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ON THE GTSOR-LIKE METHOD FOR THE AUGMENTED SYSTEMS

H. Nasabzadeh*

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Abstract. In this paper, by using SOR-Like method that introduced by Golub, Wu and Yuan and generalized Taylor expansion method for solving linear systems [F. Toutounian, H. Nasabzadeh, A new method based on the generalized Taylor expansion for computing a series solution of linear systems, Appl. Math. Comput. 248 (2014) 602-609], the GTSOR-Like method is proposed for augmented systems. The convergence analysis and the choice of the parameters of the new method are discussed. While there is no guarantee the SOR-Like method converges for the negative parameter, ω additional parameters of the new method can be adjusted for the corresponding GTSOR-Like method to converge. Finally, numerical examples are given to show that the new method is much more efficient than the SOR-Like method.

Keywords: Linear system, SOR-Like method, Taylor expansion, Augmented systems.
2020 MSC: 65f10.

1. Introduction

Consider the following linear system of equations with 2×2 block structure,

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a matrix of full column rank, m and n with $m \geq n$ are two positive integers, $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ are two given vectors. B^T denotes the transpose of the matrix B. These system of linear equations are called saddle point problems, which arise in many scientific computing and engineering applications such as optimization [27, 28], Stokes equations and Maxwell equation, computational fluid dynamics [2, 8, 15, 16], weighted least squares problems [1, 30], optimal control [9] and absolute value equations [13, 18, 21] and so on.
the system of linear equation (1) is also termed as a Karush-Kuhn-Tucker (KKT) system, or an augmented system or an equilibrium system. When the matrix blocks A and B are large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problem (1).

*ORCID: 0000-0002-8163-7569
E-mail: h.nasabzadeh@ub.ac.ir
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Many efficient iterative methods, for example Uzawa-type methods [3,7,11,14], HSS iteration methods [4,19], Preconditioned Krylov subspace iteration methods [10, 20], restrictively preconditioned conjugate gradient methods [5, 29], have been proposed. Golub et al [17], proposed the SOR-like method and considered the optimum choice for iteration parameters, Li et al [22] proposed the GAOR method, also Bai et al [6] proposed the GSOR method on the basis of the SOR-like method. The SSOR-Like methods presented by the symmetric processing technique [12,23,25,26].
In this paper, based on the generalized Taylor expansion for solving the linear systems that introduced by F. Toutounian and H. Nasabzadeh [24], and by the SOR-like method for saddle point problem (1), we present a new method. We call the new method, GTSOR-like method. By choosing suitable parameters the new method is faster than the corresponding SOR-Like method, also from Theorem 2.2 it is clear that there is no guarante that the SOR-Like method converges for the negative parameter, ω, while we show that additional parameters of the corresponding GTSOR-Like method can be adjusted to converge. The numerical experiments show that GTSOR-Like method work quit well.
The outline of this paper is as follow. In Section2, we introduce the GTSOR-Like method, in Section3 we give some Theorems and Lemmas to provide the convergence property of the new method and choices of the parameters are discussed. In Section4, numerical examples are given to show that the new method is much more efficient that the SOR-Like method. Finally, we make some conclusions and outlook in Section5.

2. The GTSOR-like Method

In this section, we introduce our new iteration method, for this purpose, first we describe the SOR-Like method that introduced by Golub, Wu and Yuan [17].

Method 1. (The SOR-like method)
Let $Q \in \mathbb{R}^{n\times n}$ be a nonsingular and symmetric matrix. Given initial vectors $x^{(0)} \in \mathbb{R}^m$ and $y^{(0)} \in \mathbb{R}^n$ and a relaxation factor $\omega > 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{({x^{(k)}}^T, {y^{(k)}}^T)\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(p - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \omega Q^{-1}(B^T x^{(k+1)} - q). \end{cases}$$

Here, Q is an approximate (preconditioning) matrix of the schur complement matrix $B^T A^{-1} B$.

We can see [17], that

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = M_\omega \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \omega(D - \omega L)^{-1} \begin{pmatrix} p \\ -q \end{pmatrix} \tag{2}$$

where

$$M_\omega = \begin{pmatrix} A & 0 \\ -\omega B^T & Q \end{pmatrix}^{-1} \begin{pmatrix} (1-\omega)A & -\omega B \\ 0 & Q \end{pmatrix}$$

(3)
$$= \begin{pmatrix} A^{-1} & 0 \\ \omega Q^{-1}B^T A^{-1} & Q^{-1} \end{pmatrix} \begin{pmatrix} (1-\omega)A & -\omega B \\ 0 & Q \end{pmatrix}$$

$$= \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B \\ \omega(1-\omega)Q^{-1}B^T & -\omega^2 Q^{-1}B^T A^{-1}B + I \end{pmatrix}$$

and

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix},\ L = \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix},\ U = \begin{pmatrix} 0 & -B \\ 0 & Q \end{pmatrix},$$

so

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \equiv D - L - U.$$

Lemma 2.1. *[17].*
Suppose that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$. If λ satisfies

(4)
$$(\lambda - 1)(1 - \omega - \lambda) = \lambda\omega^2\mu,$$

then λ is an eigenvalue of M_ω. Conversely, if λ is an eigenvalue of M_ω such that $\lambda \neq 1$ and $\lambda \neq 1-\omega$, and μ satisfies (4), then μ is a nonzero eigenvalue of $Q^{-1}B^TA^{-1}B$.

Theorem 2.2. *[17].*
suppose that B has full rank and A is symmetric and positive definite. Assume that all eigenvalues μ_i of $Q^{-1}B^TA^{-1}B$ are real, then if $\mu_i > 0,\ i = 1, 2, ..., n$. The SOR-like method is convergent for all ω such that

(5)
$$0 < \omega < \frac{4}{1 + \sqrt{4\mu_{max} + 1}}$$

where $\mu_{max} = \max_{i=1}^{n}(\mu_i)$.

Now, put

(6)
$$G = M_\omega = \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B \\ \omega(1-\omega)Q^{-1}B^T & -\omega^2 Q^{-1}B^T A^{-1}B + I, \end{pmatrix}$$

taking

(7)
$$G_{\alpha,\hbar} = \frac{\hbar G - \alpha(\hbar + 1)I}{\hbar - \alpha(\hbar + 1)},$$

where α and $\hbar$ are real parameters, see [24], so

(8)
$$G_{\alpha,\hbar} = \frac{1}{\hbar - \alpha(\hbar+1)} \begin{pmatrix} (\hbar(1-\omega) - \alpha(\hbar+1))I & -\omega\hbar A^{-1}B \\ \omega\hbar(1-\omega)Q^{-1}B^T & -\omega^2\hbar Q^{-1}B^TA^{-1}B + (\hbar - \alpha(\hbar+1))I \end{pmatrix}.$$

Let $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ be an initial approximation to the exact solution $\begin{pmatrix} u \\ v \end{pmatrix}$ of the system (1), put:

$$\begin{aligned} &\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \frac{-\hbar}{\hbar - \alpha(\hbar+1)}[(I-G)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \omega(D-\omega L)^{-1}\begin{pmatrix} p \\ -q \end{pmatrix}] \\ &\begin{pmatrix} u_i \\ v_i \end{pmatrix} = G_{\alpha,\hbar}\begin{pmatrix} u_{i-1} \\ v_{i-1} \end{pmatrix},\ i = 2,3,..., \end{aligned} \tag{9}$$

then, $\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{i=0}^{\infty}\begin{pmatrix} u_i \\ v_i \end{pmatrix}$.

Method 2. (The GTSOR-like method) Let $Q \in \mathbb{R}^{n\times n}$ be a nonsingular and symmetric matrix. Given initial vectors $u_0 \in \mathbb{R}^m$ and $v_0 \in \mathbb{R}^n$ and $\alpha, \hbar \in \mathbb{R}$ where, $\hbar - \alpha(\hbar+1) \neq 0$ and a relaxation factor $\omega \neq 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\sum_{j=0}^{k}(u_j^T, v_j^T)$ is convergent, compute

$$\begin{aligned} u_1 &= \frac{-\omega\hbar}{\hbar-\alpha(\hbar+1)}[u_0 + A^{-1}(Bv_0 - p)], \\ v_1 &= \omega Q^{-1}B^T u_1 + \frac{\omega\hbar}{\hbar-\alpha(\hbar+1)}Q^{-1}[B^T u_0 - q], \\ u_i &= \frac{1}{\hbar-\alpha(\hbar+1)}[(\hbar(1-\omega)-\alpha(\hbar+1))u_{i-1} - \omega\hbar A^{-1}Bv_{i-1}], \\ v_i &= \omega Q^{-1}B^T u_i + v_{i-1} + \frac{\omega\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}Q^{-1}B^T u_{i-1}, \quad i = 2,3,4,.... \end{aligned} \tag{10}$$

3. Convergece analysis of GTSOR-Like method

Here, we give some Theorems and Lemmas to provide the convergence property of the GTSOR-Like method.

Lemma 3.1. *Suppose that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real and positive then:*

(1) *G has not real simple eigenvalue if :*

$$\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) < \omega < \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}). \tag{11}$$

(2) *G has not complex eigenvalue if :*

$$\omega \in (-\infty, \min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i})] \cup [\max_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}), +\infty). \tag{12}$$

Proof. From Lemma 2.1 the eigenvalues of G satisfies (4) or equivalently

$$\lambda^2 + (\omega^2\mu + \omega - 2)\lambda + 1 - \omega = 0. \tag{13}$$

We can easily see if:

$$\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) < \omega < \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}),$$

then

$$\triangle = (\omega^2\mu + \omega - 2)^2 - 4(1-\omega) \leq 0.$$

So the roots of equation (13) are not real simple. In similar way, also we can see if

$$\omega \leq \min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) \text{ or } \omega \geq \max_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}),$$

then $\triangle \geq 0$ so the roots of equation (13) are real. □

Remark 3.2. The function $f_1(\mu) = \dfrac{-2\sqrt{\mu}-1}{\mu}$ for $\mu > 0$ is an increasing function, so

$$\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) = \frac{-2\sqrt{\mu_{max}}-1}{\mu_{max}}$$

and

$$\min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) = \frac{-2\sqrt{\mu_{min}}-1}{\mu_{min}}$$

where $\mu_{max} = \max_i(\mu_i)$ and $\mu_{min} = \min_i(\mu_i)$.

Remark 3.3. The function $f_2(\mu) = \dfrac{2\sqrt{\mu}-1}{\mu}$ has the following properties:

(i) It has only one root, $\mu = \dfrac{1}{4}$.
(ii) Its maximum point is $(1, 1)$.
(iii) $\lim_{\mu \longrightarrow +\infty} f_2(\mu) = 0$.
(iv) $\lim_{\mu \longrightarrow 0^+} f_2(\mu) = -\infty$.

Corollary 3.4. *If $\mu_i \geq \frac{1}{4}$ for $i = 1, 2, ..., n$ then the interval*

$$(\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}), \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}))$$

is non-empty.

Now, let $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$, $i = 1, 2, ..., m+n$ are the eigenvalues of G and $Re_{min}(\lambda_i) = \min_{i=1}^{n+m}(Re(\lambda_i))$ and $Re_{max}(\lambda_i) = \max_{i=1}^{n+m}(Re(\lambda_i))$, so

Lemma 3.5. *Suppose that all eigenvalues μ_i of $Q^{-1}B^TA^{-1}B$ are real and positive.*
If $\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) < \omega < \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i})$ then:

(1) $Re_{max}(\lambda_i) < 1$ *if*

$$\omega \in \mathcal{K} = ((-\infty, -\frac{1}{\mu_{min}}) \cup (0, +\infty)) \cap (\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}), \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}))$$

(2) $Re_{max}(\lambda_i) > 1$ *if*

$$\frac{-1}{\mu_{max}} < \omega < 0$$

Proof. Since $\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}) < \omega < \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i})$, from Lemma 3.1, G has not real simple eigenvalue so by (13), we have

$$Re(\lambda_i) = -\frac{1}{2}(\omega^2\mu_i + \omega - 2).$$

By simple computation, we can show that if
$\omega \in \mathcal{K} = ((-\infty, -\frac{1}{\mu_{min}}) \cup (0, +\infty)) \cap [\max_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i}), \min_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i})]$,
then $Re_{max}(\lambda_i) < 1$ and if $\frac{-1}{\mu_{max}} < \omega < 0$ then $Re_{max}(\lambda_i) > 1$. □

Lemma 3.6. *Suppose that all eigenvalues μ_i of $Q^{-1}B^TA^{-1}B$ are real and positive.*
If $\omega \in (-\infty, \min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i})] \cup [\max_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}), +\infty)$, then $Re_{max}(\lambda_i) < 1$.

Proof. Clearly, if $\omega \in (-\infty, \min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i})] \cup [\max_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}), +\infty)$. then from Lemma 3.1, G has not complex eigenvalue, so $Re(\lambda_i) = \lambda_i$. From (13), with few computation we can see that

$$\pm\sqrt{\omega^4\mu_i^2 + 2\omega^2(\omega-2)\mu_i} < \omega^2\mu_i + \omega, \ i = 1, 2, ..., n+m$$

and this leads to $\lambda_i < 1$ for $i = 1, 2, ..., n+m$ so $Re_{max}(\lambda_i) < 1$. □

Theorem 3.7. *Suppose that all eigenvalues μ_i of $Q^{-1}B^TA^{-1}B$ are real and positive and let $\lambda_i = Re(\lambda_i) + iIm(\lambda_i)$, $i = 1, 2, ..., m+n$ are the eigenvalues of G and $Re_{min}(\lambda_i) = \min_{i=1}^{n+m}(Re(\lambda_i))$ and $Re_{max}(\lambda_i) = \max_{i=1}^{n+m}(Re(\lambda_i))$ and let $\theta_i = \frac{|\lambda_i|^2-1}{Re(\lambda_i)-1}$, $i = 1, 2, \ldots, n+m$, and $\theta_{min} = \min_{i=1}^{n+m}(\theta_i)$ and $\theta_{max} = \max_{i=1}^{n+m}(\theta_i)$. Suppose that $\alpha \neq \frac{Re_{min}(\lambda_i) + Re_{max}(\lambda_i)}{2}$. Let*

$$\mathcal{L} = \mathcal{K} \cup (-\infty, \min_{\mu_i}(\frac{-2\sqrt{\mu_i}-1}{\mu_i})] \cup [\max_{\mu_i}(\frac{2\sqrt{\mu_i}-1}{\mu_i}), +\infty).$$

Then the GTSOR-like method converges if the parameters α, $\hbar$ and ω take any values from their domains, as these are defined and given in the Table 1.

Proof. From Theorem 4.3 in [24] and lemmas (3.5) and (3.6) it is trival. □

Table 1: The possible domains of the parameters ω, α and $\hbar$

case	ω- domain	α- domain	$\hbar$- domain
1	$\frac{-1}{\mu_{max}} < \omega < 0$	$\frac{\theta_{max}}{2} < \alpha$	$(-\infty, \frac{2\alpha}{\theta_{max} - 2\alpha}) \cup (0, +\infty)$
2	$\frac{-1}{\mu_{max}} < \omega < 0$	$0 < \alpha < \frac{\theta_{min}}{2}$	$(0, \frac{2\alpha}{\theta_{max} - 2\alpha})$
3	$\frac{-1}{\mu_{max}} < \omega < 0$	$\alpha < 0$	$(\frac{2\alpha}{\theta_{max} - 2\alpha}, 0)$
4	$\omega \in \mathcal{L}$	$0 < \alpha$ and $\frac{\theta_{max}}{2} < \alpha$	$(\frac{2\alpha}{\theta_{min} - 2\alpha}, 0)$
5	$\omega \in \mathcal{L}$	$\frac{\theta_{max}}{2} < \alpha < 0$	$(0, \frac{2\alpha}{\theta_{min} - 2\alpha})$
6	$\omega \in \mathcal{L}$	$0 < \alpha < \frac{\theta_{min}}{2}$	$(-\infty, 0) \cup (\frac{2\alpha}{\theta_{min} - 2\alpha}, +\infty)$
7	$\omega \in \mathcal{L}$	$\alpha < \frac{\theta_{min}}{2}$ and $\alpha < 0$	$(-\infty, \frac{2\alpha}{\theta_{min} - 2\alpha}) \cup (0, +\infty)$

4. **Numerical Example**

In this section we give an example to compare the SOR-Like method and the GTSOR-Like method.

Example 4.1. *[10] Consider the augmented system (1) in which*

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2} \text{ and } \mathrm{B} = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2\mathrm{p}^2 \times \mathrm{p}^2}$$

and $T = \frac{1}{h^2}.tridiag(-1, 2, -1) \in \mathbb{R}^{p \times p}$, $F = \frac{1}{h}.tridiag(-1, 1, 0) \in \mathbb{R}^{p \times p}$ *with* $\otimes$ *being the kronecker product symbol and* $h = \frac{1}{p+1}$ *being the mesh size.*

Here, we set $m = 2p^2$ and $n = p^2$. We choose the matrix Q the identity matrix (as an approximation of the matrix $B^T A^{-1} B$).
In our computations, the initial guesses are set with the zero vector and terminated if the current iteration satisfy $EER < 10^{-9}$, where

$$EER = \frac{\sqrt{\| x^{(k)} - x^* \|_2^2 + \| y^{(k)} - y^* \|_2^2}}{\sqrt{\| x^{(0)} - x^* \|_2^2 + \| y^{(0)} - y^* \|_2^2}}$$

also, we choose the right hand side vector $(p^T, -q^T)^T \in \mathbb{R}^{m+n}$, such that the exact solution of (1) is $((x^*)^T, (y^*)^T)^T = (1, 1, ..., 1)^T \in \mathbb{R}^{m+n}$.
All the computations results are show in MATLAB R2015a and performed on a PC with Intel(R) Core(TM) i3-2330M Processor/ 2.20 GHz and RAM 4GBz.

In Tables 2-4, we list the values ω and $\hbar$ and we report the numerical results, IT (the iteration step), CPU (the CPU time in seconds)and ERR, when $p = 8$, $p = 16$ and $p = 24$.

Table 2: $p = 8$ and $\alpha = 1$

Method	ω	$\hbar$	IT	ERR	CPU
SOR-Like	1.0585	–	127	$8.8776e-10$	0.251176
GTSOR-Like	1.0585	-1.1	114	$9.3871e-10$	0.197615
GTSOR-Like	1.0585	-1.3	95	$8.2062e-10$	0.171492
SOR-Like	1.2	–	185	$8.8994e-10$	0.322125
GTSOR-Like	1.2	-0.9	123	$8.5986e-10$	0.239337
GTSOR-Like	1.2	-0.8	139	$9.8915e-10$	0.252605

Tables 2-4 indicate that the GTSOR-Like method is much more effective than the SOR-Like method, since the GTSOR-Like requires much less iteration steps and CPU times than the SOR-Like method. Table 5 show that the SOR-Like method diverges for negative ω while the GTSOR-Like converges.

Table 3: $p = 16$ and $\alpha = 1$

Method	ω	$\hbar$	IT	ERR	CPU
SOR-Like	1.03	–	232	$9.4364e-10$	10.126101
GTSOR-Like	1.03	-1.2	191	$9.6420e-10$	8.321967
GTSOR-Like	1.03	-1.5	155	$7.7704e-10$	6.763611
SOR-Like	0.731	–	331	$9.7599e-10$	14.101262
GTSOR-Like	0.731	-1.6	202	$9.7255e-10$	8.788048
GTSOR-Like	0.731	-2.1	154	$9.4438e-10$	6.838050

Table 4: $p = 24$ and $\alpha = 1$

Method	ω	$\hbar$	IT	ERR	CPU
SOR-Like	0.731	–	475	$9.5808e-10$	238.767499
GTSOR-Like	0.731	-1.8	258	$9.6540e-10$	129.184637
GTSOR-Like	0.731	-2.1	219	$9.8753e-10$	110.570769

Table 5: $p = 8$ and $\alpha = -1$

Method	ω	$\hbar$	IT	ERR	CPU
SOR-Like	-0.5	–	–	–	–
GTSOR-Like	-0.5	-0.256	753	$8.9664e-10$	1.266614

5. Conclusions

In this paper, by using SOR-Like method [17] and generalized Taylor expansion method for solving linear systems [24], the GTSOR-Like method is introduced for augmented systems. The convergence property of the new method is derived and choices of the parameters are discussed in detail. Numerical results verified the effectiveness of the proposed method.
However, the proposed method involves three iteration parameters, ω, $\hbar$ and α, the choice of the optimal parameters was not discussed in this work, how to determine these three optimal parameters should be a direction for future research.

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Hamideh Nasabzadeh
Orcid number: 0000-0002-8163-7569
Department of Mathematics, Faculty of Basic Sciences
University of Bojnord, P. O. Box 9453155111
Bojnord, Iran
E-mail address: h.nasabzadeh@ub.ac.ir