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DYNAMICS OF A HARMONIC OSCILLATOR PERTURBED BY A NON-SMOOTH VELOCITY-DEPENDENT DAMPING FORCE

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Dedicated to sincere professor Mehdi Radjabalipour on turning 75

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Abstract. This paper studies the dynamics of a non-smooth vibrating system of the Filippov type. The main focus is on investigating the stability and bifurcation of a simple harmonic oscillator subjected to a non-smooth velocity-dependent damping force. In this way, we can analyze the effects of damping on the system's vibrations. For this purpose, we will find a parametric region for the existence of generalized Hopf bifurcation, in order to compute a branch of periodic orbits for the system. The tool for our purpose is the theoretical results about generalized Hopf bifurcation for planar Filippov systems. Some numerical simulations as examples are given to illustrate our theoretical results. Our theoretical and numerical findings indicate that the harmonic oscillator can experience different kinds of vibrations, in the presence of a non-smooth damping.

Keywords: Velocity-dependent damping, Non-smooth dynamical systems, Generalized Hopf bifurcation, Vibrations, Nonlinear oscillator
2020 MSC: 74H99, 34C23, 74H45, 34C15

1. Introduction

Harmonic oscillators have been studying by many researchers due to their widespread applications in different research areas, see [1–3, 5, 9, 10, 12, 14, 19, 25, 26, 33, 43] and the references therein. Most of these studies demonstrate the importance of damping as an inevitable perturbation for an oscillator. It is observed that the existence of a non-smooth nonlinearity (such as damping) may result in dramatic changes in the systems' dynamics. We note that non-smooth dynamical systems is almost a young research area, and the theory of such systems has not been completely studied yet. Moreover, using non-smooth dynamical systems is essential for the modelling of many phenomena as it allows them to be modelized in a more precise and realistic way [32–36]. Especially, considering such systems is very advantageous for many engineering

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models like mechanical systems that are dealing with frictions. One important type of non-smooth systems is Flippov systems. Vector field of these systems is discontinuous but the trajectories of these systems are continuous with respect to the time t. Filippov in [24] showed that many results in the classical theory of differential equation are also valid for non-smooth differential equations with discontinuous right hand sides.

In some cases, a non-smooth substitute for assumed smooth damping enables us to obtain more comprehensive results [32, 33, 36]. In particular, some oscillators may vibrate in different environments and enter from one environment to another one, during their motion. In such situations, the oscillatory systems encounter to different velocity dependent dampings in each area. This gives rise to a non-smooth damping for these systems. For example let us consider an airfoil system, as the cross section of a body such as an aircraft wing, that is located in an airstream. The motion of a two-dimensional airfoil is described by two positions, plunge (bending) and pitch (torsion). The plunge, is the position along the y-axis measured positive down. In fact, the plunge motion is a harmonic oscillation and a linear spring is used to simulate this motion; for more details see [21, 33]. When an airfoil is moving up through the air, it may enter from the air to the clouds. Hence, a non-smooth damping imposes on the plunge oscillations of the airfoil.

There are papers concerning the effect of smooth damping on the harmonic oscillators, see [3, 4, 31, 42]. But, here we are interested in investigating the effects of a non-smooth damping on pendulum. It is supposed that the non-smooth damping force is a function depending upon velocity. A natural consequence of using a non-smooth model for damping is to consider a more suitable and real model, for damping, than the smooth one.

As mentioned above, non-smooth dynamical systems are observed in many phenomena such as turbine blade and friction oscillators. The friction contact between some surfaces widely are seen in mechanical models. The study on this problem started with the work of Den Hartog [20] in 1930 on the forced vibrations with Coulomb and viscous damping. A spring mass system with combined Coulomb and viscous damping is investigated by Levitan [26] in 1960. Four years later, Filippov stated concepts on non-smooth differential equations, [15]. After twenty years, in 1988, he presented the summary for such systems. In 1996, Oestreich et al. [37] presented their work on the dynamical behavior of a non-smooth friction oscillator. After one year, Kunze et al. used KAM theory to non-smooth systems for analyzing the periodic motion of a forced oscillator. Küpper et al. [25] and Zuo et al. [45], in 2001 and 2006, surveyed generalized Hopf bifurcation in planar non-smooth systems. Additionally, Cid and Sanchez [7] investigated a system with non-smooth restoring forced in 2003. Furthermore, periodic solutions of friction oscillators have been studied by many researchers, see [1, 2, 8, 27, 29–31, 38–40, 43]. Besides, for the sake of the significant role of sliding bifurcations, analysing Filippov systems has become a hot topic for many researchers, [5, 16, 18, 22, 23, 33, 35].

A key point of this paper is that a velocity-dependent damping is considered to model a non-smooth harmonic oscillator. To the best of our knowledge, damping is a challenge in science and engineering and it exists in many models with different scales; for example, in a protein motor with nanometers scale or in earthquakes with kilometers scale. Furthermore, damped oscillators have been more popular in recent years as a research topic; see [6, 13, 14, 17, 28, 44]. The current study aims to examine the effects of a non-smooth velocity-dependent damping on a harmonic oscillator. First, we present our mathematical model. Afterwards, the generalized Hopf bifurcation theorem, in non-smooth systems, is applied to obtain a parametric region for existence of periodic oscillations for the oscillator. Finally, some analytical results and numerical simulations are given to better comprehend the complicated dynamics of the model.

2. Preliminaries

In this section, we briefly explain some definitions and theorems of non-smooth dynamical systems which are going to be used in the next section. For more details see [33, 45].

2.1. **Non-smooth dynamical systems.** Let $U \subseteq \mathbb{R}^2$ be an open subset centered about the origin and M be an open interval in R containing 0. Moreover, suppose that

($\mathbf{H_1}$) $f^+ : U \times M \longrightarrow \mathbb{R}^2$, $f^- : U \times M \longrightarrow \mathbb{R}^2$ are C^k $(k \geq 2)$ functions.

Furthermore, consider the smooth function $H : U \longrightarrow \mathbb{R}$, with $H(x, y) = x$, where $(x, y)^T \in U$. The scaler function H has nonvanishing gradient $\nabla H(x, y) = (1, 0)^T$. We define

$$S^+ = \{(x, y)^T \in U \subseteq \mathbb{R}^2;\ H(x, y) > 0\}, \tag{1}$$

$$S^- = \{(x, y)^T \in U \subseteq \mathbb{R}^2;\ H(x, y) < 0\}, \tag{2}$$

and the discontinuity boundary Σ separating the two regions S^+ and S^- as

$$\Sigma = \{(x, y)^T \in U \subseteq \mathbb{R}^2;\ H(x, y) = 0\}. \tag{3}$$

Now consider the following non-smooth system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} f^+(x, y, \mu); & (x, y)^T \in S^+, \\ f^-(x, y, \mu); & (x, y)^T \in S^-, \end{cases} \tag{4}$$

such that $f^{\pm}(x, y, \mu) = (f_1^{\pm}(x, y, \mu), f_2^{\pm}(x, y, \mu))$ and

($\mathbf{H_2}$) $f^{\pm}(0, 0, \mu) \equiv (0, 0)$ for all $\mu \in M$, i.e., the $(0, 0)$ is a stationary point for the system (4) for every $\mu \in M$.

Let us define the set-valued extension of (4) to a differential inclusion

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in F(x, y, \mu), \tag{5}$$

where

(6)

$$F(x,y,\mu) = \begin{cases} f^{+}(x,y,\mu); & (x,y)^T \in S^{+} \\ \{(1-\lambda)f^{-}(x,y,\mu) + \lambda f^{+}(x,y,\mu), \forall\lambda \in [0,1]\}; & (x,y)^T \in \Sigma \\ f^{-}(x,y,\mu); & (x,y)^T \in S^{-} \end{cases}.$$

By Filippov Theorem, differential equations (4) and (6) have a solution in the sense of Filippov definition of the solution; see [24].

Since $f^{\pm}$ satisfies conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$), it has the following Taylor expansion at $(0,0)$

$$f^{\pm}(x,y,\mu) = A^{\pm}(\mu)(x,y)^T + g^{\pm}(x,y,\mu), \tag{7}$$

where $g^{\pm}(x,y,\mu) = (g_1^{\pm}(x,y,\mu), g_2^{\pm}(x,y,\mu))$ is C^k and $|g^{\pm}(x,y,\mu)| = O(x^2 + y^2)$ as $(x,y) \longrightarrow (0,0)$. Therefore, the piecewise linearization of the system (4) at the origin is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^{+}(\mu)(x,y)^T; & (x,y)^T \in S^{+} \\ A^{-}(\mu)(x,y)^T; & (x,y)^T \in S^{-} \end{cases}. \tag{8}$$

In order to study the nonlinear system (4), it is necessary to first investigate the piecewise linear system (8). Now, consider the linearized system (8) and assume that

($\mathbf{H_3}$) the matrix $A^{\pm}$ has a pair of complex eigenvalues $\alpha^{\pm}(\mu) \pm i\beta^{\pm}(\mu)$ with $\beta^{\pm}(\mu) > 0$.

In this paper, we investigate orbits that transversally cross the discontinuity boundary Σ. If

$$A^{\pm}(\mu) = \begin{pmatrix} a_{11}^{\pm}(\mu) & a_{12}^{\pm}(\mu) \\ a_{21}^{\pm}(\mu) & a_{22}^{\pm}(\mu) \end{pmatrix}, \tag{9}$$

then the condition

($\mathbf{H_4}$) $a_{12}^{\pm}(\mu_0) > 0$ (or $a_{12}^{\pm}(\mu_0) < 0$) for some $\mu_0 \in M$,

implies that the flow of system (4) crosses Σ transversally and clockwise (or counter-clockwise), see [45].

Finally, we assume that

$(\mathbf{H_5})$

$$\frac{\alpha^-(\mu_0)}{\beta^-(\mu_0)} + \frac{\alpha^+(\mu_0)}{\beta^+(\mu_0)} = 0, \tag{10}$$

and

$$\frac{\partial}{\partial\mu}\left(\frac{\alpha^-(\mu)}{\beta^-(\mu)} + \frac{\alpha^+(\mu)}{\beta^+(\mu)}\right)|_{\mu=\mu_0} \neq 0. \tag{11}$$

Now we state the following theorem which there is in [45], to investigate the generalized Hopf bifurcation of non-smooth system (4).

Theorem 2.1. *Consider the non-smooth system* (4) *and assume* ($\boldsymbol{H_1}$)-($\boldsymbol{H_5}$) *holds for that. Then, generalized Hopf bifurcation occurs as the parameter* μ *passes through* μ_0*. In this case, at* $\mu = \mu_0$ *there bifurcates a continuous branch of periodic orbits from the stationary point* $(0,0)$*; i.e., there exists a constant* $\delta_0 > 0$ *and a uniquely determined continuous function* $\mu^* : (-\delta_0, \delta_0) \longrightarrow \mathbb{R}$ *satisfying* $\mu^*(0) = \mu_0$*, such that for each* $y \in (-\delta_0, \delta_0)$ *there is a periodic orbit of system* (4) *crosses* $(0, y)$ *at the parameter* $\mu = \mu^*(y)$ *with the period* $\tilde{T}(y, \mu^*(y))$*. The function* $\tilde{T}$ *is continuous and satisfies*

$$\tilde{T}(0,\mu_0) = \frac{\pi}{\beta^-(\mu_0)} + \frac{\pi}{\beta^+(\mu_0)}. \tag{12}$$

Moreover, there is no other periodic orbit of system (4) *locally near* $(0,0)$ *at* $\mu = \mu_0$*.*

Proof. see [45]. □

Theorem 2.2. *Suppose that,*

$$B(\mu) = \exp\left[\pi\left(\frac{\alpha^-(\mu)}{\beta^-(\mu)} + \frac{\alpha^+(\mu)}{\beta^+(\mu)}\right)\right]. \tag{13}$$

Then, bifurcating periodic orbit through $(0, y, \mu)$ *is asymptotically stable from the interior if* $|B(\mu)| < 1$ *and unstable if* $|B(\mu)| > 1$*.*

Proof. see [45]. □

3. Generalized Hopf bifurcation of a harmonic oscillator with a non-smooth velocity-dependent damping

Any real harmonic oscillator deals with different kinds of frictions and external forces, as it interacts with its environment. These frictions and external forces impose positive and negative damping on the system. Hence in this section, we study the dynamical behavior of a non-smooth harmonic oscillator which is subjected to a damping force. For this purpose, consider a system of damped harmonic oscillator and without loss of generality suppose that the system has the mass $m = 1$. Due to the Newton's 2nd Law, the equation of the motion of this system can be stated by the following mathematical model

$$\ddot{x} + \omega^2 x + \epsilon f_{damping} = 0, \tag{14}$$

where $\epsilon^{\pm}$ is a small parameter and ω is the natural frequency of harmonic oscillator. Moreover, let the damping force $f_{damping}$ depend on the relative velocity $v_{rel} = \dot{x} - v_0$, i.e., $f_{damping} = f(v_{rel})$. Then, system (14) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 x - \epsilon f(v_{rel}). \end{cases} \tag{15}$$

It should be mentioned that, for a nonlinear damping force $f(v_{rel})$ proportional to the relative velocity and for $v_0 = 0$, we have

$$f(v_{rel}) = \begin{cases} a_n \dot{x}^n, & \text{if } n \text{ is odd,} \\ a_n |\dot{x}^{n-1}| \dot{x}, & \text{if } n \text{ is even,} \end{cases} \tag{16}$$

for more details see [12]. Now, we assume that the velocity dependent damping force $f^{\pm}(v_{rel})$, in system (15), is an analytic function in $U^{\pm} \subseteq \mathbb{R}^2$ which contains 0. Furthermore $f^{\pm}(0) = 0$, as with no motion there is no damping. Also for $v_0 = 0$, it has the following Taylor expansions about 0:

$$f(v_{rel}) \; = \; a_1 y + a_3 y^3 + a_5 y^5 + \mathcal{O}(y^7). \tag{17}$$

In general, damping is positive and the energy of the system is always absorbed by dampers. However, if the system receives energy from some source, the amplitude of its vibrations will increase by the time. In this case, the system will lead to an unstable state and we say that it is negatively damped; for more information see [41]. For instance, suspension bridges under the action of uniform wind flow at critical speeds are negatively damped systems, [41].
Note that, in equation (17) if $a_i > 0$, then $f(v_{rel})$ incurs a positive damping on the system (15). Whereas for $a_i < 0$, the system (15) exhibits a negative damping.

Now, suppose that the mentioned damped oscillator vibrates in two different environments and enters from one environment to another. In this case, it experiences different kinds of frictions and external forces, and so different kinds of damping forces, in each environment. Thus, the equation of motion of this system can be stated as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 x - \epsilon^{\pm} f^{\pm}(v_{rel}), \end{cases} \qquad for \quad \pm x > 0, \tag{18}$$

where

(19) $$f^{\pm}(v_{rel}) = \left[a_1^{\pm}y + a_3^{\pm}y^3 + a_5^{\pm}y^5 + \mathcal{O}(y^7)\right].$$

Furthermore due to relations (1)-(3) in section 2, the sets S^+, S^- and Σ for system (18) are defined by the following relations :

(20) $$S^- = \{(x,y)^T \in \mathbb{R}^2;\ H(x,y) = x < 0\},$$

(21) $$\Sigma = \{(x,y)^T \in \mathbb{R}^2;\ H(x,y) = x = 0\},$$

(22) $$S^+ = \{(x,y)^T \in \mathbb{R}^2;\ H(x,y) = x > 0\}.$$

Now by means of theorem 2.1, we will present a detailed mathematical analysis for the system (18).

Theorem 3.1. *Consider the system* (18) *and let for* $\epsilon^{\pm} \neq 0$*, the following assumptions are given*

(23) $$\left[\epsilon^- a_1^-\right]\left[\epsilon^+ a_1^+\right] < 0, \quad \left|\epsilon^- a_1^-\right| \neq \left|\epsilon^+ a_1^+\right|$$

Then at $\omega = 0$*, the generalized Hopf bifurcation occurs for system* (18)*. In this case, at* $\omega = 0$ *there bifurcates a continuous branch of periodic orbits from the origin; i.e., there is a constant* $\delta_0 > 0$ *and an uniquely determined continuous function* $\omega^* : (-\delta_0, \delta_0) \longrightarrow \mathbb{R}$ *with* $\omega^*(0) = 0$*, such that for each* $y \in (-\delta_0, \delta_0)$ *there is a periodic orbit of system* (18) *crosses* $(0, y)$ *at the parameter* $\omega = \omega^*(y)$ *with the period* $\tilde{T}(y, \omega^*(y))$*. Moreover,* $\tilde{T}$ *is continuous and satisfies*

(24) $$\tilde{T}(0,0) = \frac{2\pi}{\epsilon^- a_1^-} + \frac{2\pi}{\epsilon^+ a_1^+}.$$

Moreover, for $\omega = 0$*, there does not exist any other periodic orbit of system* (18) *locally near the origin.*

Proof. It suffices to check the conditions of theorem 2.1 for system (18).

Obviously, condition $(\mathbf{H_1})$ is true for (18). Moreover, for $\epsilon^{\pm} \neq 0$ the Jaconian matrices of the perturbed system (18) in S^+ and S^- are:

(25) $$J^{\pm}(\omega) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\epsilon^{\pm}\dfrac{\partial f^{\pm}(y)}{\partial y} \end{pmatrix}.$$

Due to the assumption $f^{\pm}(0) = 0$, the point $(0, 0, \omega)$ is a stationary point of system (18) for all $\omega \in M$, that is $(\mathbf{H_2})$ holds too.

Furthermore, for $\epsilon^{\pm} \neq 0$, the Jacobian matrix $J^{\pm}(\omega)$ at $(0, 0, \omega)$ has a pair of complex conjugate eigenvalues

(26) $$\alpha^{\pm} \pm i\beta^{\pm}(\omega) = -\frac{1}{2}\epsilon^{\pm}a_1^{\pm} \pm \frac{i}{2}\sqrt{(\epsilon^{\pm}a_1^{\pm})^2 - 4\omega^2},$$

such that $(\epsilon^{\pm}a_1^{\pm})^2 - 4\omega^2 > 0$; thus condition $(\mathbf{H_3})$ is satisfied for (18).
In addition, for $J^{\pm}(\omega)$, the element $a_{12}^{\pm}(\omega) = 1 > 0$, for every $\omega \in M$. So $(\mathbf{H_4})$ holds for every $\omega \in M$. Therefore the flow of system (18) crosses the discontinuity boundary Σ transversally and clockwise.

Now the first part of $(\mathbf{H_5})$ is true for $\omega = \omega_0$, if and only if

$$\frac{\epsilon^- a_1^-}{\epsilon^+ a_1^+} = -\sqrt{\frac{(\epsilon^- a_1^-)^2 - 4\omega_0^2}{(\epsilon^+ a_1^+)^2 - 4\omega_0^2}}. \tag{27}$$

Also

(28)
$$\frac{\partial}{\partial\omega}\left(\frac{\alpha^-}{\beta^-(\omega)} + \frac{\alpha^+}{\beta^+(\omega)}\right)\Bigg|_{\omega=\omega_0} = 2\left[\frac{\epsilon^- a_1^-}{\sqrt{\left((\epsilon^- a_1^-)^2 - 4\omega_0^2\right)^3}} + \frac{\epsilon^+ a_1^+}{\sqrt{\left((\epsilon^+ a_1^+)^2 - 4\omega_0^2\right)^3}}\right],$$

which implies the second part of $(\mathbf{H_5})$ is satisfied for $\omega = \omega_0$ provided that

$$\frac{\epsilon^- a_1^-}{\epsilon^+ a_1^+} \neq -\left[\sqrt{\frac{(\epsilon^- a_1^-)^2 - 4\omega_0^2}{(\epsilon^+ a_1^+)^2 - 4\omega_0^2}}\,\right]^3. \tag{29}$$

Then, condition $(\mathbf{H_5})$ of theorem 2.1 holds for (18) at all stationary points $(0, 0, \omega)$, $\omega \in M$, if both relations (27) and (29) hold; or equivalently relation (27) holds and $\dfrac{\epsilon^- a_1^-}{\epsilon^+ a_1^+} \neq -1, 0$. The last assertion satisfies if and only if $\omega = 0$, $\left|\epsilon^- a_1^-\right| \neq \left|\epsilon^+ a_1^+\right|$, and

$$\begin{cases} \epsilon^- a_1^- > 0, \epsilon^+ a_1^+ < 0, \\ \text{or} \\ \epsilon^- a_1^- < 0, \epsilon^+ a_1^+ > 0, \end{cases} \tag{30}$$

which means condition (23) satisfies. In this case by theorem 2.1 and relation (26), it is easy to see that

$$\tilde{T}(0, 0) = \frac{\pi}{\beta^-(0)} + \frac{\pi}{\beta^+(0)} = \frac{2\pi}{\epsilon^- a_1^-} + \frac{2\pi}{\epsilon^+ a_1^+}. \tag{31}$$

According to theorem 2.1, this completes the proof. □

By the aid of theorem 3.1 we can obtain a parametric region for the existence of generalized Hopf bifurcations for (18). Indeed in this parametric region, periodic oscillations can occur for the damped harmonic oscillator (18).

Proposition 3.2. *Suppose that the perturbed system* (18) *satisfies in the conditions of theorem 3.1, and further*

$$\left[\frac{\epsilon^- a_1^-}{\sqrt{(\epsilon^- a_1^-)^2 - 4\omega^2}} + \frac{\epsilon^+ a_1^+}{\sqrt{(\epsilon^+ a_1^+)^2 - 4\omega^2}}\right] < 0. \tag{32}$$

Then, the bifurcating periodic orbit through $(0, y, \omega)$ *is asymptotically stable from the interior; and if*

$$\left[\frac{\epsilon^- a_1^-}{\sqrt{(\epsilon^- a_1^-)^2 - 4\omega^2}} + \frac{\epsilon^+ a_1^+}{\sqrt{(\epsilon^+ a_1^+)^2 - 4\omega^2}}\right] > 0, \tag{33}$$

it is unstable.

Proof. It can be shown that

$$\pi\left(\frac{\alpha^-(\omega)}{\beta^-(\omega)} + \frac{\alpha^+(\omega)}{\beta^+(\omega)}\right) = \pi\left[\frac{\epsilon^- a_1^-}{\sqrt{(\epsilon^- a_1^-)^2 - 4\omega^2}} + \frac{\epsilon^+ a_1^+}{\sqrt{(\epsilon^+ a_1^+)^2 - 4\omega^2}}\right]. \tag{34}$$

So, by theorem 2.2, the proof is clear. □

4. Numerical simulations

Here, some numerical simulations are performed to investigate the effects of a non-smooth velocity-dependent damping on the vibrations of system (18). These numerical simulations are in good agreement with our theoretical results. Moreover, our theoretical and numerical findings reveal that the non-smooth damped harmonic oscillator (18) can observe different kinds of vibrations.

Example 4.1. *Consider a model which is simulated by a mass* m *that is connected to springs at both ends. Assume that these two springs have the same stiffness, and the mass is oscillating on a surface horizontally under the action of two springs; see figure 1. Suppose further that* x *represents the displacement*

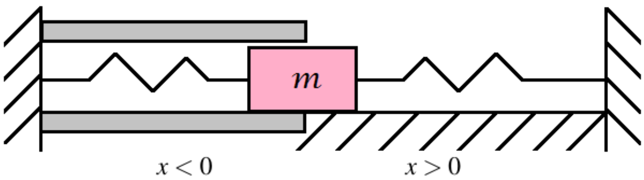


FIGURE 1. A mass m which is controlled by two springs

of the mass, such that $x = 0$ *corresponds to the position that both springs are unloaded. For* $x > 0$*, only the spring on the right acts on the mass such that the friction between the mass and the surface causes a linear damping force* $f^+(v_{rel}) = a_1^+ y$, $a_1^+ > 0$*. And for* $x < 0$*, just the spring on the left takes* *action on the mass. Also let the left surface be smooth enough in the sense that the friction between the mass and the surface can be ignored. Furthermore, suppose that in the left side there is an external force field exciting the mass with an extra force* $f^-(v_{rel}) = a_1^- y, a_1^- < 0$. *In this case, the oscillations of the mass are influenced by a positive damping on the right side and a negative damping on the left. In addition, for* $m = 1$, *the equation of the motions of the mentioned system is*

$$(35) \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 x - \epsilon f(v_{rel}), \end{cases}$$

in which $f^{\pm}(v_{rel}) = a_1^{\pm} y$, *where* $a_1^+ > 0$ *and* $a_1^- < 0$.

Now consider system (35) *and let* $a_1^- = -2, a_1^+ = 1.5, \epsilon^- = 0.1$, *and* $\epsilon^+ = 0.13$. *These values of parameters satisfies in the conditions of theorem 3.1. So by theorem 3.1, at* $\omega = 0$, *the generalized Hopf bifurcation happens for system* (35). *Especially, at* $\omega = 0$ *there bifurcates a continuous branch of periodic orbits from the origin. Using the software Matlab, we have shown that for* $\omega = 1$ *and* $y_1 = -2.35$ *the system has a period-one orbit* $\mathcal{O}_{y_1}$ *crossing* $(0, y_1) = (0, -2.35)$. ***This means in the presence of a non-smooth linear damping, there is a periodic vibration for the harmonic oscillator with the frequency*** $\omega = 1$. *The corresponding periodic orbit in the displacement-velocity coordinates is plotted in figure 2,* (a).

Moreover, the conditions of theorem 3.1 imply that ***for the damping forces*** $f^{\pm}(v_{rel})$***, just the coefficients of linear terms, i.e.*** $a_1^{\pm}$***, are responsible for existence of periodic orbits****. To show this, suppose that a cubic damper is added to the right side of figure 1. Therefore, the damping force in the space* S^+ *changes into* $f^+(v_{rel}) = a_1^+ y + a_3^+ y^3, a_3 > 0$. *Then as shown in figure 2,* (b), ***by adding a cubic term to the damping force (in the space*** S^+***), the periodic orbit*** $\mathcal{O}_{y_1}$ ***will change to a stable fused focus. In this case, the amplitude of the vibration vanishes by the time. Namely the amplitude of the oscillations is reduced when cubic damping is added to the system****. Also if we add another cubic damper to the left side of figure 1, then the amplitude of the vibrations will vanish faster; see figure 2,* (c).

Example 4.2. *Assume that for system* (35), $f^{\pm}(v_{rel}) = a_1^{\pm} y$ *and* $a_1^- = -1, a_1^+ = 0.91, \epsilon^{\pm} = 0.1$. *Then based on theorem 3.1, the generalized Hopf bifurcation will occur for the system at* $\omega = 0$. *Figure 3,* (a) *indicates that for* $\omega = 2$ *and* $y = -4.25$ *there exists a period-one orbit* $\mathcal{O}_{y_2}$ *passing through* $(0, y_2) = (0, -4.25)$. ***This implies the existence of a periodic motion for the damped harmonic oscillator, with the frequency*** $\omega = 2$; *see figure 3.*

Now, suppose that two cubic dampers are added to both left and right sides

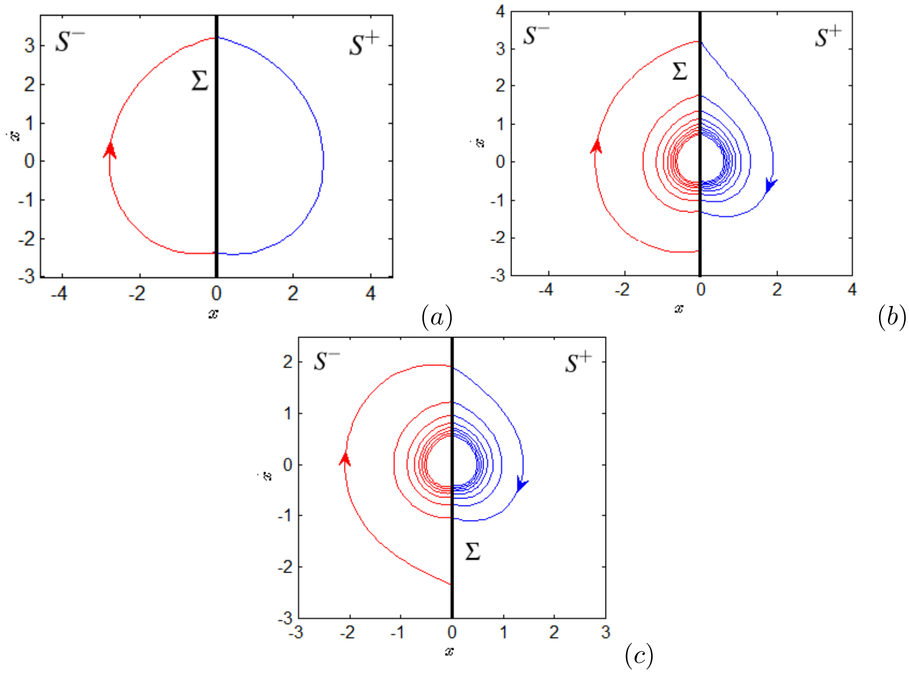


FIGURE 2. Example of a period-one orbit and stable fused focuses of system (35) in the phase space $x - \dot{x}$ with the initial point $X_0 = (0, -2.35)$. The values of parameters are: $a_1^- = -2, a_1^+ = 1.5, \epsilon^- = 0.1, \epsilon^+ = 0.13, \omega = 1$ and (a): $a_3^\pm = 0$; (b): $a_3^- = 0, a_3^+ = 1$; (c): $a_3^- = 1, a_3^+ = 1$

of figure 1. As one can see in figure 3, ***by adding a cubic term to the damping forces*** $f^\pm(v_{rel})$***, the periodic orbit*** $\mathcal{O}_{y_2}$ ***becomes a stable fused focus. That means the amplitude of the vibration tends to zero by passing the time****; see figure 3.*

Example 4.3. *Consider the model of a mass* m *controlled by two springs which was mentioned in example 4.1. Let in both cases* $x < 0$ *and* $x > 0$*, there exist a friction between the mass and the surface which causes a positive linear damping in each case. Hence, the damping forces in* $S^\pm$ *are* $f^\pm(v_{rel}) = a_1^\pm y$*,* $a_1^\pm > 0$*. Assume further that the friction coefficient in* S^- *is bigger than in* S^+*, i.e.,* $a_1^- > a_1^+$*; see figure 4. If* $m = 1$*, and we choose the values of parameters of the system* (35) *as:* $\omega = 2, a_1^- = 2, a_1^+ = 1, \epsilon^\pm = 0.1$*, then numerical findings demonstrate that:*

There is a stable fused focus for system (35)***, in the presence of a non-smooth linear damping which is positive in both spaces*** $S^\pm$*; see*

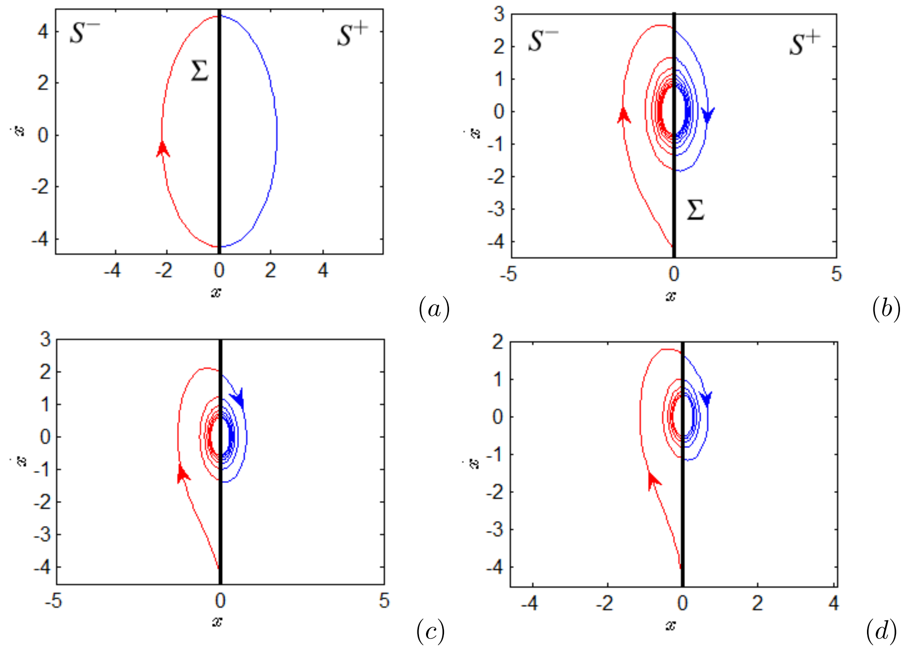


FIGURE 3. Example of a period-one orbit and stable fused focuses of system (35) in the phase space $x-\dot{x}$ with the initial point $X_0 = (0, -4.25)$. The values of parameters are: $a_1^- = -1, a_1^+ = 0.91, \epsilon^{\pm} = 0.1$, $\omega = 2$ and (a): $a_3^{\pm} = 0$; (b): $a_3^{\pm} = 1$; (c): $a_3^{\pm} = 2$; (d): $a_3^{\pm} = 3$.

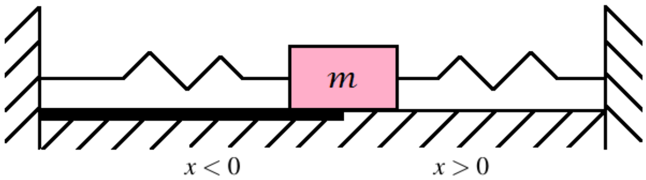
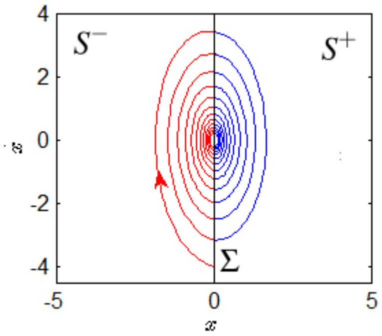
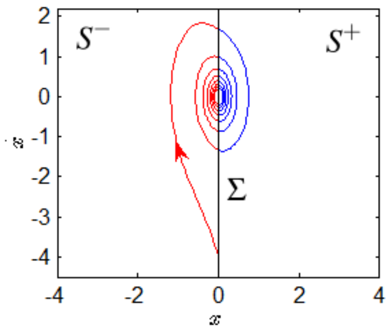


FIGURE 4. A mass m which is controlled by two springs.

figure 3. ***This means there exists a stable behavior in the vibrations of system*** (35) ***(with the frequency*** $\omega = 2$***). In this situation, the amplitude of the oscillations will vanish under the mentioned non-smooth damping. Also adding two cubic dampers in both left and right sides of the mass, i.e. considering cubic damping terms in*** $f^{\pm}(v_{rel})$***, incurs higher damping in the system. Hence, the vibrations of the system decay more quickly****; see figure 5,* (b).



(a)



(b)

FIGURE 5. A stable fused focus of system (35), in the phase space $x-\dot{x}$, corresponding to a stable vibration of the system. The initial point is $X_0 = (0, -4)$ and the values of the parameters are $\omega = 2, a_1^- = 2, a_1^+ = 1, \epsilon^{\pm} = 0.1$ and (a): $a_3^{\pm} = 0$; (b): $a_3^- = 2, a_3^+ = 1$.

Example 4.4. *Consider the oscillatory model described in example 4.1, but suppose that the positions of the left and right surfaces are swapped (figure 6). Then, the vibrations of this model can be stated by system* (35) *where*

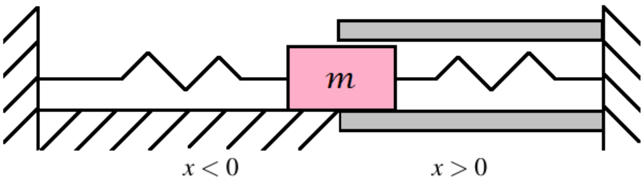
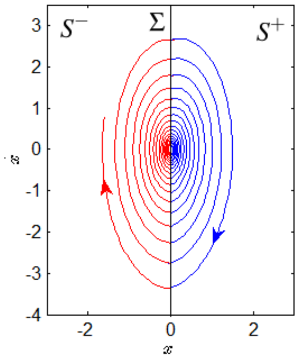


FIGURE 6. A mass m which is controlled by two springs

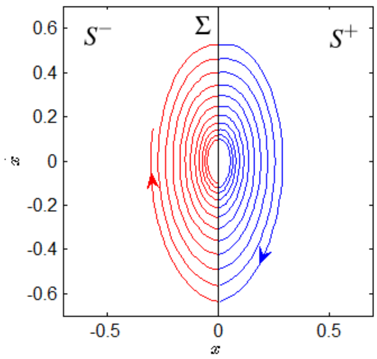
$f^{\pm}(v_{rel}) = a_1^{\pm}y$*, but* $a_1^+ < 0$*, and* $a_1^- > 0$*. If* $\omega = 2, a_1^- = 0.5, a_1^+ = -3, \epsilon^{\pm} = 0.1$*, numerical results display that:*

There is an unstable fused focus for system (35) ***with*** $f^{\pm}(v_{rel}) = a_1^{\pm}y$***,*** $a_1^+ < 0$ ***and*** $a_1^- > 0$*; see figure 7.* ***This implies that there is an unstable vibration (with the frequency*** $\omega = 2$***) such that its amplitude gradually increases by passing the time. In this case, the system may lead to an oscillation with high amplitude which can be dangerous for the structure.*** *Also,* ***adding cubic terms to the damping forces*** $f^{\pm}(v_{rel})$ ***causes the growth of the vibrations' amplitude to occur more slowly*** *(figure 7,* (b)*).*

At last, we point out the results of this section as follows:



(a)



(b)

FIGURE 7. An unstable fused focus of the described system in example 4.4, with $\omega = 2, a^- = 0.5, a^+ = -3, \epsilon^\pm = 0.1$, and the initial point $X_0 = (0, -0.1)$ for (a): $c^\pm = 0$; (b): $c^\pm = 2$.

Results (i) *For $\omega = 1$ and $y_1 = -2.35$ the system (35) has a period-one orbit $\mathcal{O}_{y_1}$ crossing $(0, -2.35)$. That is in the presence of a non-smooth linear damping, there is a periodic vibration for the harmonic oscillator (35) with the frequency $\omega = 1$. Moreover, by adding a cubic term to the damping force (in the space S^+), the periodic orbit $\mathcal{O}_{y_1}$ will change into a stable fused focus. In such a case, the amplitude of the vibration vanishes by the time. Also if we add cubic damping terms to both $f^-(v_{rel})$ and $f^+(v_{rel})$, then the amplitude of the vibrations will vanish faster.*

(ii) *For $\omega = 2$ and $y = -4.25$ there exists a period-one orbit $\mathcal{O}_{y_2}$ for the system (35), passing through $(0, -4.25)$. This implies the existence of a periodic motion for the damped harmonic oscillator (35), with the frequency $\omega = 2$. Furthermore, by adding a cubic term to the damping forces $f^\pm(v_{rel})$, the periodic orbit $\mathcal{O}_{y_2}$ becomes a stable fused focus. Namely the amplitude of the vibration tends to zero by passing the time.*

(iii) *There is a stable fused focus for system (35), in the presence of a non-smooth linear damping which is positive in both spaces $S^\pm$. That is there exists a stable behavior in the vibrations of system (35) (with the frequency $\omega = 2$). In this situation, the amplitude of the oscillations will vanish under the mentioned non-smooth damping. Also, considering cubic damping terms in $f^\pm(v_{rel})$ incurs higher damping in the system. And so the vibrations of the system decay more quickly.*

(iv) *There is an unstable fused focus for system* (35) *with* $f^{\pm}(v_{rel}) = a_1^{\pm}y$, $a_1^{+} < 0$ *and* $a_1^{-} > 0$. *This indicates there is an unstable vibration (with the frequency* $\omega = 2$*) such that its amplitude gradually increases by passing the time. In this case, the system may lead to an oscillation with high amplitude which can be dangerous for the structure. Also, adding cubic terms to the damping forces* $f^{\pm}(v_{rel})$ *causes the growth of the vibrations' amplitude to occur more slowly*

5. Conclusion

In this paper, we analyzed the dynamical behavior a of the harmonic oscillator subjected to a non-smooth damping. In fact using the theory of Filippov systems, we could obtain a parametric region for the existence of generalized Hopf bifurcation for the system (18). Moreover, it has been shown that in this parametric region there are periodic oscillations for (18). To the best of our knowledge, it is the first time to find such a region for the existence of generalized Hopf bifurcation and so the existence of periodic oscillations for the nonlinear damped oscillator (18). Furthermore, some numerical simulations have been performed which are in good agreement with our theoretical results. These numerical investigations explored that system (18) can undergo different kinds of vibrations, in the presence of the non-smooth damping. Here our theoretical and numerical findings can provide useful information for analysing the vibrations of the harmonic oscillators dealing with a non-smooth damping.

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