Journal of Mahani Mathematical Research Center
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ON EXTENDED REAL VALUED QUASI-CONCAVE FUNCTIONS

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Dedicated to sincere professor Mehdi Radjabalipour on turning 75

Article type: Research Article
(Received: 29 July 2021, Accepted: 06 October 2021)
(Communicated by S.N. Hosseini)

Abstract. In this paper, we first study the non-positive decreasing and inverse co-radiant functions defined on a real locally convex topological vector space X. Next, we characterize non-positive increasing, co-radiant and quasi-concave functions over X. In fact, we examine abstract concavity, upper support set and superdifferential of this class of functions by applying a type of duality. Finally, we present abstract concavity of extended real valued increasing, co-radiant and quasi-concave functions.

Keywords: Abstract concavity, Duality, Co-radiant function, Quasi-concave function, Upper support set, Superdifferential.
2020 MSC: 26A51, 26A48,26B25.

1. Introduction

Increasing and co-radiant (ICR) functions are among the main objects in monotonic analysis [12]. Monotonic analysis is one of the advanced topics in so-called abstract convex analysis which is a natural generalization of classical convex analysis. In convex analysis one of the main results asserts that every lower semi-continuous convex function can be expressed as a point-wise supremum of a family of affine functions majorized by it. Many results in convex analysis can easily follow this fact. It is well known that similar results hold in quasi-convex analysis: each lower semi-continuous quasi-convex function can be represented as the upper envelope of a set of quasi-affine functions [6]. Abstract convexity has found many applications in mathematical analysis and optimization problems (see [7,12]). Functions which can be represented as upper envelopes of subsets of a set H of sufficiently simple (elementary) functions, are studied in this theory (for more details see [11,12,14]).

It is well-known that some classes of increasing functions are abstract convex. For example, the class of increasing and positively homogeneous (IPH) functions (see [3,4,9]) and the class of increasing and convex-along-rays (ICAR)

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DOI: 10.22103/jmmrc.2021.17976.1158

How to cite: S. Mirzadeh, S. Bahrami, *On Extended Real Valued Quasi-Concave Functions*, J. Mahani Math. Res. Cent. 2021; 10(2): 163-180.

functions are abstract convex (see [10, 13]). The class of increasing and co-radiant (ICR) functions is another class of increasing functions which is abstract convex. Abstract convexity of ICR functions has been investigated in [1, 2, 5]. Also, abstract concavity of non-negative increasing, co-radiant and quasi-concave functions defined on $\mathbb{R}^n_+$ has been characterized in [7]. Abstract concavity of non-negative increasing, co-radiant and quasi-concave functions are defined on a real locally convex topological vector space which has been investigated in [8]. In this paper, we are going to extend the results obtained in [8] for extended real valued increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space. In fact, we find an infimal generator for this class of functions.

The structure of the paper is as follows: In Section 2, we collect some preliminaries and definitions. In Section 3, we first discuss abstract convexity and abstract concavity of non-positive decreasing and inverse co-radiant functions which are defined on X. Next, we define a type of duality for non-positive function. Moreover, we characterize non-positive increasing, co-radiant and quasi-concave functions over X by applying this duality. Characterizations of the extended real valued increasing, co-radiant and quasi-concave functions and abstract concavity of this class of functions are illustrated in Section 4. Finally, the conclusion is stated in Section 5.

2. **Preliminaries**

Let X be a real locally convex topological vector space with the dual space X^*. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$. In addition, we consider an order relation on X^* by $x_1^* \leq x_2^*$ if and only if $\langle x, x_1^* \rangle \leq \langle x, x_2^* \rangle$ for all $x \in X$. Also, suppose that X^* is equipped with the weak-star topology and let $S^* := \{x^* \in X^* : \ \langle x, x^* \rangle \geq 0, \ for\ all\ x \in S\}$.

The following definitions are well-known (see [12]).

(1) A function $f : X \longrightarrow [-\infty, +\infty]$ is called co-radiant if $f(\lambda x) \geq \lambda f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is easy to see that f is co-radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in X$ and all $\lambda \geq 1$.

(2) The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$.

(3) A function $f : X \longrightarrow [-\infty, +\infty]$ is called inverse co-radiant if $f(\lambda x) \leq \frac{1}{\lambda} f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is easy to see that f is inverse co-radiant if $f(\lambda x) \geq \frac{1}{\lambda} f(x)$ for all $x \in X$ and all $\lambda \geq 1$.

Remark 2.1. Let $f : X \longrightarrow [-\infty, 0]$ be an ICR (increasing and co-radiant) function. Then it is clear that $f(x) = 0$ for all $x \in S$.

Definition 2.2. ([12]) Let X be a non-empty set, L be a non-empty set of functions $l : X \longrightarrow [-\infty, +\infty]$ and $f : X \longrightarrow [-\infty, +\infty]$ be a function.

(1) The upper support set of f with respect to L is defined by

$$supp_u(f, L) := \{l \in L : l(x) \geq f(x), \ \forall\ x \in X\}.$$

(2) The function f is called abstract concave with respect to L (or L-concave) if there exists a subset $\triangle$ of L such that

$$f(x) = \inf_{l \in \triangle} l(x), \ \forall \ x \in X.$$

(3) Let $x_0 \in X$ be such that $-\infty < f(x_0) < +\infty$. The superdifferential of the function f at x_0 with respect to L (or L-superdifferential of f) is defined by

$$\partial_L^+ f(x_0) := \{l \in L : \ l(x_0) \in \mathbb{R}, \ l(x) - l(x_0) \geq f(x) - f(x_0), \ \forall \ x \in X\}.$$

The set L in Definition 2.2 is called the set of elementary functions.

Definition 2.3. ([8], Definition 4.1) Let $\alpha \in \mathbb{R}_+$ be arbitrary and consider a function $f : X \longrightarrow [0, +\infty]$. Then the α-dual function $f_\alpha^* : X^* \longrightarrow [0, +\infty]$ of f is defined on X^* by

$$f_\alpha^*(x^*) := \sup\{f(x) : x \in X, \ \langle x, x^* \rangle + \alpha \leq 1\},$$

(we use the conventions $\sup \emptyset = 0$ and $f^* := f_0^*$).

Now, consider the function $h : X \times S^* \times \{0, 2\} \times \mathbb{R}_{++} \longrightarrow [0, +\infty]$ is defined by:

$$h(x, y^*, \alpha, \beta) := \inf\{q_{(y^*,\beta)}(x^*) \ : \ x^* \in S^*, \ \langle x, x^* \rangle + \alpha \leq 1\},$$

where $q_{(y^*,\beta)}(x^*) := \inf\{\lambda : \lambda \geq \beta, \ \lambda x^* \geq y^*\}$, (we use the convention $\inf \emptyset = +\infty$). This function was introduced and examined in [8].
Let $y^* \in S^*$, $\alpha \in \{0, 2\}$ and $\beta \in \mathbb{R}_{++}$ be arbitrary. Define the function $h_{(y^*,\alpha,\beta)} : X \longrightarrow [0, +\infty]$ by $h_{(y^*,\alpha,\beta)}(x) := h(x, y^*, \alpha, \beta)$ for all $x \in X$. Also, let $H := \{h_{(y^*,\alpha,\beta)} : y^* \in S^*, \ \alpha \in \{0, 2\}, \ \beta \in \mathbb{R}_{++}\}$ be the set of elementary functions.

Consider the set U_{iq}^+ of all non-negative increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space X. We recall the following results from [8].

Proposition 2.4. *([8], Proposition 5.1) Let $(y^*, \alpha, \beta) \in S^* \times \{0, 2\} \times \mathbb{R}_{++}$ be arbitrary. Then the function $h_{(y^*,\alpha,\beta)} : X \longrightarrow [0, +\infty]$ is in U_{iq}^+.*

Remark 2.5. ([8], Remark 5.1) The function $h_{(y^*,\alpha,\beta)}$ has a simpler form. Let $x \in X$, $y^* \in S^*$, $\alpha \in \{0, 2\}$ and $\beta \in \mathbb{R}_{++}$ be arbitrary. If $\alpha = 0$, then $h_{(y^*,0,\beta)}(x) = \max\{\langle x, y^* \rangle, \beta\}$, and if $\alpha = 2$, then

$$h_{(y^*,2,\beta)}(x) = \begin{cases} \beta, & \langle x, \frac{y^*}{\beta} \rangle \leq -1, \\ +\infty, & \langle x, \frac{y^*}{\beta} \rangle > -1. \end{cases}$$

Theorem 2.6. *([8], Theorem 5.1) Let $f : X \longrightarrow [0, +\infty]$ be an upper semi-continuous function. Then, f is a function in U_{iq}^+ if and only if there exists a non-empty set $A \subseteq S^* \times \{0, 2\} \times \mathbb{R}_{++}$ such that*

$$f(x) = \inf_{(y^*,\alpha,\beta) \in A} h_{(y^*,\alpha,\beta)}(x), \ (x \in X).$$

In this case, one can take $A := \{(y^*, \alpha, \beta) \in S^* \times \{0, 2\} \times \mathbb{R}_{++} : f_\alpha^*(\frac{y^*}{\beta}) \leq \beta\}$*. Hence,* $f \in U_{iq}^+$ *if and only if* f *is* H*-concave.*

Proposition 2.7. *([8], Proposition 5.2) Let* $f : X \longrightarrow [0, +\infty]$ *be a co-radiant function. Then*

$$\text{supp}\ _u(f, H) = \{h_{(y^*,\alpha,\beta)} \in H \ : \ f_\alpha^*(\frac{y^*}{\beta}) \leq \beta\}.$$

3. Abstract Concavity of Non-positive Increasing Co-radiant and Quasi-concave Functions

Some definitions related to the abstract convexity have been introduced in [12]. Consider the set U_{iq}^- of all non-positive increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space X. In this section, we first discuss abstract convexity and abstract concavity of non-positive decreasing and inverse co-radiant functions. Next, we define a type of duality for non-positive functions. Finally, we characterize functions in U_{iq}^- by applying this duality.

Consider the function $u : X \times X \times \mathbb{R}_{--} \longrightarrow [-\infty, 0]$ is defined by

$$u(x, y, \alpha) := \sup\{\lambda : \alpha \leq \lambda \leq 0, \ -\lambda x \leq y\},$$

for all $x, y \in X$ and all $\alpha < 0$ (we use the convention $\sup \emptyset = -\infty$).
Assume that $y \in X$ and $\alpha \in \mathbb{R}_{--}$ are arbitrary. Consider the function $u_{(y,\alpha)} : X \longrightarrow [-\infty, 0]$ is defined by $u_{(y,\alpha)}(x) = u(x, y, \alpha)$ for all $x \in X$. Also, let $U := \{u_{(y,\alpha)} : y \in X, \ \alpha \in \mathbb{R}_{--}\}$ be the set of elementary functions.

We can also introduce the function $v : X \times X \times \mathbb{R}_{--} \longrightarrow [-\infty, 0]$ is defined by

$$v(x, y, \beta) := \inf\{\lambda : \lambda \leq \beta, \ -\lambda x \geq y\},$$

for all $x, y \in X$ and all $\beta < 0$ (with the convention $\inf \emptyset = 0$).
Let $y \in X$ and $\beta \in \mathbb{R}_{--}$ be arbitrary. Consider the function $v_{(y,\beta)} : X \longrightarrow [-\infty, 0]$ is defined by $v_{(y,\beta)}(x) = v(x, y, \beta)$ for all $x \in X$. Also, let $V := \{v_{(y,\beta)} : y \in X, \ \beta \in \mathbb{R}_{--}\}$ be the set of elementary functions.

Remark 3.1. It is easy to see that the functions $u_{(y,\alpha)}$ and $v_{(y,\beta)}$ are non-positive decreasing and inverse co-radiant for all $y \in X$ and all $\alpha, \beta \in \mathbb{R}_{--}$. And also, the functions $u_{(y,\alpha)}$ and $v_{(y,\beta)}$ have a relation as p and q in Remark 3.2 in [8].

The proofs of the following theorems are similar to the ones from Theorems 3.2 and 3.3 in [8].

Theorem 3.2. *Let* $g : X \longrightarrow [-\infty, 0]$ *be a function. Then the following assertions are equivalent:*

(1) g *is decreasing and inverse co-radiant.*
(2) $-\lambda g(y) \leq g(x)$ *for all* $x, \ y \in X$ *and all* $\lambda \in [-1, 0)$ *such that* $-\lambda x \leq y$*.*

(3) $\alpha g(x) \leq u_{(y,\alpha)}(x)g(\frac{-y}{\alpha})$ *for all* $x,\ y \in X$ *and all* $\alpha \in \mathbb{R}_{--}$*, with the convention* $0 \times (-\infty) = +\infty$.
(4) $u_{(x,\frac{1}{\alpha})}(y)g(x) \geq \frac{1}{\alpha}g(\frac{-y}{\alpha})$ *for all* $x,\ y \in X$ *and all* $\alpha \in \mathbb{R}_{--}$*, with the convention* $0 \times (-\infty) = +\infty$.
(5) $v_{(y,\alpha)}(x)g(\frac{-y}{\alpha}) \leq \alpha g(x)$ *for all* $x,\ y \in X$ *and all* $\alpha \in \mathbb{R}_{--}$*, with* $-\infty < v(x, y, \beta) < 0$.

Theorem 3.3. *Let* $g : X \longrightarrow [-\infty, 0]$ *be a function. Then*
(1) *g is decreasing and inverse co-radiant if and only if it is V-concave.*
(2) *g is decreasing and inverse co-radiant if and only if it is U-convex.*

Remark 3.4. Note that, if we consider functions $u_{(y^*,\alpha)}$ and $v_{(y^*,\beta)}$ on X^*. Let $U := \{u_{(y^*,\alpha)} : y^* \in X^*,\ \alpha \in \mathbb{R}_{--}\}$ and $V := \{v_{(y^*,\beta)} : y^* \in X^*,\ \beta \in \mathbb{R}_{--}\}$. Then we get the of all obtained results in the above theorems for each decreasing and inverse co-radiant function defined on X^*. Also, it is worth noting that we can easily see $v_{(y^*,\beta)}(x^*) > -\infty$ for all $x^*, y^* \in X^*$ and all $\beta < 0$.

In the following, we define a type of duality for non-positive functions over X.

Definition 3.5. $f : X \longrightarrow [-\infty, 0]$ be a function. Then dual function $f^\sharp : X^* \longrightarrow [-\infty, 0]$ of f is defined on X^* by

$$f^\sharp(x^*) := \sup\{f(x) : x \in X,\ \langle x, x^*\rangle \geq 1\},\ \forall\ x^* \in X^*, \tag{1}$$

(we use the convention $\sup \emptyset = -\infty$).

Remark 3.6. It follows directly from (1) that $f^\sharp : X^* \longrightarrow [-\infty, 0]$ is a decreasing and quasi-convex function for an arbitrary function $f : X \longrightarrow [-\infty, 0]$.

Theorem 3.7. *Let* $f : X \longrightarrow [-\infty, 0]$ *be a co-radiant function. Then* $f^\sharp$ *is an inverse co-radiant.*

Proof. Let $\lambda \in (0, 1]$ and $x^* \in X^*$ be arbitrary, then

$$\begin{aligned} f^\sharp(\lambda x^*) &= \sup\{f(x) : x \in X,\ \langle x, \lambda x^*\rangle \geq 1\} \\ &= \sup\{f(x) : x \in X,\ \langle \lambda x, x^*\rangle \geq 1\} \\ &\leq \sup\{\frac{f(\lambda x)}{\lambda} : x \in X,\ \langle \lambda x, x^*\rangle \geq 1\} \\ &= \frac{1}{\lambda}\sup\{f(y) : y \in X,\ \langle y, x^*\rangle \geq 1\} \\ &= \frac{1}{\lambda} f^\sharp(x^*). \end{aligned}$$

So, $f^\sharp$ is inverse co-radiant. □

In the end of this section, we discuss on abstract concavity with respect to a certain class of functions $f \in U^-_{iq}$. We also, present a description of upper support set and superdifferential of a non-positive co-radiant function.

Consider the function $k : X \times (-S^*) \times \mathbb{R}_{--} \longrightarrow [-\infty, 0]$ is defined by:

$$k(x, y^*, \beta) := \inf\{v_{(y^*,\beta)}(x^*) \ : \ x^* \in (-S^*), \ \langle x, x^* \rangle \geq 1\},$$

for all $x \in X$, all $y^* \in (-S^*)$ and all $\beta < 0$, (we use the convention $\inf \emptyset = 0$). Let $y^* \in -S^*$, $\beta \in \mathbb{R}_{--}$ be arbitrary. Define the function $k_{(y^*,\beta)} : X \longrightarrow [-\infty, 0]$ by $k_{(y^*,\beta)}(x) := k(x, y^*, \beta)$ for all $x \in X$.

Proposition 3.8. *Let $(y^*, \beta) \in (-S^*) \times \mathbb{R}_{--}$ be arbitrary. Then, $k_{(y^*,\beta)} : X \longrightarrow [-\infty, 0]$ is in U^-_{iq}.*

Proof. Let $x_1, x_2 \in X$ be such that $x_1 \leq x_2$. Then, if $x^* \in -S^*$, we have $\{x^* \in -S^*, \ \langle x_2, x^* \rangle \geq 1\} \subseteq \{x^* \in -S^*, \ \langle x_1, x^* \rangle \geq 1\}$. So,

$$\begin{aligned} k_{(y^*,\beta)}(x_1) &= \inf\{v_{(y^*,\beta)}(x^*) \ : \ x^* \in -S^*, \ \langle x_1, x^* \rangle \geq 1\} \\ &\leq \inf\{v_{(y^*,\beta)}(x^*) \ : \ x^* \in -S^*, \ \langle x_2, x^* \rangle \geq 1\} \\ &= k_{(y^*,\beta)}(x_2). \end{aligned}$$

By taking recourse to similar argument shown in Proposition 5.1 in [8], we observe that $k_{(y^*,\beta)}$ is a co-radiant and quasi-concave function. □

Remark 3.9. In fact, the function $k_{(y^*,\beta)}$ has a simpler form. Let $x \in X$, $y^* \in -S^*$ and $\beta \in \mathbb{R}_{--}$ be arbitrary. Then

$$k_{(y^*,\beta)}(x) = \begin{cases} \langle x, -y^* \rangle, & -\langle x, y^* \rangle \leq \beta, \\ 0, & \langle x, -y^* \rangle \geq \beta. \end{cases}$$

Because, if $\langle x, -y^* \rangle \leq \beta$, then $\langle x, -y^* \rangle < 0$. By putting $x_0^* := \frac{-y^*}{\langle x, -y^* \rangle} \in -S^*$ and $\lambda_0 := \langle x, -y^* \rangle \leq \beta$, we get that $\langle x, x_0^* \rangle \geq 1$ and $-\lambda_0 x_0^* \geq y^*$. Hence, by (1) one has

$$k_{(y^*,\beta)}(x) \leq \lambda_0 = \langle x, -y^* \rangle. \tag{2}$$

On the other hand, if there exists $x^* \in -S^*$ and $\lambda \leq \beta$ such that $\langle x, x^* \rangle \geq 1$ and $-\lambda x^* \geq y^*$. Then, $\langle x, y^* \rangle \geq -\lambda \langle x, x^* \rangle$ and $-\lambda \langle x, x^* \rangle \geq -\lambda$. So, $\langle x, y^* \rangle \geq -\lambda$. Therefore, $\langle x, -y^* \rangle \leq \lambda$, this together with (1) implies that

$$\langle x, -y^* \rangle \leq k_{(y^*,\beta)}(x). \tag{3}$$

Thus by (2) and (3) we have $k_{(y^*,\beta)}(x) = \langle x, -y^* \rangle$. Now, suppose that $\langle x, -y^* \rangle > \beta$. Let $A_x := \{x^* \in -S^* \ : \ \langle x, x^* \rangle \geq 1\}$ and $A^{x^*}_{y^*,\beta} := \{\lambda : \lambda \leq \beta, \ -\lambda x^* \geq y^*\}$. Then, we obtain that $A_x = \emptyset$ or $A^{x^*}_{y^*,\beta} = \emptyset$ for each $x^* \in A_x$. Hence, $k_{(y^*,\beta)}(x) = 0$.

Example 3.10. *Let $X := \mathbb{R}^2$ and $S := \mathbb{R}^2_+$. Let $x \in \mathbb{R}^2$ be arbitrary and put $y^* = (0, -1)$ and $\beta = -2$. Then by Remark 3.9, we have*

$$k_{(y^*,\beta)}(x) = \begin{cases} x_2, & x_2 \leq -2, \\ 0, & x_2 > -2, \end{cases}$$

for each $x = (x_1, x_2) \in \mathbb{R}^2$.

Example 3.11. *Let* $X := C([0,1])$ *be the Banach space of all real valued continuous functions defined on* $[0,1]$ *and* $S := \{f \in X : f(x) \geq 0, \forall x \in [0,1]\}$*. Then,* S *is a closed convex pointed cone in* X*. For each* $t_0 \in [0,1]$*, consider the function* $L_{t_0} : X \longrightarrow \mathbb{R}$ *is defined by* $L_{t_0}(f) := f(t_0)$ *for all* $f \in X$*, Clearly* $L_{t_0} \in S^*$*. Now, let* $y^* := -L_{t_0}$*,* $\beta := -1$*. Then by Remark 3.9, one has*

$$k_{(y^*,\beta)}(f) = \begin{cases} f(t_0), & f(t_0) \leq -1, \\ 0, & f(t_0) > -1, \end{cases}$$

for each $f \in X$*.*

Let $K := \{k_{(y^*,\beta)} : y^* \in (-S^*),\ \beta \in \mathbb{R}_{--}\}$ be the set of elementary functions. We now show that the set K is an infimal generator for the set of all functions $f \in U_{iq}^-$.

Theorem 3.12. *Let* $f : X \longrightarrow [-\infty, 0]$ *be an upper semi-continuous function. Then,* $f \in U_{iq}^-$ *if and only if there exists a non-empty set* $B \subsetneq (-S^*) \times \mathbb{R}_{--}$ *such that*

$$f(x) = \inf_{(y^*,\beta)\in B} k_{(y^*,\beta)}(x), \ (x \in X). \tag{4}$$

In this case, one can take $B := \{(y^*,\beta) \in (-S^*) \times \mathbb{R}_{--} : f^\sharp(\frac{-y^*}{\beta}) \leq \beta\}$*. Hence,* $f \in U_{iq}^-$ *if and only if* f *is* K*-concave.*

Proof. First, suppose that f has a representation of the form (4). Then, f is the point-wise infimum of a family of functions $\{k_{(y^*,\beta)}\}_{(y^*,\beta)\in B}$, which by Proposition 3.8 they are non-positive increasing, co-radiant and quasi-concave. Therefore, $f \in U_{iq}^-$.
To prove the converse, since f is co-radiant, then by Remark 3.6 and Theorem 3.7, we have $f^\sharp$ is a non-positive decreasing and inverse co-radiant function. Thus by Remark 3.4 and Theorem 3.2(5), we obtain that

$$v_{(y^*,\beta)}(x^*) f^\sharp(\frac{-y^*}{\beta}) \leq \beta f^\sharp(x^*)$$

for all $x^*, y^* \in X^*$ and all $\beta \in \mathbb{R}_{--}$, with $v_{(y^*,\beta)}(x^*) < 0$. Let $x \in X$ be arbitrary. So, by definition of $k_{(y^*,\beta)}$ and (1) one has

$$\begin{aligned} k_{(y^*,\beta)}(x) f^\sharp(\frac{-y^*}{\beta}) &= \inf\{v_{(y^*,\beta)}(x^*) \ : \ x^* \in -S^*,\ \langle x, x^*\rangle \geq 1\} f^\sharp(\frac{-y^*}{\beta}) \\ &\leq \beta \inf\{f^\sharp(x^*) \ : \ x^* \in -S^*,\ \langle x, x^*\rangle \geq 1\} \\ &= \beta \inf\{\sup\{f(t) \ : \ t \in X,\ \langle t, x^*\rangle \geq 1\} \ : \ x^* \in -S^*, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \langle x, x^*\rangle \geq 1\} \\ &\leq \beta f(x). \end{aligned}$$

Now, let $(y^*, \beta) \in B$ be arbitrary. Hence, it follows from the above inequality, $k_{(y^*,\beta)}(x) \geq f(x)$ (note that if, for some $x^* \in -S^*$ with $\langle x, x^*\rangle \geq 1$, one has

$v_{(y^*,\beta)}(x^*) = 0$, then it is easy to see that the value of the infimum in the above inequality does not change). Therefore

$$k_{(y^*,\beta)}(x) \geq f(x), \quad \forall\ (y^*,\beta) \in B. \tag{5}$$

We now consider two possible cases for x :
Case (i). Let $-\infty \leq f(x) < 0$. Since f is an ICR function, then by Remark 2.1 we have $f(0) = 0$. This implies that $f(x) < f(0) = 0$. Now, let $m \in \mathbb{R}_{--}$ be arbitrary such that $f(x) < m \leq f(0)$. Therefore, one has $f^{-1}([m, 0])$ is a non-empty subset of X which does not contain x. Since f is quasi-concave and upper semi-continuous, so this set is convex, closed and does not contain x. Thus by Hahn-Banach Theorem there exist $0 \neq x^* \in X^*$ and $0 \neq \gamma \in \mathbb{R}$ such that

$$\langle x, x^*\rangle > \gamma > \langle y, x^*\rangle, \ \forall \ \ y \in f^{-1}([m, 0]). \tag{6}$$

From $f^{-1}([m, 0])$ is a non-empty subset of X and the fact that f is increasing, one can derive, using a standard argument, that $x^* \in -S^*$. Since $0 \in f^{-1}([m, 0])$, then (6) implies that $\gamma > 0$ and

$$\langle x, \frac{1}{\gamma}x^*\rangle > 1 > \langle y, \frac{1}{\gamma}x^*\rangle, \quad \forall\ y \in f^{-1}([m, 0]). \tag{7}$$

Now, putting $y_0^* := \frac{-m}{\gamma}x^* \in -S^*$ and $\beta_0 = m$. Thus by (7) one has

$$\begin{aligned}
f^\sharp(\frac{-y_0^*}{\beta_0}) &= \sup\{f(t)\ :\ \langle t, \frac{-1}{\beta_0}y_0^*\rangle \geq 1\} \\
&= \sup\{f(t)\ :\ \langle t, \frac{1}{\gamma}x^*\rangle \geq 1\} \\
&\leq \sup\{f(t)\ :\ t \notin f^{-1}([m, 0])\} \\
&\leq m \\
&= \beta_0,
\end{aligned}$$

and hence $(y_0^*, \beta_0) \in B$. Also, with regard to Remark 3.9 and (7) we have $k_{(y_0^*,\beta_0)}(x) = -\langle x, y_0^*\rangle = \langle x, \frac{m}{\gamma}x^*\rangle \leq m = \beta_0$. Then by (5) one has

$$\begin{aligned}
f(x) = \inf_{m > f(x)} m &\geq \inf_{(y_0^*,\beta_0)} k_{(y_0^*,\beta_0)}(x) \\
&\geq \inf_{(y^*,\beta) \in B} k_{(y^*,\beta)}(x) \\
&\geq f(x).
\end{aligned}$$

So,

$$f(x) = \inf_{(y^*,\beta) \in B} k_{(y^*,\beta)}(x).$$

Case (ii). Let $f(x) = 0$. So, by (5) we get

$$f(x) = 0 = \inf_{(y^*,\beta) \in B} k_{(y^*,\beta)}(x),$$

which completes the proof. □

In the sequel, we characterize the upper support set and K-superdifferential of a non-positive co-radiant function. The proof of the following theorem is similar to the one from Theorem 5.2 in [8].

Proposition 3.13. *Let $f : X \longrightarrow [-\infty, 0]$ be a co-radiant function. Then*

$$supp_u(f, K) = \{k_{(y^*,\beta)} \in K \ : \ f^\sharp(\frac{-y^*}{\beta}) \leq \beta\}.$$

Theorem 3.14. *Let $f : X \longrightarrow [-\infty, 0]$ be a co-radiant function and $x_0 \in X$ be such that $f(x_0) \neq 0, -\infty$. Then*

$$\{k_{(y^*,\beta)} \in K \ : \ f^\sharp(\frac{-y^*}{\beta}) \leq \beta, \ k_{(y^*,\beta)}(x_0) = f(x_0)\} \subseteq \partial_K^+ f(x_0).$$

Proof. This is an immediate consequence of Proposition 3.13. □

Example 3.15. *Let $X := \mathbb{R}$ and $S := \mathbb{R}_+$. Consider the function $f : X \longrightarrow (-\infty, 0]$ is defined by*

$$f(x) = \begin{cases} x^3, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

for all $x \in \mathbb{R}$. Clearly, f is a co-radiant function. It is easy to see that

$$\begin{aligned} \text{supp }_u(f, K) &= \{k_{(y^*,\beta)} \in K : f^\sharp(\frac{-y^*}{\beta}) \leq \beta\} \\ &= \{k_{(y^*,\beta)} \in K : \ y^* = 0, \ \beta < 0\} \\ &\cup \ \{k_{(y^*,\beta)} \in K : \ 0 > y^* \geq -\beta^{\frac{2}{3}}\}. \end{aligned}$$

Now, let $x_0 \in \mathbb{R}$ be such that $f(x_0) \neq 0$, i.e. $x_0 < 0$. Put

$$O_{x_0} := \{k_{(y^*,\beta)} \in K \ : \ f^\sharp(\frac{-y^*}{\beta}) \leq \beta, \ k_{(y^*,\beta)}(x_0) = f(x_0)\}.$$

In view of Theorem 3.14, one has $O_{x_0} \subseteq \partial_K^+ f(x_0)$. Indeed, it is not difficult to see that

$$O_{x_0} = \{k_{(y^*,\beta)} \in K : \ x_0^2 = -y^*, \ 0 > y^* \geq -\beta^{\frac{2}{3}}\}, \ \forall \ x_0 < 0.$$

4. Abstract Concavity of Extended Real Valued Increasing Co-radiant and Quasi-concave Functions

Consider the set U_{iq} of all extended real valued increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space X. In this section, we are going to extend the results obtained in section 3, for a function $f : X \longrightarrow [-\infty, +\infty]$, where f is in U_{iq}. Also, in this section we assume that $f(0) = 0$.

Let $f : X \longrightarrow [-\infty, +\infty]$ be a function. We recall the positive and negative part of function f as follows $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \min\{f(x), 0\}$ for all $x \in X$. It is clear that $f^+(x) + f^-(x) = f(x)$ for all $x \in X$.

Remark 4.1. It is easy to see that f is an increasing, co-radiant and quasi-concave function if and only if f^+ and f^- are increasing, co-radiant and quasi-concave functions.

Now, consider the function $T : X \times S^* \times (-S^*) \times \{0, 2\} \times \mathbb{R}_{++} \times \mathbb{R}_{--} \longrightarrow [-\infty, +\infty]$ defined by

$$T(x, y^*, z^*, \alpha, \beta, \beta') := \begin{cases} h_{(y^*,\alpha,\beta)}(x), & x \in S \\ k_{(z^*,\beta')}(x), & x \notin S, \end{cases} \tag{8}$$

for all $x \in X$, all $y^*, -z^* \in S^*$, all $\alpha \in \{0, 2\}$, all $\beta \in \mathbb{R}_{++}$and all $\beta' \in \mathbb{R}_{--}$.

We are going to introduce a class of elementary functions such that the extended real valued increasing, co-radiant and quasi-concave functions are infimally generated. Let $(y^*, z^*, \alpha, \beta, \beta') \in S^* \times (-S^*) \times \{0, 2\} \times \mathbb{R}_{++} \times \mathbb{R}_{--}$ be arbitrary. Define the function $T_{(y^*,z^*,\alpha,\beta,\beta')} : X \longrightarrow [-\infty, +\infty]$ by $T_{(y^*,z^*,\alpha,\beta,\beta')}(x) := T(x, y^*, z^*, \alpha, \beta, \beta')$ for all $x \in X$.

Proposition 4.2. *Let $(y^*, z^*, \alpha, \beta, \beta') \in S^* \times (-S^*) \times \{0, 2\} \times \mathbb{R}_{++} \times \mathbb{R}_{--}$ be arbitrary. Then, the function $T_{(y^*,z^*,\alpha,\beta,\beta')} : X \longrightarrow [-\infty, +\infty]$ is in U_{iq}.*

Proof. Let $x_1, x_2 \in X$ be such that $x_1 \leq x_2$. If $x_1 \in S$, then, since S is a convex cone, it follows that $x_2 \in S$. So, by (8) and Proposition 2.4, we have $T_{(y^*,z^*,\alpha,\beta,\beta')}(x_1) = h_{(y^*,\alpha,\beta)}(x_1) \leq h_{(y^*,\alpha,\beta)}(x_2) = T_{(y^*,z^*,\alpha,\beta,\beta')}(x_2)$. Suppose that $x_1 \notin S$. If $x_2 \notin S$, thus by (8) and Proposition 1 we get $T_{(y^*,z^*,\alpha,\beta,\beta')}(x_1) = k_{(z^*,\beta')}(x_1) \leq k_{(z^*,\beta')}(x_2) = T_{(y^*,z^*,\alpha,\beta,\beta')}(x_2)$. If $x_2 \in S$, then one has

$$T_{(y^*,z^*,\alpha,\beta,\beta')}(x_2) = h_{(y^*,\alpha,\beta)}(x_2) \geq 0 \geq k_{(z^*,\beta')}(x_1) = T_{(y^*,z^*,\alpha,\beta,\beta')}(x_1).$$

Therefore, the function $T_{(y^*,z^*,\alpha,\beta,\beta')}$ is increasing.

Now, let $\gamma \in (0, 1]$ and $x \in X$ be arbitrary. Since S is a cone, we conclude that $\gamma x \in S$ whenever $x \in S$. If $x \notin S$, then $\gamma x \notin S$. So, by (8) and Propositions 2.4 and 1, we get $T_{(y^*,z^*,\alpha,\beta,\beta')}(\gamma x) \geq \gamma T_{(y^*,z^*,\alpha,\beta,\beta')}(x)$. Therefore, $T_{(y^*,z^*,\alpha,\beta,\beta')}$ is a co-radiant function.

Finally, we prove that $T_{(y^*,z^*,\alpha,\beta,\beta')}$ is quasi-concave. Let $\lambda \in [0, 1]$ and $x, y \in X$ be arbitrary. We consider four possible cases.

Case (i). If $x, y \in S$, then, since S is a convex cone, it follows that $\lambda x+(1-\lambda)y \in S$. So, by (8) and Proposition 2.4, we get

$$\begin{aligned} T_{(y^*,z^*,\alpha,\beta,\beta')}(\lambda x + (1 - \lambda)y) &= h_{(y^*,\alpha,\beta)}(\lambda x + (1 - \lambda)y) \\ &\geq \min\{h_{(y^*,\alpha,\beta)}(x), h_{(y^*,\alpha,\beta)}(y)\} \\ &= \min\{T_{(y^*,z^*,\alpha,\beta,\beta')}(x), T_{(y^*,z^*,\alpha,\beta,\beta')}(y)\}. \end{aligned}$$

Case (ii). Suppose that $x, y \notin S$. If $\lambda x + (1 - \lambda)y \notin S$, then by (8) and Proposition 3.8, we have

$$\begin{aligned} T_{(y^*,z^*,\alpha,\beta,\beta')}(\lambda x + (1 - \lambda)y) &= k_{(z^*,\beta')}(\lambda x + (1 - \lambda)y) \\ &\geq \min\{k_{(z^*,\beta')}(x), k_{(z^*,\beta')}(y)\} \\ &= \min\{T_{(y^*,z^*,\alpha,\beta,\beta')}(x), T_{(y^*,z^*,\alpha,\beta,\beta')}(y)\}. \end{aligned}$$

If $\lambda x + (1-\lambda)y \in S$, hence by (8) we conclude that

$$\begin{aligned} T_{(y^*,z^*,\alpha,\beta,\beta')}(\lambda x + (1-\lambda)y) &= h_{(y^*,\alpha,\beta)}(\lambda x + (1-\lambda)y) \\ &\geq 0 \\ &\geq \min\{k_{(z^*,\beta')}(x), k_{(z^*,\beta')}(y)\} \\ &= \min\{T_{(y^*,z^*,\alpha,\beta,\beta')}(x), T_{(y^*,z^*,\alpha,\beta,\beta')}(y)\}. \end{aligned}$$

Case (iii). If $x \in S$ and $y \notin S$. Suppose that $\lambda x + (1-\lambda)y \in S$. So by (8) one has

$$\begin{aligned} T_{(y^*,z^*,\alpha,\beta,\beta')}(\lambda x + (1-\lambda)y) &= h_{(y^*,\alpha,\beta)}(\lambda x + (1-\lambda)y) \\ &\geq 0 \\ &\geq k_{(z^*,\beta')}(y) \\ &= \min\{h_{(y^*,\alpha,\beta)}(x), k_{(z^*,\beta')}(y)\} \\ &= \min\{T_{(y^*,z^*,\alpha,\beta,\beta')}(x), T_{(y^*,z^*,\alpha,\beta,\beta')}(y)\}. \end{aligned}$$

If $\lambda x + (1-\lambda)y \notin S$, then by (8), Proposition 3.8 and Remark 2.1 we get

$$\begin{aligned} T_{(y^*,z^*,\alpha,\beta,\beta')}(\lambda x + (1-\lambda)y) &= k_{(z^*,\beta')}(\lambda x + (1-\lambda)y) \\ &\geq \min\{k_{(z^*,\beta')}(x), k_{(z^*,\beta')}(y)\} \\ &= \min\{0, k_{(z^*,\beta')}(y)\} \\ &= k_{(z^*,\beta')}(y) \\ &= \min\{k_{(z^*,\beta')}(y), h_{(y^*,\alpha,\beta)}(x)\} \\ &= \min\{T_{(y^*,z^*,\alpha,\beta,\beta')}(x), T_{(y^*,z^*,\alpha,\beta,\beta')}(y)\}. \end{aligned}$$

Case (iv). If $x \notin S$ and $y \in S$. The proof of this case is similar to the case (iii), and therefore we omit it. Hence the proof is complete. □

In the sequel, define the set Δ by

(9) $\quad \Delta := \{T_{(y^*,z^*,\alpha,\beta,\beta')} : y^*, -z^* \in S^*, \quad \alpha \in \{0,2\}, \ \beta \in \mathbb{R}_{++}, \ \beta' \in \mathbb{R}_{--}\}.$

The set Δ is called a set of elementary functions defined by (8). In view of Proposition 4.2, we have each element of Δ is an increasing, co-radiant and quasi-concave function.

In the following, we characterize the upper support set of an extended real valued co-radiant function.

Theorem 4.3. *Let $f : X \longrightarrow [-\infty, +\infty]$ be a co-radiant function such that $f(x) \leq 0$ for all $x \in X \setminus S$. Then*

$$\text{supp}_{\,u}(f, \Delta) = \Omega,$$

where Ω defined as follows

$$\Omega := \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta : (f^+)^*_\alpha(\frac{y^*}{\beta}) \leq \beta, \ (f^-)^\sharp(\frac{-z^*}{\beta'}) \leq \beta'\}.$$

Notice that the set Δ is defined by (9).

Proof. Let $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Omega$ be arbitrary. So, $(f^+)^*_\alpha(\frac{y^*}{\beta}) \le \beta$ and $(f^-)^\sharp(\frac{-z^*}{\beta'}) \le \beta'$. Since f^+ and f^- are co-radiant functions, hence by Proposition 2.7 and Proposition 3.13, we have

$$h_{(y^*,\alpha,\beta)}(x) \ge f^+(x),\ \forall x \in X \tag{10}$$

and

$$k_{(z^*,\beta')}(x) \ge f^-(x),\ \forall x \in X. \tag{11}$$

Now, if $x \in S$, then (8) and (10) implies that $T_{(y^*,z^*,\alpha,\beta,\beta')}(x) = h_{(y^*,\alpha,\beta)}(x) \ge f^+(x) \ge f(x)$. If $x \notin S$, so by hypothesis $f(x) \le 0$. Hence $f^-(x) = f(x)$. It follows from (8) and (11) that $T_{(y^*,z^*,\alpha,\beta,\beta')}(x) = k_{(z^*,\beta')}(x) \ge f^-(x) = f(x)$. Therefore

$$T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \ge f(x),\ \forall x \in X.$$

So, $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \text{supp}\ _u(f,\Delta)$. For the converse, let $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \text{supp}\ _u(f,\Delta)$ be arbitrary. Thus,

$$T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \ge f(x), \quad \forall x \in X. \tag{12}$$

If $(f^+)^*_\alpha(\frac{y^*}{\beta}) = 0$, then it is clear that $(f^+)^*_\alpha(\frac{y^*}{\beta}) \le \beta$. Suppose that $(f^+)^*_\alpha(\frac{y^*}{\beta}) > 0$. Hence by (8), (12) and Remark 2.5, one has

$$\begin{aligned}
(f^+)^*_\alpha(\frac{y^*}{\beta}) &= \sup\{f^+(t) \ :\ \langle t, \frac{1}{\beta}y^*\rangle + \alpha \le 1\} \\
&= \sup\{f(t) \ :\ \langle t, \frac{1}{\beta}y^*\rangle + \alpha \le 1\} \\
&\le \sup\{T_{(y^*,z^*,\alpha,\beta,\beta')}(t) \ :\ \langle t, \frac{1}{\beta}y^*\rangle + \alpha \le 1\} \\
&\le \sup\{h_{(y^*,\alpha,\beta)}(t) \ :\ \langle t, \frac{1}{\beta}y^*\rangle + \alpha \le 1\} \\
&= \beta.
\end{aligned}$$

Also, since $f^- \le f$, it follows from (8), (12) and Remark 3.9 that

$$
\begin{aligned}
(f^-)^\sharp(\frac{-z^*}{\beta'}) &= \sup\{f^-(t) \ : \ \langle t, \frac{-1}{\beta'}z^*\rangle \geq 1\} \\
&\leq \sup\{f(t) \ : \ \langle t, \frac{-1}{\beta'}z^*\rangle \geq 1\} \\
&\leq \sup\{T_{(y^*,z^*,\alpha,\beta,\beta')}(t) \ : \ \langle t, \frac{-1}{\beta'}z^*\rangle \geq 1\} \\
&\leq \sup\{T_{(y^*,z^*,\alpha,\beta,\beta')}(t) \ : \ t \notin S, \ \langle t, \frac{-1}{\beta'}z^*\rangle \geq 1\} \\
&= \sup\{k_{(z^*,\beta')}(t) \ : \ \langle t, \frac{-1}{\beta'}z^*\rangle \geq 1\} \\
&= \sup\{\langle t, -z^*\rangle \ : \ \langle t, -z^*\rangle \leq \beta'\} \\
&\leq \beta'.
\end{aligned}
$$

Therefore $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Omega$. This completes the proof. □

In the following, we show that each function in U_{iq} is a Δ-concave function. Note that, if f is an upper semi-continuous function, then f^+ and f^- are upper semi-continuous functions.

Theorem 4.4. *Let $f : X \longrightarrow [-\infty, +\infty]$ be an upper semi-continuous function such that $f(x) \leq 0$ for all $x \in X \setminus S$. Then, f is a function in U_{iq} if and only if there exists a non-empty set $\Omega \subseteq \Delta$ such that*

(13) $$f(x) = \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x), \quad \forall x \in X.$$

In this case, one can take

$$\Omega := \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta : (f^+)^*_\alpha(\frac{y^*}{\beta}) \leq \beta, \ (f^-)^\sharp(\frac{-z^*}{\beta'}) \leq \beta'\}.$$

Proof. We only show that every extended real valued function in U_{iq} is Δ-concave. Let $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Omega$ be arbitrary. Then, in view of Theorem 4.3 we conclude that supp $_u(f, \Delta) = \Omega$. So, $T_{(y^*,z^*,\alpha,\beta,\beta')} \geq f$ on X. Therefore

(14) $$\inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \geq f(x), \ (x \in X).$$

Now, let $x \in X$ be fixed. We consider five possible cases.
Case (i). $0 < f(x) < +\infty$ and the point x is a global maximum of f^+. Let $y_0^* := 0$, $\alpha_0 := 0$, $\beta_0 := f(x)$, $z_0^* := 0$ and $\beta_0' := -1$. Thus,

$$
\begin{aligned}
(f^+)^*_{\alpha_0}(\frac{y_0^*}{\beta_0}) &= \sup\{f^+(t) : t \in X, \ \langle t, \frac{y_0^*}{\beta_0}\rangle + \alpha_0 \leq 1\} \\
&= \sup\{f^+(t) \ : \ t \in X\} = f^+(x) = f(x) = \beta_0
\end{aligned}
$$

and

$$(f^-)^\sharp(\frac{-z_0^*}{\beta_0'}) = \sup\{f^-(t) \ : \ \langle t, \frac{-1}{\beta_0'}z_0^*\rangle \geq 1\} = \ sup \ \emptyset = -\infty < -1 = \beta_0'.$$

So, $T_{(y_0^*,z_0^*,\alpha_0,\beta_0,\beta_0')} \in \Omega$. Since $0 < f(x) < +\infty$, hence $x \in S$. Then by (8) and Remark 2.5, we conclude that $T_{(y_0^*,z_0^*,\alpha_0,\beta_0,\beta_0')}(x) = h_{(y_0^*,\alpha_0,\beta_0)}(x) = \max(0,\beta_0) = \beta_0 = f(x)$. This together with (14) implies (13).
Case (ii). $f(x) = 0$ and the point x is a global maximum of f^+. Let $y_1^* := 0$, $\alpha_1 := 0$, $z_1^* := 0$, $\beta_1' := -1$ and $\beta_1 > 0$ be arbitrary. Therefore,

$$\begin{aligned}(f^+)^*_{\alpha_1}(\frac{y_1^*}{\beta_1}) &= \sup\{f^+(t) : t \in X,\ \langle t, \frac{y_1^*}{\beta_1}\rangle + \alpha_1 \leq 1\}\\ &= \sup\{f^+(t)\ :\ t \in X\} = f^+(x) = f(x) = 0 < \beta_1\end{aligned}$$

and

$$(f^-)^\sharp(\frac{-z_1^*}{\beta_1'}) = \sup\{f^-(t)\ :\ \langle t, \frac{-1}{\beta_1'}z_1^*\rangle \geq 1\} = \ sup\ \emptyset = -\infty < -1 = \beta_1'.$$

So, $T_{(y_1^*,z_1^*,\alpha_1,\beta_1,\beta_1')} \in \Omega$ for all $\beta_1 > 0$. Now, if $x \notin S$, then by (8) and Remark 3.9, we get $T_{(y_1^*,z_1^*,\alpha_1,\beta_1,\beta_1')}(x) = k_{(z_1^*,\beta_1')}(x) = 0 = f(x)$. This together with (14) implies (13). If $x \in S$. In view of (8) and Remark 2.5, one has $T_{(y_1^*,z_1^*,\alpha_1,\beta_1,\beta_1')}(x) = h_{(y_1^*,\alpha_1,\beta_1)}(x) = \max(0,\beta_1) = \beta_1$. Thus by (14) we conclude that

$$\begin{aligned}f(x) = 0 = \inf_{\beta_1>0}\beta_1 &= \inf_{(y_1^*,,z_1^*,\alpha_1,\beta_1,\beta_1')} T_{(y_1^*,z_1^*,\alpha_1,\beta_1,\beta_1')}(x)\\ &\geq \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \geq f(x).\end{aligned}$$

Hence (13) holds.
Case (iii). $0 \leq f(x) < +\infty$ and the point x is not a global maximum of f^+. So, $0 \leq f^+(x) < +\infty$. Then in view of the proof of Theorem 5.1(case (iii)) in [8], we conclude that there exists $y_2^* \in S^*$, $\alpha_2 \in \{0,2\}$ and $\beta_2 \in \mathbb{R}_{++}$ such that

$$(f^+)^*_{\alpha_2}(\frac{y_2^*}{\beta_2}) \leq \beta_2 \tag{15}$$

and

$$f(x) = f^+(x) = \inf_{(y_2^*,\alpha_2,\beta_2)} h_{(y_2^*,\alpha_2,\beta_2)}(x). \tag{16}$$

Now, putting $z_2^* := 0$ and $\beta_2' = -1$. Thus, one has $(f^-)^\sharp(\frac{-z_2^*}{\beta_2'}) = \ sup\ \emptyset = -\infty < -1 = \beta_2'$. This together with (15) implies $T_{(y_2^*,z_2^*,\alpha_2,\beta_2,\beta_2')} \in \Omega$. If $x \in S$, then by (8), (14) and (16) we get

$$\begin{aligned}f(x) &= \inf_{(y_2^*,\alpha_2,\beta_2)} h_{(y_2^*,\alpha_2,\beta_2)}(x) = \inf_{(y_2^*,z_2^*,\alpha_2,\beta_2,\beta_2')} T_{(y_2^*,z_2^*,\alpha_2,\beta_2,\beta_2')}(x)\\ &\geq \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \geq f(x).\end{aligned}$$

So, (13) holds. If $x \notin S$, then by (8), (14) and Remark 3.9 we have

$$T_{(y_2^*,z_2^*,\alpha_2,\beta_2,\beta_2')}(x) = k_{(z_2^*,\beta_2')}(x) = 0 \leq f(x) \leq \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x).$$

Hence (13) holds.

Case (iv). Let $-\infty \leq f(x) < 0$. Then $f^-(x) = f(x)$. So, in view of the proof of Theorem 3.12(case (i)), there exist $z_3^* \in -S^*$ and $\beta_3' \in \mathbb{R}_{--}$ such that

$$(f^-)^\sharp(\frac{-z_3^*}{\beta_3'}) \leq \beta_3' \tag{17}$$

and

$$f(x) = f^-(x) \geq \inf_{(z_3^*,\beta_3')} k_{(z_3^*,\beta_3')}(x). \tag{18}$$

Set $y_3^* := 0$, $\alpha_3 := 2$, and $\beta_3 := 1$. Thus, we obtain

$$(f^+)^*_{\alpha_3}(\frac{y_3^*}{\beta_3}) = \sup\{f^+(t) : t \in X,\ \langle t, \frac{y_3^*}{\beta_3}\rangle \leq -1\} = \ sup\ \emptyset = 0 < 1 = \beta_3.$$

This together with (17) implies $T_{(y_3^*,z_3^*,\alpha_3,\beta_3,\beta_3')} \in \Omega$. Since $-\infty \leq f(x) < 0$, hence $x \notin S$. Then by (8), (14) and (18) we conclude that

$$\begin{aligned} f(x) \geq \inf_{(z_3^*,\beta_3')} k_{(z_3^*,\beta_3')}(x) &= \inf_{(y_3^*,z_3^*,\alpha_3,\beta_3,\beta_3')} T_{(y_3^*,z_3^*,\alpha_3,\beta_3,\beta_3')}(x) \\ &\geq \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \geq f(x). \end{aligned}$$

Therefore (13) holds.
Case (v). Finally, assume that $f(x) = +\infty$. So, by (14) we get

$$f(x) = +\infty = \inf_{\Omega} T_{(y^*,z^*,\alpha,\beta,\beta')}(x).$$

This completes the proof. □

In the final part of this section, we present a characterization for the Δ-superdifferential of an extended real valued co-radiant function.

Theorem 4.5. *Let $f : X \longrightarrow [-\infty, +\infty]$ be a co-radiant function such that $f(x) \leq 0$ for all $x \in X \setminus S$ and $x_0 \in X$ be such that $f(x_0) \neq -\infty, 0, +\infty$. Then*

$$\begin{aligned} O_{x_0} := \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta \quad : \quad & (f^+)^*_\alpha(\frac{y^*}{\beta}) \leq \beta,\ (f^-)^\sharp(\frac{-z^*}{\beta'}) \leq \beta', \\ & T_{(y^*,z^*,\alpha,\beta,\beta')}(x_0) = f(x_0)\} \subseteq \partial^+_\Delta f(x_0). \end{aligned}$$

Proof. Let $T_{(y^*,z^*,\alpha,\beta,\beta')} \in O_{x_0}$ be arbitrary. So, Theorem 4.3 implies that $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \text{supp }_u(f, \Delta)$. Thus $T_{(y^*,z^*,\alpha,\beta,\beta')}(x) \geq f(x)$ for all $x \in X$. Since $T_{(y^*,z^*,\alpha,\beta,\beta')}(x_0) = f(x_0)$. Therefore $T_{(y^*,z^*,\alpha,\beta,\beta')} \in \partial^+_\Delta f(x_0)$, which completes the proof. □

Corollary 4.6. *Let $f : X \longrightarrow [-\infty, +\infty]$ be a co-radiant function such that $f(x) \leq 0$ for all $x \in X \setminus S$ and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Suppose that f has a global maximum at x_0. Then, $\partial^+_\Delta f(x_0) \neq \emptyset$.*

Proof. Since $f(0) = 0$, $f(x_0) \neq 0, +\infty$ and f has a global maximum at x_0. So, $0 < f(x_0) < +\infty$ and the point x_0 is a global maximum of f^+. Then by the proof of Theorem 4.4(case (i)), there exist $(y_0^*, z_0^*, \alpha_0, \beta_0, \beta_0') \in S^* \times (-S^*) \times \{0, 2\} \times \mathbb{R}_{++} \times \mathbb{R}_{--}$ such that $(f^+)^*_{\alpha_0}(\frac{y_0^*}{\beta_0}) \leq \beta_0$, $(f^-)^\sharp(\frac{-z_0^*}{\beta_0'}) \leq \beta_0'$ and

$T_{(y_0^*,z_0^*,\alpha_0,\beta_0,\beta_0')}(x_0) = f(x_0)$. Hence, according to Theorem 4.5, we conclude that $T_{(y_0^*,z_0^*,\alpha_0,\beta_0,\beta_0')} \in \partial_\Delta^+ f(x_0)$. □

Example 4.7. *Let $X := \mathbb{R}$ and $S := \mathbb{R}_+$. Consider the function $f : X \longrightarrow \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ x^3, & x < 0, \end{cases}$$

for all $x \in \mathbb{R}$. It is clear that f is a co-radiant function. It is not difficult to check that

$$\begin{aligned} \text{supp }_u(f,\Delta) &= \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta : (f^+)^*_\alpha(\frac{y^*}{\beta}) \leq \beta,\ (f^-)^\sharp(\frac{-z^*}{\beta'}) \leq \beta'\} \\ &= \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 0,\ y^* \geq \frac{1}{\beta},\ -(\beta')^{\frac{2}{3}} \leq z^* < 0\} \\ &\cup\ \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 0,\ y^* \geq \frac{1}{\beta},\ \beta' < 0,\ z^* = 0\} \\ &\cup\ \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 2,\ y^* \in S^*, \\ &\qquad \beta > 0,\ -(\beta')^{\frac{2}{3}} \leq z^* < 0\} \\ &\cup\ \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 2,\ y^* \in S^*, \\ &\qquad \beta > 0,\ \beta' < 0,\ z^* = 0\}. \end{aligned}$$

Now, let $x_0 \in \mathbb{R}$ be such that $f(x_0) \neq 0$, i.e. $x_0 \neq 0$. Put

$$\begin{aligned} O_{x_0} := \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta \quad : \quad & (f^+)^*_\alpha(\frac{y^*}{\beta}) \leq \beta,\ (f^-)^\sharp(\frac{-z^*}{\beta'}) \leq \beta', \\ & T_{(y^*,z^*,\alpha,\beta,\beta')}(x_0) = f(x_0)\}. \end{aligned}$$

In view of Theorem 4.5, one has $O_{x_0} \subseteq \partial_\Delta^+ f(x_0)$. Indeed, it is not difficult to see that

$$\begin{aligned} O_{x_0} &= \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta\ :\ \alpha = 0,\ \beta = \sqrt{x_0},\ y^* \geq \frac{1}{\beta},\ -(\beta')^{\frac{2}{3}} \leq z^* < 0\} \\ &\cup\ \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 0,\ \beta = \sqrt{x_0},\ y^* \geq \frac{1}{\beta},\ \beta' < 0,\ z^* = 0\}, \end{aligned}$$

for all $x_0 > 0$, and

$$\begin{aligned} O_{x_0} &= \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 0,\ y^* \geq \frac{1}{\beta},\ x_0^2 = -z^*,\ -(\beta')^{\frac{2}{3}} \leq z^* < 0\} \\ &\cup\ \{T_{(y^*,z^*,\alpha,\beta,\beta')} \in \Delta :\ \alpha = 2,\ y^* \in S^*,\ \beta > 0, \\ &\qquad x_0^2 = -z^*,\ -(\beta')^{\frac{2}{3}} \leq z^* < 0\}, \end{aligned}$$

for all $x_0 < 0$.

5. Conclusion

In this paper, we found an infimal generator for the set U_{iq} of all extended real valued increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space X. We also obtained the upper support set of this class of functions. As an application of our results (will appear in the future research) we characterize the minimal elements of the upper support set of this class of functions, and by using these characterizations, we give the necessary and sufficient conditions for the global minimum of the difference of two extended real valued increasing, co-radiant and quasi-concave functions defined on a real locally convex topological vector space.

6. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

References

[1] M.H. Daryaei and H. Mohebi, *Abstract convexity of extended real valud ICR functions*, Optimization, 62(6) (2013), 835-855.

[2] A.R. Doagooei and H. Mohebi, *Monotonic Analysis over ordered topological vector spaces: IV*, Journal of Global Optimization, 45(3) (2009), 355-369.

[3] J. Dutta, J.E. Martínez-Legaz and A.M. Rubinov, *Monotonic analysis over cones: I,* Optimization, 53(2) (2004), 165-177.

[4] J. Dutta, J.E. Martínez-Legaz and A.M. Rubinov, *Monotonic analysis over cones: II,* Optimization, 53(5-6) (2004), 529-547.

[5] J. Dutta, J.E. Martínez-Legaz and A.M. Rubinov, *Monotonic analysis over cones: III,* Journal of Convex Analysis, 15(3) (2008), 581-592.

[6] J.E. Martínez-Legaz, *Quasi-convex duality theory by generalized conjugation methods,* Optimization, 19(5) (1988), 603-652.

[7] J.E. Martínez-Legaz, A.M. Rubinov and S. Schaible, *Increasing quasi-concave co-radiant functions with applications in mathematical economics,* Mathematical Methods in Operation Research, 61(2) (2005), 261-280.

[8] S. Mirzadeh and H. Mohebi, *Abstract concavity of increasing co-radiant and quasi-concave functions with applications in mathematical economics,* Journal of Optimization Theory and Applications, 169(2) (2016), 443- 472.

[9] H. Mohebi and H. Sadeghi, *Monotonic analysis over ordered topological vector spaces: I,* Optimization, 56(3) (2007), 305-321.

[10] H. Mohebi and H. Sadeghi, *Monotonic analysis over ordered topological vector spaces: II,* Optimization, 58(2) (2009), 241-249.

[11] R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[12] A.M. Rubinov, *Abstract convexity and global optimization,* Kluwer Academic Publishers, Dordrecht-Boston-London, 2000.

[13] A.M. Rubinov and B.M. Glover, *Increasing convex-along-rays functions with application to global optimization*, Journal of Optimization Theory and Applications, 102(3) (1999), 615-642.

[14] I. Singer, *Abstract convex analysis*, Wiley-Interscience, 1997.

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