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EIGENVALUES FOR TRIDIAGONAL 3-TOEPLITZ MATRICES

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Dedicated to sincere professor Mehdi Radjabalipour on turning 75

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Abstract. In this paper, we study the eigenvalues of real tridiagonal 3-Toeplitz matrices of different order. When the order of a tridiagonal 3-Toeplitz matrix is $n = 3k+2$, the eigenvalues were found explicitly. Here, we consider the distribution of eigenvalues for a tridiagonal 3-Toeplitz matrix of orders $n = 3k$ and $n = 3k + 1$. We explain our method by finding roots of a combination of Chebyshev polynomials of the second kind. This distribution solves the eigenproblem for integer powers of such matrices.

Keywords: 3-Toeplitz matrix, Chebyshev Polynomials, Eigenvalue.
2020 MSC: 15B05; 15A18; 65F15.

1. Introduction

The purpose of this paper is the spectral analysis of an $n \times n$ real tridiagonal 3-Toeplitz matrices, of the form

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & \\ c_1 & a_2 & b_2 & & & & & \\ & c_2 & a_3 & b_3 & & & & \\ & & c_3 & a_1 & b_1 & & & \\ & & & c_1 & a_2 & b_2 & & \\ & & & & c_2 & a_3 & b_3 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{1}$$

k-Toeplitz matrices are tridiagonal matrices of the form $A = [a_{i,j}]_{i,j=1}^{n}$ (with $n \geq k$) such that $a_{i+k,k+j} = a_{i,j}$ $(i, j = 1, 2, ..., n - k)$, so that they are k-periodic along the diagonals parallel to the main diagonal. A Toeplitz matrix is a k-Toeplitz matrix when $k = 1$. The interest of the study of k-Toeplitz matrices appears to be very important not only in linear algebra or numerical analysis, but also in applications such as solving the inverse of a matrix, systems of linear equations, stability of difference approximations to differential equations, chain models in quantum physics, sound propagation problems, etc [7, 12].

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In [13, 17, 20], some explicit expressions for eigenvalues of a tridiagonal 2 and 3-Toeplitz matrices were introduced.
In this paper, we extend those analysis for tridiagonal 3-Toeplitz matrices with different ranks. Tridiagonal k-Toeplitz matrices have application in Applied Mathematics, Orthogonal Polynomials and as well as in Quantum Physics, hence we hope that our results would be useful for some researchers working in these fields [2, 17, 18]. This paper is organized as follows:
In the next section, we overview some preliminaries for tridiagonal 3-Toeplitz matrices in the analysis of orders $n = 3k + 2$, $n = 3k$ and $n = 3k + 1$.
Finally, we conclude some remarks on tridiagonal 3-Toeplitz matrices.

2. The main theorem in the analysis

In this section, we analysis and summary some main results that will be useful throughout the work.

Theorem 2.1. *(Marcella'n and Petronilho [17]). Let A_n, $n = 3, 4, 5, ...$, be the irreducible tridiagonal 3-Toeplitz matrix given by (1), where b_1, b_2, b_3, c_1, c_2 and c_3 are positive numbers. Define the sequence $\{S_n\}_{n\geq 0}$ of orthogonal polynomials associated with the matrices A_n as*

$$S_{3k}(x) = (b_1b_2b_3)^{-k}\{P_k(\pi_3(x)) + b_3c_3(x - a_2)P_{k-1}(\pi_3(x))\}, \tag{2}$$

$$S_{3k+1}(x) = b_1^{-1}(b_1b_2b_3)^{-k}\{(x - a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\}, \tag{3}$$

$$S_{3k+2}(x) = (b_1b_2)^{-1}(b_1b_2b_3)^{-k}(x - \xi_1)(x - \xi_2)P_k(\pi_3(x)),\ k = 0, 1, \dots, \tag{4}$$

where ξ_1 and ξ_2 are the roots of the polynomial

$$(x - a_1)(x - a_2) - b_1c_1, \tag{5}$$

and

$$\pi_3(x) := \begin{vmatrix} x - a_1 & 1 & 1 \\ b_1c_1 & x - a_2 & 1 \\ b_3c_3 & b_2c_2 & x - a_3 \end{vmatrix}. \tag{6}$$

Then the eigenvalues $\lambda_{n,m}$ of A_n are the zeros of S_n, and the corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by

$$\mathbf{v_{n,m}} = \begin{pmatrix} S_0(\lambda_{n,m}) \\ S_1(\lambda_{n,m}) \\ \vdots \\ S_{n-1}(\lambda_{n,m}) \end{pmatrix}, \quad m = 1, 2, \dots, n. \tag{7}$$

Define

$$P_n(x) = (b_1b_2b_3c_1c_2c_3)^{n/2}U_n\left(\frac{x - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right),\ n = 0, 1, 2, \dots, \tag{8}$$

where $U_n(x)$ is the Chebyshev polynomial of degree n of the second kind with $n \in \mathbb{N} \cup \{-1, 0\}$.

All Chebyshev polynomials, among them $U_n(x)$, satisfy the three-term recurrence relations [6, 14]:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad (U_{-1}(x) = 0,\ U_0(x) = 1,\ U_1(x) = 2x).$$

Note that since the sequence $\{S_k\}_k$ is an orthogonal polynomial sequence corresponding to a positive definite case, then the zeros are simple and interlace [1, 2], i.e., if $\{x_{k,j}\}_{j=1}^{k}$ denotes the zeros of the polynomial S_k, then $x_{k,j} < x_{k-1,j} < x_{k,j+1}$, $\ j = 1, 2, ..., k-1$. Using this fact, we obtain bounds for the eigenvalues of the corresponding matrices.

2.1. **Tridiagonal 3-Toeplitz matrix of order** $n = 3k + 2$**.** In particular, when $n = 3k + 2$, from Equation (4), the eigenvalues $\lambda_{3k+2,m}$ of A_{3k+2} ($m = 1, 2, ..., 3k + 2$) are $\lambda_{3k+2,1} = \xi_1,\ \lambda_{3k+2,2} = \xi_2$ in the solutions of the cubic equations

(9)
$$Q(\lambda) := \pi_3(\lambda) - \left[b_1c_1 + b_2c_2 + b_3c_3 + 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}\right] = 0,\ i = 1, \dots, k.$$

From (6)

$$\begin{aligned}\pi_3(\lambda) &= (\lambda - a_1)(\lambda - a_2)(\lambda - a_2) - (b_1c_1 + b_2c_2 + b_3c_3)(\lambda - a_3)\\ &\quad + b_2c_2(a_1 - a_3) + b_3c_3(a_2 - a_3) + b_1c_1 + b_2c_2 + b_3c_3,\end{aligned} \tag{10}$$

and from Shengjin formulas are given in [22], we compute the roots of the cubic Equation (9).
Denote the coefficients $Q(\lambda)$ of Equation (9) with

$q_1 = 1,\ q_2 = -(a_1 + a_2 + a_3),\ q_3 = a_1a_2 + a_2a_3 + a_1a_3 - b_1c_1 - b_2c_2 - b_3c_3,$
$q_4 = a_3b_1c_1 + a_1b_2c_2 + a_2b_3c_3 - a_1a_2a_3 - 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}.$

Let

$$\Delta_1 = q_2^2 - 3q_1q_3,\ \Delta_2 = q_2q_3 - 9q_1q_4,\ \Delta_3 = q_3^2 - 3q_2q_4,\ \Delta_4 = \Delta_2^2 - 4\Delta_1\Delta_3.$$

Then we have

(1) If $\Delta_1 = \Delta_2 = 0$, $Q(\lambda)$ has only one real triple root;
(2) If $\Delta > 0$, $Q(\lambda)$ has one real root and a pair of conjugate imaginary roots;
(3) If $\Delta = 0$, $Q(\lambda)$ has three real roots: one simple and the other double;
(4) If $\Delta < 0$, $Q(\lambda)$ has three different real roots.

The corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by (7).

2.2. **Tridiagonal 3-Toeplitz matrix of order** $n = 3k+1$**.** When $n = 3k+1$, in Equation (3), the eigenvalues $\lambda_{3k+1,m}$ of A_{3k+1}

$(m = 1, 2, ..., 3k+1)$ are the roots x of $S_{3k+1}(x)$ satisfy the equation

$$b_1^{-1}(b_1b_2b_3)^{-k}\{(x-a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\} = 0. \quad (11)$$

With following (8) in Equation (11), we have $s = \frac{\sqrt{b_1b_3c_1c_3}}{\sqrt{b_2c_2}}$.

If x is not a common root of $U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_1 - x$, then we conclude

$$\frac{U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{s}{a_1 - x}. \quad (12)$$

Note 1. Let $\eta_0 < \xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n < \eta_n$ with $\eta_0 = -\infty$, $\eta_n = \infty$, where $\xi_1, \ \xi_2, \ldots, \ \xi_n$ are the roots of $U_n(x)$, and $\eta_1, \ \eta_2, \ldots, \ \eta_{n-1}$ are the roots of $U_{n-1}(x)$ in Equation (12).

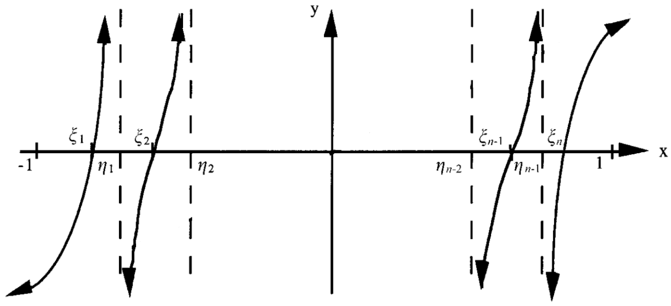

FIGURE 1. $p_{n,n-1}(x)$

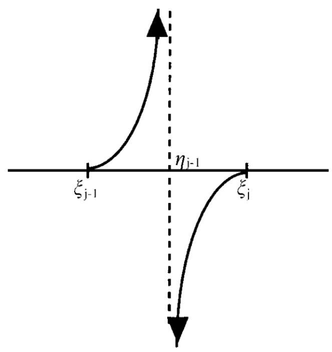

FIGURE 2. $p_{n,n-1}(x)$

Let $U_n(x)/U_{n-1}(x) = p_{n,n-1}(x)$, $n \geq 1$ and $p_{0,-1}(x) = 1$.
Next we denote $g(x) = s/(a_1 - x)$ that here $s = \frac{\sqrt{b_1 b_3 c_1 c_3}}{\sqrt{b_2 c_2}}$.

The graph of $p_{n,n-1}(x)$ is shown in Fig. 1. Also Fig. 2 shows $p_{n,n-1}(x)$ in the interval (ξ_{j-1}, ξ_j).
Under the above notations, we have the following theorem:

Theorem 2.2. *If $s > 0$, for some i in Equation (12) and Note 1, then there are precisely two additional roots, exactly one lying in each of the intervals*

$$(\eta_{i-1}, a_1) \text{ and } (a_1, \eta_i).$$

If $s < 0$, then one or two additional roots of Equation (12) can be zero, in the interval (η_{i-1}, η_i).
Finally, the next elseif $s = 0$, the problem is solved easily by finding roots of $U_n(x)$.

Proof. If a_1 coincides with one of the η_i's, it is a root of (11). Otherwise, we call the interval (η_{i-1}, η_i) the distinguished interval if $\eta_{i-1} < a_1 < \eta_i$ for some i,
there is exactly one root of (12) in each of the $n-1$ intervals (η_{j-1}, η_j) where $j \neq i,\ 1 \leq j \leq n$.
Let δ_1 be the part of the graph of $g(x)$ for $x < a_1$ and δ_2 be the part of the graph of $g(x)$ for $x > a_1$. We observe that if $\eta_{i-1} < a_1 < \eta_i$, from Fig. 1 and $s > 0$, then we see that δ_1 meets each component of the graph $y = U_n(x)/U_{n-1}(x)$ once in the $i-2$ intervals on the left of (η_{i-1}, η_1), and δ_2 meets each component in $n-i+1$ intervals once on the right of (η_i, η_n), producing $n-1$ roots of (12). This holds, if $s > 0$, then $y = g(x)$ is decreasing on each interval $(-\infty, a_1)$ and (a_1, ∞) as depicted in Fig. 3;

If $s < 0$, then $y = g(x)$ is increasing on each interval $(-\infty, a_1)$ and (a_1, ∞) as depicted in Fig. 4 and the component of the graph of $y = U_n(x)/U_{n-1}(x)$ in the distinguished interval (η_{i-1}, η_i), meets both δ_1 and δ_2, and we have two additional roots of (12). Namely for $s < 0$, the graph of $y = g(x)$ is increasing on $(-\infty, a_1)$ and (a_1, ∞), with a_1 fixed, $s < 0$ and each of the three illustrations in Fig. 4.
Elseif $s = 0$ then it is sufficient to find roots of $U_n(x)$, but here is impossible, because we let the irreducible tridiagonal 3-Toeplitz matrix in (1). □

Note that, here $s > 0$. Then by the results of Theorem 2.2, the function (12) has the same roots as

(13)
$$h(x) \equiv (a_1 - x) U_n\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right) - sU_{n-1}\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right),$$

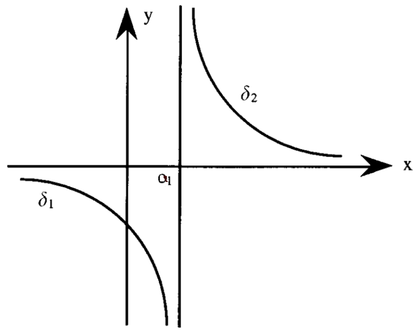

FIGURE 3. $p_{n,n-1}(x)$

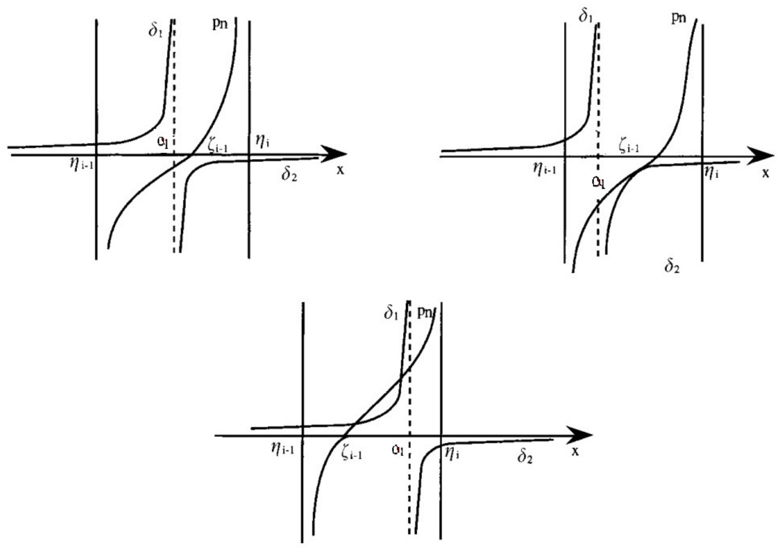

FIGURE 4. Intersections when $s < 0$.

for more details see [16].
Therefore, we can compute the roots of the following function by to approximate function $h(x)$ by Chebyshev interpolation for every interval (η_{i-1}, η_i), $i = 1, \ldots, n$, then apply Chebyshev companion matrix to find roots.
To increase the accuracy, we can increase the degree of the Chebyshev approximation. Namely, if we need already very accurate roots, it is preferable to polish the zeros by Newton's iteration or the secant iteration. The first approximated root for starting can be selected by $\xi_{i-1} \in (\eta_{i-1}, \eta_i)$.

From Equation (13), we have a polynomial with degree $3n+1$ that numerically computed roots found by Chebfun. Chebfun finds roots with a global rootfinding capability by a method that shows in [8]. In Chebfun, if the degree greater than about 50, it is broken into smaller pieces recursively then the zeros are found as eigenvalues of the analogue for Chebyshev polynomials of a companion matrix for monomials on each small piece [6].

2.3. **Tridiagonal 3-Toeplitz matrix of order $n = 3k$.** When $n = 3k$, from Equation (2), the eigenvalues $\lambda_{3k,m}$ of A_{3k} $(m = 1, 2, ..., 3k)$ are the roots x of $S_{3k}(x)$ satisfy the equation

$$(b_1b_2b_3)^{-k}\{P_k(\pi_3(x)) + b_3c_3(x - a_2)P_{k-1}(\pi_3(x))\} = 0. \tag{14}$$

If x is not a common root of $U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_2 - x$, then we conclude

$$\frac{U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}(a_2 - x)}. \tag{15}$$

Here, suppose $U_{n-1}(x)/U_n(x) = p_{n-1,n}(x)$, $n \geq 1$ and $g(x) = s/(a_2 - x)$ that $s = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}}$.
Where $\xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n$.
$\xi_1, \ \xi_2, \ldots, \ \xi_n$ are the roots of $U_n(x)$ and $\eta_1, \ \eta_2, \ldots, \ \eta_{n-1}$ are the roots of $U_{n-1}(x)$.
Here, we use of Theorem 2.2 for finding eigenvalues of matrix (1) when $n = 3k$.

We want to emphasize another way to the problem concerning the study of the eigenvalues of the sequences of matrices dened by (1), based on some results in [3, 5, 9, 10]. We will consider the case when the order $n = 3k$ of the matrix A_n in (1). Then A_n is the block Toeplitz matrix

$$A_n = \begin{pmatrix} B_0 & B_1 & & & & & \\ B_{-1} & B_0 & B_1 & & & & \\ & B_{-1} & B_0 & B_1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & B_1 \\ & & & & & B_{-1} & B_0 \end{pmatrix},$$

generated by the 3×3 matrix valued polynomial

$f(x) := B_0 + B_1e^{ix} + B_{-1}e^{-ix}$

with

$$B_0 = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} 0 & 0 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since, $f(x)$ is not hermitian generally then not very much can be said on the eigenvalues. However, from Theorem 2.1, we know $b_1,\ b_2,\ b_3,\ c_1, c_2$ and c_3 are positive numbers and so it is well-known that, under such conditions, A_n is similar to the block Toeplitz matrix $\hat{A}_n$ by diagonal transformations, that is generated by the 3×3 matrix valued polynomial

$$\hat{f}(x) := \hat{B}_0 + \hat{B}_1 e^{ix} + \hat{B}_{-1} e^{-ix}$$

with

$$\hat{B}_0 = \begin{pmatrix} a_1 & \sqrt{b_1 c_1} & 0 \\ \sqrt{b_1 c_1} & a_2 & \sqrt{b_2 c_2} \\ 0 & \sqrt{b_2 c_2} & a_3 \end{pmatrix}, \quad \hat{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{b_3 c_3} & 0 & 0 \end{pmatrix}, \quad \hat{B}_{-1} = \begin{pmatrix} 0 & 0 & \sqrt{b_3 c_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similar considerations remain true for the generalized case of a tridiagonal k-Toeplitz matrix, because the result holding for $n = tk$ can be deduce from a tridiagonal $t - block \times t - block$ matrix that every block is a $k \times k$ matrix.
There are some papers for the Evaluation of the Eigenvalues of a Banded Toeplitz Block Matrix, for more details see [3, 4, 15, 21].
The subject for future work involve, the result holding for $n = 3k$ can be extended to the cases $n = 3k + 1; 3k + 2$ by using the Theorem 4.3 in [19].
We end this section by observing that Theorem 4.3 can be extended to sequences of non-Hermitian matrices and related sequences of principal submatrices, when replacing the eigenvalue distribution with the singular value one. See the following Definition 2.6 and Corollary 4.4 of [19].

3. **Some Results**

In this paper, analysis reviews is devoted to reviewing recent works for a tridiagonal 3-Toeplitz matrix for the cases $n = 3k + 2$, $n = 3k + 1$ and $n = 3k$ with some details on (explicitly or implicitly) ways.

In order to find the eigenvalues of a $n \times n$ tridiagonal 3-Toeplitz matrices (if any order by the results of the last sections), we deduce matrix V is defined by

$$V = [\mathbf{v_{n,1}}\ \mathbf{v_{n,2}}\ \ldots\ \mathbf{v_{n,n}}],$$

that

$$\mathbf{v_{n,m}} = \begin{pmatrix} S_0(\lambda_{n,m}) \\ S_1(\lambda_{n,m}) \\ \vdots \\ S_{n-1}(\lambda_{n,m}) \end{pmatrix}, \quad m = 1, 2, \ldots, n.$$

Then by inverse of matrix V, we have

$$(A_n)^l = VJ^lV^{-1}, \quad l \in \mathbb{N}, \tag{16}$$

that $\lambda_{n,k},\ 1 \leq k \leq n$ are the eigenvalues of A_n and $J = diag(\lambda_{n,1},\ \lambda_{n,2}, \ldots, \lambda_{n,n})$ is the Jordan form of the matrix A_n.

Therefore, we deduce Proposition 3.1 of [20] that works well for the tridiagonal 3-Toeplitz matrix of every order A_n in (1). Namely, if $f(x)$ is any function defined on spectrum of A_n, then

$$f(A_n) = V diag(f(\lambda_{n,1}), \ldots, f(\lambda_{n,n}))V^{-1}.$$

The expressions that we derived can be applied for computing negative integer powers in Equation (16), when all eigenvalues of A_n are non zero. Namely the condition of non singularity of the matrix is satisfied.

When A_n in (1) is Hermitian tridiagonal 3-Toeplitz matrix, we may generalize the results in [9, 11], in this special case.

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