



## SOME CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF UNIVALENT FUNCTIONS INVOLVING PASCAL DISTRIBUTION SERIES

R. VALIZADE, SH. NAJAFZADEH\*, A. RAHIMI, AND B. DARABY

Article type: Research Article

(Received: 12 May 2021, Accepted: 19 November 2021)

(Communicated by S.N. Hosseini)

**ABSTRACT.** The main object of this paper is to define a new class of univalent functions and two subclasses of this class along with the Pascal distribution associated with convolution and subordination structures. We obtained a number of useful properties such as, coefficient bound, convolution preserving and some other geometric properties.

*Keywords:* Univalent function, Pascal distribution, Subordination.

*2020 MSC:* 30C45, 30C50.

### 1. Introduction

Let  $\mathcal{A}$  denote the family of functions  $f$  of the type

$$(1) \quad f(z) = z + \sum_{k=2}^{+\infty} a_k z^k,$$

which are analytic in the open unit disk

$$(2) \quad \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Also  $\mathcal{N}$  be the main subclass of  $\mathcal{A}$  consisting the functions of the type

$$(3) \quad f(z) = z - \sum_{k=2}^{+\infty} a_k z^k, \quad (a_k \geq 0, z \in \mathbb{U}).$$

See [2]. For parameters  $r, p$  and  $k \in \{0, 1, 2, \dots\}$  we consider a non-negative discrete random variable  $X$  with a Pascal probability generating function

$$(4) \quad P(x = k) = \binom{k+r-1}{r-1} p^k (1-p)^r.$$

Now we consider a power series whose coefficients are probabilities of the Pascal distribution as follows:

$$(\mathcal{P}_p^r)^r(z) = z + \sum_{k=2}^{+\infty} \binom{k+r-2}{r-1} p^{k-1} (1-p)^r z^k, \quad (r \geq 1, 0 \leq p \leq 1, z \in \mathbb{C}).$$

\*Corresponding author, ORCID: 0000-0002-8124-8344

E-mail: najafzadeh1234@yahoo.ie

DOI: 10.22103/jmmrc.2021.17574.1149

How to cite: R. Valizadeh, S. Najafzadeh, A. Rahimi and B. Daraby, *Some Connections Between Various Subclasses of Univalent Functions Involving Pascal Distribution Series*, J. Mahani Math. Res. Cent. 2022; 11(1): 35-42.



© the Authors

It is easy to see that by using ration test, the radius of convergence of the power series given in (5) is infinity. For more details see [1, 4–6].

For  $f$  given by (1) and  $g(z) = z + \sum_{k=2}^{+\infty} b_k z^k$ , the Hadamard product (or convolution) of  $f$  and  $g$  denote by  $f * g$  is defined

$$(6) \quad (f * g) = z + \sum_{k=2}^{+\infty} a_k b_k z^k = (g * f)(z).$$

Now we consider the function

$$(7) \quad \mathcal{P}_f(z) = [(2z - \mathcal{P}_p^r(z)) * f](z),$$

where  $\mathcal{P}_p^r(z)$  and  $f$  given by (5) and (3) respectively.

A function  $f \in \mathcal{N}$  is a member of  $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$  if

$$(8) \quad \left| \frac{z(\mathcal{P}_f(z))^n}{z\lambda(\mathcal{P}_f(z))' - \alpha(1+\beta)\lambda} \right| < \gamma,$$

where  $\alpha, \beta, \gamma \in [0, 1)$  and  $\mathcal{P}_f(z)$  is given by (7).

## 2. Main Results

In this section we obtain a sharp coefficient bound for functions in the class  $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ . Also convolution preserving properties are investigated.

**Theorem 2.1.** *Let  $f \in \mathbb{N}$ , then  $f \in Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$  if and only if*

$$(9) \quad \sum_{k=2}^{+\infty} k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k \leq \lambda\gamma(2-\alpha(1+\beta)).$$

The result is sharp for the function

$$(10) \quad F(z) = z - \frac{\lambda\gamma(2-\alpha(1+\beta))}{2r(1+2\lambda\gamma)p(1-p)^r} z^2.$$

*Proof.* Let the inequality (9) holds true and suppose  $z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ . Then we obtain

$$\begin{aligned} & |z(\mathcal{P}_f(z))'' - \gamma| - \gamma |2\lambda(\mathcal{P}_f(z))' - \alpha(1+\beta)\lambda| \\ &= \left| -\sum_{k=2}^{+\infty} k(k-1) \binom{k+r-\lambda}{r-1} p^{k-1} (1-p)^r a_k z^{k-1} \right| \\ & \quad - \gamma \left| 2\lambda - 2\lambda \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k z^{k-1} - \alpha(1+\beta)\lambda \right| \\ &= \sum_{k=2}^{+\infty} k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k - \lambda\gamma(2-\alpha(1+\beta)) \leq 0. \end{aligned}$$

Hence by maximum modulus theorem, we conclude that  $f \in Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ .

conversely, let  $f$  be in the class  $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ , so the condition (8) yields

$$\left| \frac{z(\mathcal{P}_f(z))''}{z\lambda(\mathcal{P}_f(z))' - \alpha(1 + \beta)\lambda} \right| = \left| \frac{\sum_{k=2}^{+\infty} k(k-1) \binom{k+r-2}{r-1} p^{k-r} (1-p)^r a_k z^{k-1}}{\lambda(2 - \alpha(1 + \beta)) - 2\lambda \sum_{k=2}^{+\infty} k \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k z^{k-1}} \right| < \gamma.$$

Since for any  $z$ ,  $|\operatorname{Re} z| < |z|$ , then

$$\operatorname{Re} \left\{ \frac{\sum_{k=2}^{+\infty} k(k-1) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k z^{k-1}}{\lambda(2 - \alpha(1 + \beta)) - 2\lambda \sum_{k=2}^{+\infty} k \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k z^{k-1}} \right\} < \gamma,$$

by letting  $z \rightarrow 1$  through real values, we have

$$\begin{aligned} & \sum_{k=2}^{+\infty} k(k-1) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k \\ & \leq \lambda\gamma(2 - \alpha(1 + \beta)) - 2\lambda\gamma \sum_{k=2}^{+\infty} k \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k, \end{aligned}$$

and this completes the proof. □

**Theorem 2.2.** Let  $f(z) = z - \sum_{k=2}^{+\infty} a_k z^k$  and  $g(z) = z - \sum_{k=2}^{+\infty} b_k z^k$  belong to  $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ . Then

(i)  $(f * g)(z)$  belong to  $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta_0, \gamma)$ , where

$$(11) \quad \beta_0 \leq \frac{2}{a} - \left( 1 + \frac{\lambda\gamma(2 - \alpha(1 + \beta))^2}{\alpha k(k-1 + 2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r} \right).$$

(ii)  $(f * g)(z)$  belong to  $Y_{\mathcal{P}}^{\lambda}(\alpha_0, \beta, \gamma)$ , where

$$(12) \quad \alpha_0 \leq \frac{2}{1 + \beta} - \frac{\lambda\gamma(2 - \alpha(1 + \beta))^2}{(1 + \beta)k(k-1 + 2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r}.$$

*Proof.* (i). It is sufficient to show that

$$\frac{\sum_{k=2}^{+\infty} k(k-1 + 2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r}{\lambda\gamma(2 - \alpha(1 + \beta))} a_k b_k \leq 1.$$

By using Cauchy-Schwartz inequality from (9), we obtain

$$\sum_{k=2}^{+\infty} \frac{k(k-1 + 2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r}{\lambda\gamma(2 - \alpha(1 + \beta))} \sqrt{a_k b_k} \leq 1.$$

Hence we find the largest  $\beta_0$  such that

$$\begin{aligned} & \sum_{k=2}^{+\infty} \frac{k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r}{\lambda\gamma(2-\alpha(1+\beta))} a_k b_k \\ & \leq \sum_{k=2}^{+\infty} \frac{k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r}{\lambda\gamma(2-\alpha(1+\beta))} \sqrt{a_k b_k} \leq 1, \end{aligned}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{2-\alpha(1+\beta_0)}{2-\alpha(1+\beta)}.$$

This inequality holds if

$$\frac{\lambda\gamma(2-\alpha(1+\beta))}{k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r} \leq \frac{2-\alpha(1+\beta_0)}{2-\alpha(1+\beta)},$$

or equivalently

$$\beta_0 \leq \frac{2}{\alpha} - \left( 1 + \frac{\lambda\gamma(2-\alpha(1+\beta))^2}{\alpha k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r} \right).$$

(ii) With a same calculation of (i), we obtain the result, hence the details are omitted.  $\square$

### 3. Geometric properties of subclasses of $Y_{\mathcal{P}}^\lambda(\alpha, \beta, \gamma)$

In this section we introduce two subclasses of  $Y_{\mathcal{P}}^\lambda(\alpha, \beta, \gamma)$  and conclude some geometric properties.

Let  $f$  and  $g$  be analytic in  $\mathbb{U}$ . Then  $f$  is said to be subordinate to  $g$  written  $f \prec g$  or  $f(z) \prec g(z)$  if there exists a function  $\omega$  analytic in  $\mathbb{U}$ , with  $\omega(0) = 0$  and  $|\omega(z)| < 1$ , such that  $f(z) = g(\omega(z))$  (see [3]).

If  $g$  is univalent, then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subset g(U)$ .

Let  $U(m, n, s)$  consists of all analytic functions  $g(z)$  in  $\mathbb{U}$  for which  $g(0) = 1$  and

$$(13) \quad g(z) \prec \frac{1 + (n + (m-n)(1-s))z}{1+nz},$$

where  $-1 \leq m < n \leq 1$ ,  $0 < n \leq 1$  and  $0 \leq s < 1$ .

Let  $V(m, n, s)$  denote the class of all functions  $f(z) \in Y_{\mathcal{P}}^\lambda(\alpha, \beta, \gamma)$  for which

$$(14) \quad \frac{z(\mathcal{P}_f(z))'}{\mathcal{P}_f(z)} \in U(m, n, s).$$

**Theorem 3.1.**  $f(z) \in V(m, n, s)$  if and only if

$$(15) \quad \sum_{k=2}^{+\infty} \left[ 1 + \frac{(n+1)(k-1)}{(n-m)(1-s)} \right] \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k < 1.$$

*Proof.* Let  $f \in V(m, n, s)$ . Then by (8), (13) and (14) we have

$$\left| \frac{\mathcal{P}_f(z) - z(\mathcal{P}_f(z))'}{nz(\mathcal{P}_f(z))' - (n + (m - n)(1 - s))(\mathcal{P}_f(z))} \right| < 1,$$

which implies that

$$\left| \frac{\sum_{k=2}^{+\infty} (k - 1) \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}}{(n - m)(1 - s) - \sum_{k=2}^{+\infty} [n(k - 1) + (n - m)(1 - s)] \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}} \right| < 1.$$

Since  $\operatorname{Re} z < |z|$  for all  $z \in \mathbb{U}$ , so we conclude that

$$\operatorname{Re} \left\{ \frac{\sum_{k=2}^{+\infty} (k - 1) \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}}{(n - m)(1 - s) - \sum_{k=2}^{+\infty} [n(k - 1) + (n - m)(1 - s)] \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}} \right\} < 1.$$

Choose the values of  $z$  on the real axis and letting  $z \rightarrow 1^-$ , we obtain

$$\frac{\sum_{k=2}^{+\infty} (k - 1) \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k}{(n - m)(1 - s) - \sum_{k=2}^{+\infty} [n(k - 1) + (n - m)(1 - s)] \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k} < 1,$$

after a simple calculation, we obtain the result.

Conversely, assume that the condition (15) holds true. We must show that  $f \in V(m, n, s)$ , or equivalently

$$\begin{aligned} \mathcal{L} &= \left| \frac{\mathcal{P}_f(z) - z(\mathcal{P}_f(z))'}{nz(\mathcal{P}_f(z))' - (n + (n - m)(1 - s))\mathcal{P}_f(z)} \right| < 1 \\ &= \left| \frac{\sum_{k=2}^{+\infty} (k - 1) \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}}{(n - m)(1 - s) - \sum_{k=2}^{+\infty} [n(k - 1) + (n - m)(1 - s)] \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k z^{k-1}} \right| \\ &< \frac{\sum_{k=2}^{+\infty} (k - 1) \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k}{(n - m)(1 - s) - \sum_{k=2}^{+\infty} [n(k - 1) + (n - m)(1 - s)] \binom{k+r-2}{r-1} p^{k-1} (1 - p)^r a_k}. \end{aligned}$$

But by applying (15), we conclude that  $\mathcal{L} < 1$ , so the proof is complete.  $\square$

**Theorem 3.2.** Let  $n \neq 1$ ,  $f \in V(m, n, s)$  and  $W = x + iy = \frac{z(\mathcal{P}_f(z))'}{\mathcal{P}_f(z)}$ . Then the values of  $W$  are in the circle

*Proof.* By (13) and (14), we have

$$W = x + iy = \frac{1 + (n + (m - n)(1 - s))J(z)}{1 + nJ(z)}, \quad |J(z)| < 1.$$

Then

$$(x + iy)(1 + nJ(z)) = 1 + (n + (m - n)(1 - s))J(z),$$

or

$$(x - 1)^2 + y^2 < (n + (m - n)(1 - s) - xn)^2 + y^2n^2.$$

After a simple calculation, we obtain

$$\left(x - \frac{1 - (n^2 + n(m - n)(1 - s))}{1 - n^2}\right)^2 + y^2 < \left(\frac{(n - m)(1 - s)}{1 - n^2}\right)^2.$$

Hence the values of  $W$  lie in the circle with center at  $\left(\frac{1 - (n^2 + n(m - n)(1 - s))}{1 - n^2}, 0\right)$

and radius  $\frac{(n - m)(1 - s)}{1 - n^2}$ .  $\square$

**Theorem 3.3.** Let  $f \in V(m, n, s)$ , then

$$\mathcal{P}_f(z) = \exp\left(\int_0^z \frac{1 - (n + (m - n)(1 - s))H(t)}{t(1 - nH(t))} dt\right),$$

where  $|H(z)| < 1$ .

*Proof.* Set  $W = \frac{z(\mathcal{P}_f(z))'}{\mathcal{P}_f(z)}$ , since  $f \in V(m, n, s)$ , so

$$\left|\frac{W - 1}{Wn - (n + (m - n)(1 - s))}\right| < 1,$$

therefore

$$\frac{W - 1}{Wn - (n + (m - n)(1 - s))} = H(z), \quad |H(z)| < 1.$$

Hence, we can write

$$\frac{(\mathcal{P}_f(z))'}{\mathcal{P}_f(z)} = \frac{1 - (n + (m - n)(1 - s))H(z)}{z(1 - nH(z))}.$$

After integration, we conclude the required result.  $\square$

**Theorem 3.4.** Let  $0 \leq s_2 \leq s_1 < 1$ , then  $V(m, n, s_1) \subset V(m, n, s_2)$ .

*Proof.* Suppose that  $f \in V(m, n, s_1)$ . Then by (1) we have

$$\sum_{k=2}^{+\infty} \left[1 + \frac{(n+1)(k-1)}{(n-m)(1-s_1)}\right] \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k < 1.$$

We have to prove

$$\sum_{k=2}^{+\infty} \left[ 1 + \frac{(n+1)(k-1)}{(n-m)(1-s_2)} \right] \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k < 1.$$

But the last inequality holds true if

$$1 + \frac{(n+1)(k-1)}{(n-m)(1-s_2)} \leq 1 + \frac{(n+1)(k-1)}{(n-m)(1-s_1)},$$

and this inequality by  $0 \leq s_2 < s_1 < 1$  definitely holds true.  $\square$

#### 4. Acknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

#### References

- [1] El-Deeb, M. Sheeza M, T. Bulboaca, and J. Dziok, *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Mathematical Journal 59, no. 2 (2019) 301-314.
- [2] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften 259, Springer-Verlag New York, NY, USA, 1983.
- [3] S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*. CRC Press, 2000.
- [4] S. Najafzadeh, *On Certain Class of Analytic Functions Based on Poisson Distribution*, Journal of Quality Measurement and Analysis JQMA 16, no. 1 (2020) 91-98.
- [5] S. Najafzadeh, *Fractional  $q$ -differintegral operator related to univalent functions with negative coefficients*, Journal of Mahani Mathematical Research Center vol., 9 no. 2 (2020) 69-77.
- [6] H. Rahimpour, P. Arjomandinia, *Some Generalized Results Based on Differential Subordinations of Analytic Functions*, Journal of Mahani Mathematical Research Center vol: 10 no 1 (2021) 11-25.

ROHOLLAH VALIZADE  
ORCID NUMBER: 0000-0003-1327-2745  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARAGHEH  
P. O. BOX 55136-553  
MARAGHEH, IRAN  
*E-mail address: alirezas6611@gmail.com*

SHAHRAM NAJAFZADEH  
ORCID NUMBER: 0000-0002-8124-8344  
DEPARTMENT OF MATHEMATICS  
PAYAME NOOR UNIVERSITY  
P. O. BOX 19395-3697  
TEHRAN, IRAN  
*E-mail address: najafzadeh1234@yahoo.ie*

ASGHAR RAHIMI  
ORCID NUMBER: 0000-0003-2095-6811  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARAGHEH  
P. O. BOX 55136-553  
MARAGHEH, IRAN  
*E-mail address: rahimi@maragheh.ac.ir*

BAYAZ DARABY  
ORCID NUMBER: 0000-0001-6872-8661  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARAGHEH  
P. O. BOX 55136-553  
MARAGHEH, IRAN  
*E-mail address: bdaraby@maragheh.ac.ir*