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PERRON-FROBENIUS THEORY ON THE HIGHER-RANK NUMERICAL RANGE FOR SOME CLASSES OF REAL MATRICES

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Dedicated to sincere professor Mehdi Radjabalipour on turning 75

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Abstract. We present an extension of Perron-Frobenius theory to the higher-rank numerical range of real matrices. We define a new type of the rank-k numerical radius for real matrices, i.e., the *sign-real rank-k numerical radius,* and derive some properties of it. In addition, we extend Issos' results on the higher-rank numerical range of nonnegative matrices to real matrices. Finally, we give some examples that are used to illustrate our theoretical results.

Keywords: Sign-real rank-k numerical radius, Sign-real spectral radius, Perron–Frobenius theory, Higher-rank numerical range.
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1. Introduction

For nonnegative matrices A, the well known Perron–Frobenius theory studies the spectral radius $\rho(A)$. Perron–Frobenius theory has numerous applications in many branches of mathematics, various fields of science and technology [2,11,13,18]. The main results of the Perron–Frobenius theory asserts that if a nonnegative irreducible matrix has h maximal characteristic roots, then these roots are equally spaced around a circle with center at the origin and one of the roots locates on the positive real axis [6]. In [15] a new quantity for real matrices, the sign-real spectral radius, is defined, which is a generalization of this theory. Let $M_n(\mathbb{R})$ be the set of $n \times n$ real matrices. For $A \in M_n(\mathbb{R})$, the real spectral radius of A is defined by $\rho_0(A) = \max\{|\lambda| : \lambda \textit{ a real eigenvalue of } A\}$. Note that $\rho_0(A) := 0$ if A has no real eigenvalues. A signature matrix is a diagonal matrix with diagonal entries $+1$ or -1. Note that there are 2^n signature matrices of dimension n. Let φ denote the set of signature matrices. The sign-real spectral radius of a real matrix A is defined by

$$\rho^{\mathbb{R}}(A) = \max_{S\in\varphi} \rho_0(SA).$$

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It has also been applied to engineering problems (see, for example, [16, 17] and the references therein).

In [4], Choi et al. introduced the notion of the rank-k numerical range of $A \in M_n$ defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} \ : \ PAP = \lambda P \ \textit{for some rank-k orthogonal projection} \ \ P \ \},$$

where $1 \leq k < n$ is a positive integer, and rank-k orthogonal projection P is an orthogonal projection of $\mathbb{C}^n$ onto any k-dimensional subspace $\mathcal{K}$ of $\mathbb{C}^n$. Equivalently,

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} \ : \ X^*AX = \lambda I_k \ \textit{for some} \ X \in M_{n,k}(\mathbb{C}) \ \textit{with} \ X^*X = I_k\}.$$

When $k = 1$, then $\Lambda_k(A)$ reduces to the classical numerical range defined and denoted by

$$\Lambda_1(A) = W(A) = \{x^*Ax : \ x \in \mathbb{C}^n, \ x^*x = 1\},$$

which is a useful concept in studying matrices and operators; see [8, Chapter 1]. It is clear that

$$W(A) = \Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_k(A). \tag{1}$$

For $k > 1$, the sets $\Lambda_k(A)$ are generally called higher-rank numerical range and used in the study of quantum error correction; see [5]. The rank-k numerical radius of $A \in M_n$ is defined as

$$\omega_k(A) = \max\{|\alpha| \ : \ \alpha \in \Lambda_k(A)\}.$$

By convention [12], $\omega_k(A) = -\infty$ if $\Lambda_k(A) = \emptyset$. Apparently, for $k = 1$, $\omega_k(A)$ yields the classical numerical radius of a matrix A, i.e., $\omega_1(A) = \omega(A) = \max\{|z| : z \in W(A)\}$.

The Perron-Frobenius theory has been extended to the numerical range of a nonnegative matrix by Issos in his unpublished Ph.D. thesis [9]. The main result of Issos [9, Theorem 7] asserts that for a nonnegative irreducible matrix A, the set of points of $W(A)$, with modulus $\omega(A)$ consists precisely of $\omega(A)$ times all the h-th roots of unity. The main result of Issos for real matrices is given in [19, 20]. In [1], Aretaki et al. extended some results of Issos on the higher-rank numerical range of a nonnegative irreducible matrix A. In the course of proving the main theorem of Issos on the higher-rank numerical range of a real matrix A, the *sign-real rank-k numerical radius* occurs.

Definition 1.1. For $A \in M_n(\mathbb{R})$ the sign-real rank-k numerical radius is defined by

$$\omega_k^{\mathbb{R}}(A) = \max_{S \in \varphi} \omega_k^0(SA),$$

where $\omega_k^0(A) := \max\{\,|z| \ : \ z \in \Lambda_k(A) \cap \mathbb{R}\,\}$. By convention [12], $\omega_k^{\mathbb{R}}(A) = -\infty$ if $\Lambda_k(SA) = \emptyset$ for some $S \in \varphi$. When $k = 1$, we call $\omega_k^{\mathbb{R}}(A)$, the sign-real numerical radius [19].

Since the literature on higher-rank numerical range analogs of the Perron-Frobenius theory is scanty, we also think it is worthwhile to offer a general results for real matrices here. In Section 3, we obtain some properties of the sign-real rank-k numerical radius. In Section 4, we show that under some conditions on the matrix A, the set of points of $\Lambda_k(SA)$, with modulus $\omega_k^{\mathbb{R}}(A)$ for some $S \in \varphi$, consists precisely of $\omega_k^{\mathbb{R}}(A)$ times all the h-th roots of unity.

2. Preliminaries

We always use $A = (a_{ij}) \in M_n(\mathbb{R})$ to denote an $n \times n$ real matrix. The following notation will be adopted:

$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$;
$\sigma(A)$	the spectrum of A;
$\rho(A)$	the spectral radius of A;
$\rho^{\mathbb{R}}(A)$	the sign-real spectral radius of real matrix A;
$W(A)$	the (classical) numerical range of A;
$\omega(A)$	the numerical radius of A;
$\omega_1^{\mathbb{R}}(A)$	the sign-real numerical radius of real matrix A;
$\Lambda_k(A)$	the higher-rank numerical range of A;
$\omega_k(A)$	the rank-k numerical radius of A;
$\omega_k^{\mathbb{R}}(A)$	the sign-real rank-k numerical radius of real matrix A;
A^t	the transpose of A;
A^*	the conjugate transpose of A;
$\|A\|$	the matrix $(\|a_{rs}\|)$ for all r, s;
$\|x\|$	the vector $(\|x_1\|, \|x_2\|, \dots, \|x_n\|)^T$;
$\|A\| \leq \|B\|$	$\|a_{rs}\| \leq \|b_{rs}\|$ for all r, s;
$arg(z)$	the argument of the complex number z.

For a vector $x \in \mathbb{C}^n$, we denote by $\|x\|$ the Euclidean norm of x, i.e., $\|x\| = (x^*x)^{1/2}$. For a matrix $A \in M_n$, we denote by $\|A\|$ the operator norm of A, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$, where $\|\cdot\|$ is the vector norm.

We call a matrix $A \in M_n$ irreducible if $n = 1$, or $n \geq 2$ and there does not exist a permutation matrix P such that

$$P^t A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are nonempty square submatrices. We call a square matrix $h-$cyclic (the super diagonal $(n_1, n_2, \ldots, n_h)$-block form) if it is permutationally similar to a matrix of the form

$$
\begin{pmatrix} 0 & A_{12} & 0 & \ldots & 0 & 0 \\ 0 & 0 & A_{23} & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & A_{h-1,h} \\ A_{h,1} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{2}
$$

where the block $A_{i,i+1}$ is $n_i \times n_{i+1},\ i = 1, 2, \ldots, h-1$, and $A_{h,1}$ is $n_h \times n_1$. The largest positive integer h for which a matrix A is $h-$cyclic is called the cyclic index of A. Throughout this paper, we assume that $h \geq 2$.

Given $A, B \in M_n$, A is said to be diagonally similar to B if there exists a nonsingular diagonal matrix D such that $A = D^{-1}BD$; if, in addition, D can be chosen to be unitary, then we say A is unitarily diagonally similar to B.

3. Perron-Frobenius theory for real matrices

Rump [15] has offered a way to generalize the Perron-Frobenius theory to arbitrary real matrices. In this section, we generalize Wielandt's lemma and Perron- Frobenius theorem from nonnegative matrices to real matrices.

In the following lemma, we investigate some properties of the sign-real rank-k numerical radius.

Lemma 3.1. *Let $A \in M_n(\mathbb{R})$, signature matrices $S_1, S_2, T \in \varphi$, a real diagonal matrix D, a real orthogonal matrix U and a permutation matrix P be given. Then*

(a) $\omega_k^0(TA) = \omega_k^0(AT)$;
(b) $\omega_k^{\mathbb{R}}(A) = \omega_k^{\mathbb{R}}(S_1AS_1) = \omega_k^{\mathbb{R}}(S_1AS_2) = \omega_k^{\mathbb{R}}\left(A^T\right) = \omega_k^{\mathbb{R}}(P^TAP)$;
(c) $\omega_k^{\mathbb{R}}(\alpha A) = |\alpha|\omega_k^{\mathbb{R}}(A)$ *for all* $\alpha \in \mathbb{R}$;
(d) $\omega_k^{\mathbb{R}}(UD) = \omega_k^{\mathbb{R}}(DU)$;
(e) $\omega_k^{\mathbb{R}}(A) \leq \|A\|$;
(f) *if there exists a matrix* $C \in M_n(\mathbb{R}),\ rank(C) = 1$ *with* $c_{ij} = sign\,(a_{ij})$ *if* $a_{ij} \neq 0$, *and* $c_{ij} \in \{-1, 1\}$ *if* $a_{ij} = 0$, *then* $\omega_k^{\mathbb{R}}(A) = \omega_k^{\mathbb{R}}(|A|)$.

Proof. (a) For all $T \in \varphi$ we have

$$
\begin{aligned}
\Lambda_k(TA) &= \{\lambda \in \mathbb{C} : X^*TAX = \lambda I_k \text{ for some } X \in M_{n,k}(\mathbb{C}) \text{ with } X^*X = I_k\} \\
&= \{\lambda \in \mathbb{C} : X^*TATTX = \lambda I_k \text{ for some } X \in M_{n,k}(\mathbb{C}) \text{ with } X^*X = I_k\} \\
&= \{\lambda \in \mathbb{C} : Y^*ATY = \lambda I_k \text{ for some } Y \in M_{n,k}(\mathbb{C}) \text{ with } Y^*Y = I_k\} \\
&= \Lambda_k(AT).
\end{aligned}
$$

Therefore $\omega_k^0(TA) = \omega_k^0(AT)$, and then, for all $S_1 \in \varphi$,

$$
\omega_k^0\left(S_1AS_1\right) = \omega_k^0(A). \tag{3}
$$

(b) By using the equation (3), for all $S_1 \in \varphi$, we see that

$$\begin{aligned} \omega_k^{\mathbb{R}}(S_1AS_1) &= \omega_k^0(T_1S_1AS_1) = \omega_k^0(S_1T_1AS_1) = \omega_k^0(T_1A) \leq \omega_k^{\mathbb{R}}(A) \\ &= \omega_k^0(T_2A) = \omega_k^0(T_2S_1AS_1) \leq \omega_k^{\mathbb{R}}(S_1AS_1), \end{aligned} \tag{4}$$

for some $T_1, T_2 \in \varphi$. Thus, $\omega_k^{\mathbb{R}}(A) = \omega_k^{\mathbb{R}}(S_1AS_1)$ for all $S_1 \in \varphi$. Again, by using the equation (3), for all $S_1, S_2 \in \varphi$, we see that

$$\begin{aligned} \omega_k^{\mathbb{R}}(A) &= \max_{T\in\varphi} \omega_k^0(TA) = \max_{T\in\varphi} \omega_k^0(S_2(S_1S_1TA)S_2) \\ &= \max_{T_1\in\varphi} \omega_k^0(T_1(S_1AS_2)) = \omega_k^{\mathbb{R}}(S_1AS_2). \end{aligned}$$

Similar to part (a), for every unitary matrix $W \in M_n$, we have

$$\Lambda_k(AW) = \Lambda_k(W(AW)W^*) = \Lambda_k(WA). \tag{5}$$

Therefore, for every permutation matrix P, we have $\omega_k^0(AP) = \omega_k^0(PA)$. Since PSP^T is a signature matrix for every signature matrix S, and then by using the same method in (4), we conclude that $\omega_k^{\mathbb{R}}(A) = \omega_k^{\mathbb{R}}(P^TAP)$. Also, $\omega_k^{\mathbb{R}}\left(A^T\right) = \omega_k^{\mathbb{R}}(A)$, since $\omega_k^0(A^T) = \omega_k^0(A)$.

(c) It is trivial.

(d) In view of (5), we have $\omega_k^0(AU) = \omega_k^0(UA)$, and thus we obtain

$$\begin{aligned} \omega_k^{\mathbb{R}}(DU) &= \omega_k^0(T_1DU) = \omega_k^0(DT_1U) = \omega_k^0(T_1UD) \leq \omega_k^{\mathbb{R}}(UD) \\ &= \omega_k^0(T_2UD) = \omega_k^0(DT_2U) = \omega_k^0(T_2DU) \leq \omega_k^{\mathbb{R}}(DU), \end{aligned}$$

for some $T_1, T_2 \in \varphi$. So $\omega_k^{\mathbb{R}}(UD) = \omega_k^{\mathbb{R}}(DU)$.

(e) For any nonzero vector $x \in \mathbb{C}^n$, we have $|x^*Ax| \leq \|Ax\| \|x\|$ (Cauchy-Schwarz inequality), and by Definition 1.1, we obtain $\omega_1^0(A) \leq \|A\|$, and hence $\omega_1^{\mathbb{R}}(A) \leq \|SA\| = \|A\|$ for any $S \in \varphi$. The inclusion (1) implies that $\omega_k^{\mathbb{R}}(A) \leq \omega_1^{\mathbb{R}}(A)$ for all $k > 1$. Therefore $\omega_k^{\mathbb{R}}(A) \leq \|A\|$.

(f) $rank(C) = 1$ implies that there exists $x, y \in \mathbb{R}^n$ with $|x_i| = |y_i| = 1$ for all $i = 1, 2, \ldots, n$ such that $C = xy^T$. Hence, $S_x := diag\,(x) \in \varphi$, $S_y := diag\,(y) \in \varphi$, and $S_xAS_y = |A|$ yield $\omega_k^{\mathbb{R}}(A) = \omega_k^{\mathbb{R}}(S_xAS_y) = \omega_k^{\mathbb{R}}(|A|)$. $\square$

The following lemma is an extension of the Wielandt's lemma for real matrices [20, Lemma 3.1]:

Lemma 3.2. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Let $C \in M_n(\mathbb{C})$ be such that $|C| \leq |A|$. Then for every eigenvalue t of C we have $|t| \leq \rho^{\mathbb{R}}(A)$. Furthermore $|t| = \rho^{\mathbb{R}}(A)$ if and only if $C = e^{i\varphi}E|A|E^{-1}$, where $t = e^{i\varphi}\rho^{\mathbb{R}}(A)$, and $|E| = I$.*

The next lemma will depend on the numerical radius that is an analog of Wielandt's lemma [[20], Lemma 4.4].

Lemma 3.3. *Let $A, B \in M_n$, and assume that A is a real irreducible matrix. Let $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$ and $|B| \leq |A|$. If ε is a unit complex number such that $\varepsilon\omega_1^{\mathbb{R}}(A) \in W(B)$, then $\varepsilon F|A|F^{-1} = B$ for some unitary diagonal matrix F.*

To prove our main results in this section, we need the following theorem [20, Theorem 3.2]:

Theorem 3.4. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Let $\lambda_1, \dots, \lambda_h$ be the eigenvalues of SA of modulus $\rho^{\mathbb{R}}(A)$ for some $S \in \varphi$, and let $\rho^{\mathbb{R}}(A)$ be one of them. Then $\lambda_1, \dots, \lambda_h$ are the distinct h-th roots of $(\rho^{\mathbb{R}}(A))^h$.*

Remark 3.5. Notice that Theorem 3.4 is the fundamental theorem for the results of this paper. In [20, Example 3.3 and Example 3.4], it was shown that Theorem 3.4 fails if the assumptions $|A| = S_1AS_2$ and $\rho^{\mathbb{R}}(A) \in \sigma(SA)$ for some $S \in \varphi$, are dropped. In other word, these assumptions are necessary in this theorem and the following results.

The next theorem gives some conditions under which a real irreducible matrix is h-cyclic.

Theorem 3.6. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Let $\lambda_1, \dots, \lambda_h$ be the eigenvalues of SA of modulus $\rho^{\mathbb{R}}(A)$ for some $S \in \varphi$, and let $\rho^{\mathbb{R}}(A)$ be one of them. Then there exists some permutation matrix P such that PAP^T is in the form* (2).

Proof. $\rho^{\mathbb{R}}(A)$ is an eigenvalue of SA for some $S \in \varphi$ [15, Lemma 2.2] so by applying Lemma 3.2 with $C = SA$, we have

$$SA = E|A|E^{-1}, \ \ |E| = I. \tag{6}$$

By Theorem 3.4, $\lambda_1 = e^{i2\pi/h}\rho^{\mathbb{R}}(A)$ is an eigenvalue of SA for some $S \in \varphi$, again by applying Lemma 3.2 with $C = SA$, we have

$$SA = e^{i2\pi/h}E_1|A|E_1^{-1}, \ \ |E_1| = I. \tag{7}$$

Hence, by (6) and (7), $A = e^{i2\pi/h}D_1AD_1^{-1}$, where $D_1 = diag(\alpha_1, \dots, \alpha_n)$ and $|D_1| = I$. Let $\alpha_1, \dots, \alpha_s$ be distinct among the α_i's, $i = 1, 2, \dots, n$. Then there exists a permutation matrix P such that

$$PD_1P^T = \begin{pmatrix} \alpha_1 I_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_s I_s \end{pmatrix},$$

where $I_1, I_2, \dots, I_s$ are identity matrices of appropriate sizes. Let

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix}.$$

Clearly, $s > 1$ and we can assume that $\alpha_1 = 1$. Since $PAP^T = e^{i2\pi/h}(PD_1P^T)$ $(PAP^T)(PD_1^{-1}P^T)$ hence $A_{jk} = e^{i2\pi/h}\alpha_j A_{jk}\alpha_k^{-1}$. In particular, $A_{11} = e^{i2\pi/h}$

$\alpha_1 A_{11}\alpha_1^{-1} = e^{i2\pi/h}A_{11}$ and $e^{i2\pi/h} \neq 1$ implies that $A_{11} = 0$. Also $A_{12} = e^{i2\pi/h}\alpha_1 A_{12}\alpha_2^{-1}, \ldots, A_{1s} = e^{i2\pi/h}\alpha_1 A_{1s}\alpha_s^{-1}$. Since A is irreducible, at least one of $A_{12}, \ldots, A_{1s}$ is nonzero. Furthermore, $e^{i2\pi/h}/\alpha_2 = 1$ or $\cdots$ or $e^{i2\pi/h}/\alpha_s = 1$. Assuming that $0 = arg(\alpha_1) < arg(\alpha_2) < \cdots < arg(\alpha_s)$, we can say that $e^{i2\pi/h}/\alpha_2 = 1$. Thus $A_{13}, \ldots, A_{1s}$ are all zero blocks. Similarly, we can show that $A_{21}, A_{22}, A_{24}, \ldots, A_{2s}$ are zero, and so on. Finally, $A_{sj} = 0$ for all $j = 2, \ldots, s$. Therefore, by $s = h$, the proof is complete. $\square$

We illustrate the Theorem 3.6 with an example.

Example 3.7. *Consider the real matrix A given in [20, Example 3.3] as follows:*

$$A = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & -6 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 4 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It was shown that there signature matrices $S_1 = diag(-1,-1,1,1,-1,1,1,1)$, and $S_2 = diag(1,-1,1,1,1,1,1,1)$ such that $|A| = S_1 A S_2$. Also, there exist signature matrices

$$\begin{aligned} T_1 &= diag(+1,+1,+1,-1,+1,+1,-1,-1), \\ T_2 &= diag(-1,+1,-1,-1,+1,-1,+1,-1), \\ T_3 &= diag(-1,-1,+1,-1,+1,+1,+1,-1), \\ T_4 &= diag(-1,+1,+1,+1,-1,+1,+1,+1), \end{aligned}$$

such that $\lambda_1 = -5.9685$, $\lambda_2 = -5.9685i$, $\lambda_3 = 5.9685i$, $\lambda_4 = 5.9685$ are the eigenvalues of $T_j A$ of modulus $\rho^{\mathbb{R}}(A) = 5.9685$ for $j = 1, \ldots, 4$. By Theorem 3.6 there exists a permutation matrix P, whose entries are as follows:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

such that $B = PAP^T$ *is in the form* (2), *i.e.,* B *is a* $4-$*cyclic matrix.*

$$B = \begin{pmatrix} 0 & -7 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. The Issos' result on $\Lambda_k(A)$ for matrices with irreducible Hermitian part

In this section, we obtain the main theorem of Issos on the higher-rank numerical range of real matrices. The method of proof is based on Theorem 3.6. After proving some results, we extend the main theorem of Issos from irreducible matrices to matrices with irreducible Hermitian part. Throughout this section, we let the sign-real rank-k numerical radius be positive.

By considering the number of maximal elements in $\Lambda_k(SA)$ for some $S \in \varphi$ and their location in the complex plane, we have the following theorem.

Theorem 4.1. *Let* $A \in M_n(\mathbb{R})$ *be irreducible, and* $|A| = S_1AS_2$ *for some* $S_1, S_2 \in \varphi$. *Let* $\lambda_1, \ldots, \lambda_h$ *be the eigenvalues of* SA *of modulus* $\rho^{\mathbb{R}}(A)$ *for some* $S \in \varphi$, *and let* $\rho^{\mathbb{R}}(A)$ *be one of them. Then there exists some* $T \in \varphi$ *such that*

$$\{z \in \Lambda_j(TA) : |z| = \omega_j^{\mathbb{R}}(A)\} = \left\{\omega_j^{\mathbb{R}}(A)e^{i(\theta_j+2\pi t/h)} : t = 0, 1, \ldots, h-1\right\}, \tag{8}$$

for every $j = 1, \ldots, k$ *with* $\theta_j = 0$ *or* $\theta_j = \pi/h$.

Proof. By Theorem 3.6, there is a permutation matrix P such that $B := PAP^T$ is of the form (2). Let $\theta = 2\pi t/h$, for $t = 0, 1, \ldots, h-1$. By the proof of Theorem 6 in [9], we have

$$D^{-1}BD = e^{i\theta}B, \text{ i.e., } D^{-1}(P^TAP)D = e^{i\theta}(P^TAP), \tag{9}$$

where $D := diag(D_1, D_2, \ldots, D_h)$ is unitary diagonal with $D_j := diag(e^{i(j-1)\theta}, e^{i(j-1)\theta} \ldots, e^{i(j-1)\theta})$ and D_j is a square matrix having the same dimension as B_{jj} for $j = 1, 2, \ldots, h$. Let $\theta_j \in [0, 2\pi)$ be the principal argument such that $\omega_j^{\mathbb{R}}(A) \in \Lambda_j(e^{-i\theta_j}TA)$ for some $T \in \varphi$, and for $j = 1, \ldots, k$. Therefore, (9) implies that $\omega_j^{\mathbb{R}}(A)e^{i(\theta_j+2\pi t/h)} \in \Lambda_j(TA)$ for $t = 0, 1, \ldots, h-1$, and $j = 1, \ldots, k$. Hence, for any $j = 1, \ldots, k$, we have

$$\{\omega_j^{\mathbb{R}}(A)e^{i(\theta_j+2\pi t/h)} \ : \ t = 0, 1, \ldots, h-1\} \subseteq \{z \in \Lambda_j(TA) \ : \ |z| = \omega_j^{\mathbb{R}}(A)\}.$$

Again by (9), h is equal to the largest positive integer such that matrix A is unitarily diagonally similar to the matrix $e^{i2\pi t/h}A$ for $t = 0, 1, \ldots, h-1$. So, there does not exist $\upsilon = 2\pi/p < 2\pi/h$ such that $\Lambda_j(A) = \Lambda_j(e^{i\upsilon}A)$ for $j = 1, 2, \ldots, k$. Hence, for every $j = 1, 2, \ldots, k$ we have (8). Since $A \in M_n(\mathbb{R})$, so $\Lambda_j(A)$ is symmetric with respect to the real axis. If we consider $\theta_j \neq 0$, we obtain $2\pi - \theta_j = \theta_j + 2\pi(h-1)/h$. Hence $\theta_j = \pi/h$, and the proof is complete. $\square$

Note that [20, Theorem 4.7] is special case of Theorem 4.1 ($j = 1$).
We are going to extend Theorem 4.1 to the case when A is a real matrix with irreducible Hermitian part. For this purpose we need the following lemmas.

Lemma 4.2. *Let $A \in M_n(\mathbb{R})$, and $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. Let $H(A) = \dfrac{A+A^*}{2}$ be irreducible and there exists $\theta_j \in [0, 2\pi)$ such that $\omega_j^{\mathbb{R}}(A) \in \Lambda_j(e^{-i\theta_j}SA)$ for some $S \in \varphi$ and for every $j = 1, \ldots, k$. If there exists a real number α such that $\omega_1^{\mathbb{R}}(A)e^{i\alpha} \in W(TA)$ for some $T \in \varphi$, then $\omega_j^{\mathbb{R}}(A)e^{i\alpha} \in \Lambda_j\left(e^{-i\theta_j}SA\right)$ for every $j = 1, \ldots, k$.*

Proof. By [20, Lemma 4.10], we have $e^{i\alpha}A = D^{-1}AD$ for some unitary diagonal matrix D, i.e., A is diagonally similar to $e^{i\alpha}A$. Therefore for every $j = 1, \ldots, k$, we have $\omega_j^{\mathbb{R}}(A) \in \Lambda_j(e^{-i\theta_j}SA) = \Lambda_j(e^{-i(\theta_j+\alpha)}SA)$ for some $S \in \varphi$. Hence, $\omega_j^{\mathbb{R}}(A)e^{i\alpha} \in \Lambda_j\left(e^{-i\theta_j}SA\right)$ for every $j = 1, \ldots, k$. $\square$

Lemma 4.3. *Let $A \in M_n(\mathbb{R})$, and $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. Let $H(A)$ be irreducible and $\lambda_1, \ldots, \lambda_h$ be the eigenvalues of SA of modulus $\rho^{\mathbb{R}}(A)$ for some $S \in \varphi$, and let $\rho^{\mathbb{R}}(A)$ be one of them.*

- (*i*) *For any unit complex number ξ, there exists some $T \in \varphi$ such that the following conditions are equivalent:*
  - (*a*) *ξTA is diagonally similar to TA.*
  - (*b*) *$\xi W(TA) = W(TA)$.*
  - (*c*) *$\xi\omega_1^{\mathbb{R}}(A) \in W(TA)$.*
- (*ii*) *There exists some $T \in \varphi$ such that exactly one of the two following statements is true.*
  - (*a*) *$W(TA)$ is a circular disk with center at the origin for some $T \in \varphi$.*
  - (*b*) *$\{z \in W(TA) : |z| = \omega_1^{\mathbb{R}}(A)\} = \{\omega_1^{\mathbb{R}}(A)e^{i(\theta+2\pi t/h)} : t = 0, 1, \ldots, h-1\}$, where h is the largest positive integer such that A is diagonally similar to $e^{2\pi i/h}A$, and $\theta = 0$ or $\theta = \pi/h$.*

Proof. (i) For any $A \in M_n$, and $T_1 \in \varphi$, the set

$$G = \{\xi \in \mathbb{C} \ : \ |\xi| = 1,\ \xi T_1 A \textit{ is (unitarily) diagonally similar to } T_1A\},$$

forms a subgroup of the group of all unit complex numbers, and furthermore it is included in the set $L = \{\xi \in \mathbb{C} \ : \ |\xi| = 1,\ \xi W(T_1A) = W(T_1A)\}$. Since A is a real matrix, therefore $\omega_1^{\mathbb{R}}(A) \in W(TA)$ for some $T \in \varphi$ (see [19, Lemma 2.1]), the latter set, in turn, is included in $M = \{\xi \in \mathbb{C} \ : \ |\xi| = 1,\ \xi\omega_1^{\mathbb{R}}(A) \in W(TA)\}$. Then, in view of [20, Lemma 4.10], the three sets are all equal.

(ii) The group G may be infinite or finite. If G is infinite (has more than n elements), then the numerical range of TA contains more than n points with modulus equal to $\omega_1^{\mathbb{R}}(A)$. In this case, by [3, Theorem 2.2], $W(TA)$ is equal to the circular disk with center at the origin and radius $\omega_1^{\mathbb{R}}(A)$. Hence, G is precisely the group of all unit complex numbers. On the other hand, if G is a finite group, say, with order $h(\leq n)$, then by Lagrange's theorem in group theory, for any $\xi \in G$, we have $\xi^h = 1$, i.e., each element of G is an h-th root of unity. But the cardinality of G is h, so it follows that G is precisely the group of all h-th roots of unity. The proof that $\theta = 0$ or $\theta = \pi/h$, is similar to Theorem 4.1. □

In view of Lemma 4.2 and Lemma 4.3, we can now extend Theorem 4.1 as follows:

Theorem 4.4. *Let $A \in M_n(\mathbb{R})$ and $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. Let $H(A)$ be irreducible and $\lambda_1, \dots, \lambda_h$ be the eigenvalues of SA of modulus $\rho^{\mathbb{R}}(A)$ for some $S \in \varphi$, and let $\rho^{\mathbb{R}}(A)$ be one of them. Then there exists some $T \in \varphi$ such that for every $j = 1, \dots, k$, exactly one of the two following statements is true.*

(i) *$\mathcal{W}_j(A) := \{z \in \Lambda_j(TA) \ : \ |z| = \omega_j^{\mathbb{R}}(A)\}$ is a circular disk centered at the origin with radius $\omega_j^{\mathbb{R}}(A)$.*

(ii) *$\mathcal{W}_j(A) = \left\{\omega_j^{\mathbb{R}}(A)e^{i(\theta_j+2\pi t/h)} \ : \ t = 0, 1, \dots, h-1\right\}$, where h is the largest positive integer such that A is diagonally similar to the matrix $e^{i2\pi/h}A$, and $\theta_j = 0$ or $\theta_j = \pi/h$.*

Proof. Let $\theta_j \in [0, 2\pi)$ be the argument such that $\omega_j^{\mathbb{R}}(A) \in \Lambda_j(e^{-i\theta_j}TA)$ for some $T \in \varphi$, and for every $j = 1, \dots, k$. By Lemma 4.3, either $\mathcal{W}_1(A)$ is a circular disk centered at the origin with radius $\omega_1^{\mathbb{R}}(A)$ or $\mathcal{W}_1(A) = \{\omega_1^{\mathbb{R}}(A)e^{i(\theta_1+2\pi t/h)}$ $: t = 0, 1, \dots, h-1\}$, where h is the largest positive integer such that A is diagonally similar to the matrix $e^{i2\pi/h}A$. If $\mathcal{W}_1(A)$ is a circular disk centered at the origin with radius $\omega_1^{\mathbb{R}}(A)$, so there exists $T_1 \in \varphi$ such that $\omega_1^{\mathbb{R}}(A)e^{i\alpha} \in W(T_1A)$ for every angle $\alpha \in \mathbb{R}$. Thus by Lemma 4.2, $\mathcal{W}_j(A)$ is a circular disk centered at the origin with radius $\omega_j^{\mathbb{R}}(A)$ for every $j = 1, \dots, k$. On the other hand, let $\omega_1^{\mathbb{R}}(A)e^{i2\pi t/h} \in W(T_1A)$, and so agian by Lemma 4.2, $\omega_j^{\mathbb{R}}(A)e^{i2\pi t/h} \in \Lambda_j(e^{-i\theta_j}TA)$ for $j = 1, \cdots, k$, and $t = 0, \dots, h-1$. Thus we have

$$\{\omega_j^{\mathbb{R}}(A)e^{i(\theta_j+2\pi t/h)} \ : \ t = 0, 1, \dots, h-1\} \subseteq \mathcal{W}_j(A).$$

The equality of the sets is established similarly as in the proof of Theorem 4.1. □

In the following examples, we illustrate Theorem 4.1.

Example 4.5. *Consider the real matrix A given in Example* 3.7. *Indeed, as shown in Example* 3.7, *the conditions of Theorem 4.1 is satisfied and thus there exists some $S \in \varphi$ such that the maximal elements in $\Lambda_2(SA)$ are equally spaced around a circle with center at the origin and radius $\omega_2^{\mathbb{R}}(A) = 4.5867$. For example, for the signature matrix $S = diag(-1, 1, -1, 1\ , -1, -1, -1, -1)$, we obtain $\omega_2^{\mathbb{R}}(A) = \omega_2^0(SA) = 4.5867$, and the relation* (8) *holds with $\theta = 0$. The graph on* *the left of Figure 1, shows the boundary of* $\Lambda_2(SA)$ *such that the absolute value of* $\omega_2^{\mathbb{R}}(A)$*, i.e., the numbers* $\omega_2^{\mathbb{R}}(A)$, $e^{i\pi/2}\omega_2^{\mathbb{R}}(A)$, $e^{i\pi}\omega_2^{\mathbb{R}}(A)$ *and* $e^{i3\pi/2}\omega_2^{\mathbb{R}}(A)$ *are marked by "*". For the signature matrix* $T = diag(1, -1, 1, 1, 1, 1, 1, -1)$*, we have also the relation* (8) *with* $\theta = \pi/4$*. The graph on the right of Figure 1, shows the boundary of* $\Lambda_2(TA)$ *and the numbers* $e^{i\pi/4}\omega_2^{\mathbb{R}}(A)$, $e^{i3\pi/4}\omega_2^{\mathbb{R}}(A)$, $e^{i5\pi/4}\omega_2^{\mathbb{R}}(A)$ *and* $e^{i7\pi/4}\omega_2^{\mathbb{R}}(A)$ *are marked by "+".*

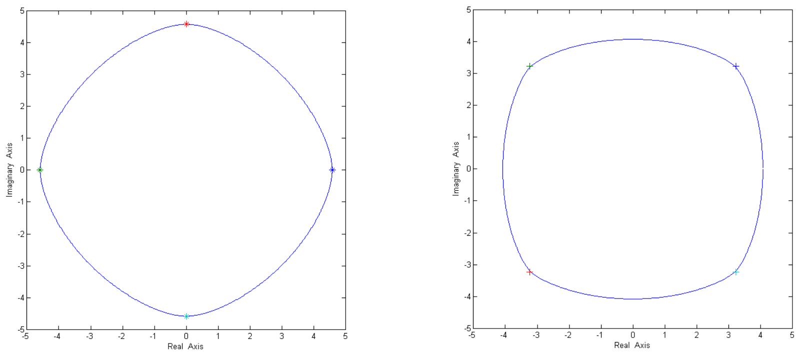

FIGURE 1. The left panel shows the boundary of $\Lambda_2(SA)$, and the one on the right shows the boundary of $\Lambda_2(TA)$.

Example 4.6. *Consider the* $5-$*cyclic matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the signature matrix $S = I$ *the relation* (8) *holds with* $\theta = 0$*, and for the signature matrix* $T = diag(1, 1, 1, 1, -1)$ *the relation* (8) *holds with* $\theta = \pi/5$*. The graph on the left of Figure 2, shows the boundary of* $\Lambda_3(SA)$ *such that the numbers* $\omega_3^{\mathbb{R}}(A)$, $e^{i2\pi/5}\omega_3^{\mathbb{R}}(A)$, $e^{i4\pi/5}\omega_1^{\mathbb{R}}(A)$, $e^{i6\pi/5}\omega_3^{\mathbb{R}}(A)\, e^{i8\pi/5}\omega_3^{\mathbb{R}}(A)$ *are belong to* $\Lambda_3(SA)$*, with modulus* $\omega_3^{\mathbb{R}}(A) = 1$*. The graph on the right, shows the boundary of* $\Lambda_3(TA)$ *such that the numbers* $e^{i\pi/5}\omega_3^{\mathbb{R}}(A)$, $e^{i3\pi/5}\omega_3^{\mathbb{R}}(A)$, $e^{i\pi}\omega_3^{\mathbb{R}}(A)$, $e^{i7\pi/5}\omega_3^{\mathbb{R}}(A)$ $e^{i9\pi/5}\omega_3^{\mathbb{R}}(A)$ *are belong to* $\Lambda_3(TA)$*, with modulus* $\omega_3^{\mathbb{R}}(A) = 1$.

5. Conclusions

In this paper, we have generalized a number of theoretical results obtained by Aretaki et al. [1] for real matrices. We propose a new definition of the rank-k numerical radius for real matrices, called the sign-real rank-k numerical radius. The main theorem of this paper show that for a real irreducible matrix A with

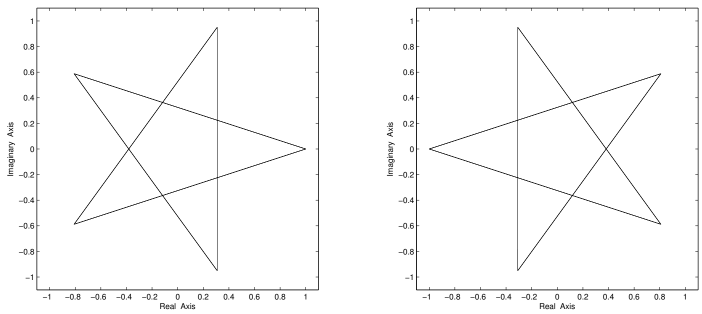

FIGURE 2. The left panel shows the boundary of $\Lambda_3(SA)$, and the one on the right shows the boundary of $\Lambda_3(TA)$.

some conditions, the set of points of $\Lambda_j(A)$ with modulus $\omega_j^{\mathbb{R}}(A)$ consists precisely of $\omega_j^{\mathbb{R}}(A)$ times all the $h-$th roots of unity (Theorem 4.1). Furthermore, a number of examples and figures are included in the paper to illustrate the type of results which is obtained.

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