}_{\alpha \in \Lambda} \subseteq \Gamma^*$, we show that $\cup_{\alpha \in \Lambda} B_\alpha \in \Gamma^*$.

(1) if $\{B_\alpha\}_{\alpha \in \Lambda} \subseteq \tau^{'}$ then clearly $\cup_{\alpha \in \Lambda} B_\alpha \in \tau^{'}$, so $\cup_{\alpha \in \Lambda} B_\alpha \in \Gamma^*$.

(2) if $\{B_\alpha\}_{\alpha \in \Lambda} \subseteq \Gamma$ then for each $\alpha \in \Lambda$ we put $B_\alpha = A_\alpha \cup P$ where $A_\alpha \in \tau^{'}$ and $cl(Y \setminus A_\alpha) \notin \mathcal{G}$. Then $\cup_{\alpha \in \Lambda} B_\alpha = (\cup_{\alpha \in \Lambda} A_\alpha) \cup P$ such that $\cup_{\alpha \in \Lambda} A_\alpha \in \tau^{'}$ and $cl(Y \setminus \cup_{\alpha \in \Lambda} A_\alpha) \subseteq cl(Y \setminus A_{\alpha_0}) \notin \mathcal{G}$ for some $\alpha_0 \in \Lambda$. Hence $\cup_{\alpha \in \Lambda} B_\alpha \in \Gamma \subseteq \Gamma^*$.

(3) if $\{B_\alpha\}_{\alpha \in \Lambda} = \{B_{\alpha_\lambda}\} \cup \{B_{\alpha_\gamma}\} \subseteq \Gamma^*$ where $\{B_{\alpha_\lambda}\} \subseteq \tau^{'}$ and $\{B_{\alpha_\gamma}\} \subseteq \Gamma$, then we have $\cup_{\alpha \in \Lambda} B_\alpha = (\cup\{B_{\alpha_\lambda}\}) \cup (\cup\{B_{\alpha_\gamma}\})$. Now, from part (1) $\cup B_{\alpha_\lambda} = A_1 \in \tau^{'}$ and from part (2) $\cup B_{\alpha_\gamma} = A_2 \cup P$ for some $A_2 \in \tau^{'}$ with $cl(Y \setminus A_2) \notin \mathcal{G}$. Thus $\cup_{\alpha \in \Lambda} B_\alpha = (A_1 \cup P) \cup A_2 = (A_1 \cup A_2) \cup P \in \Gamma \subseteq \Gamma^*$, because $A_1 \cup A_2 \in \tau$ and $cl(Y \setminus (A_1 \cup A_2)) \subseteq cl(Y \setminus A_2) \notin \mathcal{G}$.

$\square$

Definition 3.18. [6] Let X be a nonempty set and μ be a nonempty collection of subsets of X. μ is called a supratopology on X whenever $\emptyset, X \in \mu$ and $\{G_\alpha\}_{\alpha \in \Lambda} \subseteq \mu$ implies $\cup_{\alpha \in \Lambda} G_\alpha \in \mu$.

Following proposition introduces a way to extract some supratopologies from any topological space.

Proposition 3.19. *Let (X,τ) be a topological space, P be a closed subset of X and $\mathcal{G} \in \mathcal{G}(X)$. If we put $T = \{A \in \tau : P \cap A \neq \emptyset, X \setminus A \notin \mathcal{G}\}$, then*

(1) *: Each pair of elements of T has nonempty intersection.*
(2) *: $T \cup \{\emptyset\}$ is a supratopology on X.*

Proof. (1) : For any $A_1, A_2 \in T$, we have $A_1 \cap A_2 \neq \emptyset$. Because from $A_1, A_2 \in T$, we will have $X \setminus A_1, X \setminus A_2 \notin \mathcal{G}$ and therefore $X \setminus (A_1 \cap A_2) = (X \setminus A_1) \cup (X \setminus A_2) \notin \mathcal{G}$. Now the assumption $A_1 \cap A_2 = \emptyset$ gives $X \setminus (A_1 \cap A_2) = X \notin \mathcal{G}$ which is a contradiction.

(2) : Clearly $\emptyset \in T \cup \{\emptyset\}$ and also by putting $A = X$ we have, A is in τ such that $P \cap A \neq \emptyset$ and $X \setminus A = \emptyset \notin \mathcal{G}$, so we get $X \in T$.
On the other hand, if for an arbitrary indexing set K we put $\{A_k : k \in K\} \subseteq T$, then clearly $\cup_{k \in K} A_k$ is in τ such that $P \cap (\cup_{k \in K} A_k) \neq \emptyset$ and $X \setminus \cup_{k \in K} A_k = \cap_{k \in K}(X \setminus A_k) \subseteq X \setminus A_1 \notin \mathcal{G}$, so $\cup_{k \in K} A_k \in T$.

□

Definition 3.20. [2] A topological space (Y,μ) is called an extension of a space (X,τ) if (Y,μ) contains (X,τ) as a dense subspace. Also, if Y is an extension of X, then we call the subspace $Y \setminus X$ of Y, the remainder of Y.

According to the previous definition, we get next corollary from Proposition 3.16.

Corollary 3.21. *Let (X,τ) be a topological space. Then, each of spaces (X,τ_P^*) associted to each grill $\mathcal{G} \in \mathcal{G}(X)$ can be counted as an extension of the (sub)space $X \setminus P$. Moreover between them, (X,τ^*) for $\tau^* = \{A \in \tau : P \cap A = \emptyset\} \cup \{X\}$ is the smallest.*

Proof. In Corollary 3.4, we showed that $\tau_P^* = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{G}\}$ and it was also shown in Proposition 3.16 that the induced subspace topology $(\tau_P^*)_{|X \setminus P}$ from τ_P^* on $X \setminus P$ is equal to $\{A \in \tau : P \cap A = \emptyset\}$. Note $X \setminus P$ is dense in (X,τ_P^*). So by definition 3.20, the space (X,τ_P^*) is an extension of the space $(X \setminus P,\ (\tau_P^*)_{|X \setminus P})$. According to Corollary 3.13, the proof of the expression $\tau^* = \{A \in \tau : P \cap A = \emptyset\} \cup \{X\}$ is the smallest one, is straightforward. □

As the final point, it is appropriate to say that replacing the role of grills with stacks in the method of this paper, instead of changing the topology of a space leads to extract supra topologies from that topology, for this purpose see [8].

4. Aknowledgement

The author is grateful to the referee(s) for certain suggestions towards improving the manuscript.

References

[1] V. M. Babych,V. O. Pyekhtyeryev, Closed extension topology, Scientific Bulletin of Uzhhorod University: Series of Mathematics and Informatics, Vol. 2, (2015), 7-10.
[2] N. Broojerdian, A. Talabeigi, One-point λ-compactification via grills, Iran J Sci Technol Trans Sci., Vol. 41, (2017), 909-912.
[3] K. C. Chattopadhyay, W. J. Thron, Extensions of closure spaces, Can. J. Math., Vol. 29, (1977), 1277-1286.
[4] K. C. Chattopadhyay, O. Njastad and W. J. Thron, Merotopic spaces and extensions of closure spaces, Can. J. Math., Vol. 35, (1983), 613-629.
[5] G. Choquet, Sur les notions de filtre et grille, Comptes Rendus Acad. Sci. Paris, Vol. 224, (1947), 171-173.
[6] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supratopological spaces, Indian J. Pure Appl. Math., Vol. 14, (1983), 502-510.
[7] A. Talabeigi, On the Tychonoff's type theorem via grills, Bull. Iranian Math. Soc., Vol. 42, No. 1 (2016), 37-41.
[8] A. Talabeigi, Extracting some supra topologies from the topology of a topological space using stacks(Accepted in AUT Journal of Mathematics and Computing (AJMC) and to appear). doi: 10.22060/ajmc.2021.19123.1042
[9] W. J. Thron, Proximity structures and grills, Math. Ann., Vol. 206, (1973), 35-62.
[10] S. Willard, General topology, AddisonWesley publishing company, 1970.

Amin Talabeigi
Orcid number: 0000-0001-5232-5008
Department of Mathematics
Payame Noor University(PNU)
P.O.Box 19395-4697, Tehran, Iran.
E-mail address: talabeigi@pnu.ac.ir, talabeigi.amin@gmail.com