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MINIMAX RISK STRATEGY FOR TESTING CAPABILITY

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Abstract. Process capability indices are used widely throughout the world to give a quick indication of a process capability in a format that is easy to use and understand. A process capability index C_p that constructed for measuring the quality is an effective tool for assessing process capability, since this index can reflect whether a centering process is capable of reproducing items meeting the specifications limits. The minimax approach is proposed in this paper for testing capability on the basis of precision index C_p when the producer goal is avoiding the largest possible risk. Motivations and benefits of proposing minimax approach are discussed for capability test. Also, the proposed method clarified by an industrial application.

Keywords: Testing hypothesis, Process capability index, Precision index, Minimax procedure, Loss function.
2020 MSC: 62C20.

1. Introduction

Process capability indices (PCIs) can be viewed as the effective and excellent statistics for measuring product quality and process performance. They are very useful statistical analysis tools to summarize process dispersion and location by using process capability analysis [17]. The process capability analysis compares the output of a process to the specification limits by using process capability indices. This comparison is made by forming the ratio of the width of the process specification limits to the width of the natural tolerance limits which is measured by 6 process standard deviation units [19]. PCIs provide numerical measures on whether a process conforms to the defined manufacturing capability prerequisite. These have been successfully applied by companies to compete with and to lead high-profit markets by evaluating the quality and productivity performance. In the literature some PCIs such as C_p, C_{pk}, C_{pm} and C_{pmk} have been used to measure the ability of process to decide how well the process meets the specification limits.

The process can be classified as capable if the PCIs are greater than predetermined critical values. Otherwise they can be labeled as incapable.

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Cheng and Spiring proposed a Bayesian method for assessing capability index C_p [8]. Chan and Cheng applied a similar Bayesian method on C_{pm} under the assumption that the process mean is equal to the target value T [7]. Shiau et al. derived the posterior distributions for C_p^2 and C_{pm}^2 under the restriction that process mean equals to the target value T, and for C_{pm}^2 under the restriction that the process mean equals to the midpoint of specification limits (say M) with respect to non-informative prior and also gamma prior [28]. However, the restriction of $\mu = T$ or $\mu = M$ is not a practical assumption for many industrial applications. A nice Bayesian procedure for assessing index C_{pm} relaxing the restriction on the process mean proposed by Shiau et al. in [27]. They also applied a similar Bayesian approach for testing the index C_{pk} but under the restriction $\mu = M$. Note that in this case C_{pk} reduces to C_p. Pearn and Wu considered the index C_{pk} for assessing process capability without restriction on the process mean [26]. Also, they proposed a Bayesian approach for assessing C_p, C_{pk} and C_{pm} based on multiple samples [25]. Another Bayesian procedure for testing the process capability C_{pk} is proposed by Kargar et al. to derive the posterior probability p for which the process under investigation is capable [15]. In this paper, unlike these studies, the minimax procedure is presented for testing process capability index C_p based on a meaningful and flexible loss function. Interested readers can follow references [4, 13, 18, 21, 22, 30] and [1–3, 10, 12, 20] to see recent investigations about univariate and multivariate analyses with process capability indices, respectively.

The organization of this paper is as follows. In Section 2, we review some preliminaries about the estimator of C_p and its statistical distribution. In Section 3, after introducing a flexible loss function for capability test, the minimax approach is presented by a theorem on C_p. Also, some motivations of proposing such minimax approach for capability test is listed in Section 3. An applied industrial example is given in Section 4 to clarify the method. The final section is the conclusion part.

2. **Preliminaries**

The first process capability index appeared in the literature is C_p and it is called precision index [14] and defined as the ratio of specification width ($USL-LSL$) over the process spread (6σ). The specification width represents customer and/or product requirements. The allowable process variation is represented by the specification width. If the process variation is very large, the C_p value is small and it represents a low process capability [17]. C_p indicates how well the process fits within the two specification limits and it simply measures the spread of the specifications relative to the six-sigma spread in the process [19].

$$C_p = \frac{Allowable\ process\ spread}{Actual\ process\ spread} = \frac{USL - LSL}{6\sigma} \tag{1}$$

where σ is the standard deviation of the process, and USL and LSL are upper and lower specification limits, respectively. The value of index C_p gives us an opinion about process performance. For example if it is greater than 1.33 which corresponds to a percentage of nonconforming items of 63 parts per million (ppm) for a centered process, see [11]. That is because of $USL - LSL = 1.33 \times 6\sigma \simeq 4\sigma$ and so $\mu \mp 4\sigma$ is the specification limits of the centered process. Therefore

$$1 - P(\mu - 4\sigma < X < \mu + 4\sigma) = 1 - [\Phi(4) - \Phi(-4)] \simeq 63 \times 10^{-6}$$

and we can conclude that process performance is satisfactory. The quality conditions for different C_p values are summarized in Table 1 of Kaya and Kahraman (2011).

The index C_p involves only one parameter σ to be estimated. If a single sample of size n is given as $X_1, X_2, \ldots, X_n$, a natural estimator $\hat{C}_p$ of C_p will be

$$\hat{C}_p = \frac{USL - LSL}{6S} \tag{2}$$

where $S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$ is the conventional estimator of the process standard deviation σ, obtained from a stable process. Under normality assumption, Chou and Owen obtained the probability density function (p.d.f.) of the natural estimator $\hat{C}_p$ as follows [9]

$$\hat{C}_p \sim f_{C_p}(c) = \frac{2(\sqrt{\frac{n-1}{2}}C_p)^{n-1}}{\Gamma(\frac{n-1}{2})} c^{-n} e^{\frac{-(n-1)C_p{}^2}{2c^2}}, \quad c > 0. \tag{3}$$

3. Minimax Procedure

After introducing a flexible loss function for capability test at the first of Section 3, a minimax approach is presented for C_p capability index in Subsection 3.2. Moreover, some motivations of the proposed minimax test for capability test are listed discussed in Subsection 3.3.

3.1. Loss function.

The objective of this paper is introducing minimax critical value for testing $H_0 : C_p \leq c_0$ (process is not capable), v.s., $H_1 : C_p > c_0$ (process is capable). We suppose that the loss function is defined as follows: if the alternative hypothesis is accepted, the loss is $C_I(C_p) \geq 0$ for $C_p \leq c_0$ and $C_I(C_p) = 0$ otherwise; if the null hypothesis is accepted, the loss is $C_{II}(C_p) \geq 0$ for $C_p \geq c_0$ and $C_{II}(C_p) = 0$ otherwise. Furthermore, it is assumed that the function $C_I(C_p)$ is actually positive for at least one value of $C_p \leq c_0$, and $C_{II}(C_p)$ is positive for at least one value of $C_p \geq c_0$. Note that in this case $C_I(C_p)$ is related to the cost of a type I error in testing capability, the error of falsely rejecting H_0 for any $C_p \leq c_0$, and similarly $C_{II}(C_p)$ is related to the cost of a type II error, the error of falsely accepting H_0 for any $C_p \geq c_0$.

The problem to be considered is the selection of a minimax test function $\phi_{\hat{c}_p}(X_1, ..., X_n)$ under the above loss function on the basis of a random sample.

3.2. **Minimax capability test.** First, let us to briefly recall some definitions of decision theory. Let $\mathcal{A}$ be the action space and $\mathcal{L}(\theta, a) : \Theta \times \mathcal{A} \to R$ a loss function. The value $\mathcal{L}(\theta, a)$ is the loss if we take action a when θ is the true parameter value. Let $\mathcal{D}$ be the class of decision functions that map R^n into $\mathcal{A}$. The function $\mathcal{R} : \Theta \times \mathcal{D} \to R^+ \cup \{0\}$ defined by $\mathcal{R}(\theta, d) = E_\theta \mathcal{L}(\theta, d(X))$ is known as the risk function associated with d at θ. In statistical decision theory, testing hypotheses is typically modelled as the choice between actions a_0 and a_1, where a_i denotes accepting hypothesis $H_i : \theta \in \Theta_i$, with i either 0 or 1. Thus, $\mathcal{A} = \{a_0, a_1\}$ and $\Theta = \Theta_0 \cup \Theta_1$.

On the basis of the above definitions, the principle of minimax is to choose decision $d_m \in \mathcal{D}$ so that

$$\max_\theta \mathcal{R}(\theta, d_m) \leq \max_\theta \mathcal{R}(\theta, d),$$

for all d in $\mathcal{D}$. Such a rule d_m, if it exists, is called a minimax rule (decision function).

Theorem 3.1. *Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from normal distribution with mean μ and standard deviation σ parameters where $n > 1$. For testing*

$$H_0 : C_p \leq c_0,$$
$$H_1 : C_p > c_0,$$

suppose that the loss function is defined by

$$\text{(4)} \qquad \begin{cases} \mathcal{L}_{H_0}(C_p, a) = \begin{cases} C_I(C_p) \geq 0, & if \quad a = a_1 \\ 0, & if \quad a = a_0. \end{cases} \\ \mathcal{L}_{H_1}(C_p, a) = \begin{cases} C_{II}(C_p) \geq 0, & if \quad a = a_0 \\ 0, & if \quad a = a_1. \end{cases} \end{cases}$$

Then the minimax solution is to reject H_0 if and only if

$$\text{(5)} \qquad \hat{c_p} > k_0,$$

in which $\hat{c_p} = \frac{USL-LSL}{6s}$ and the constant k_0 is chosen so that

$$\begin{aligned} \max_{C_p \leq c_0} C_I(C_p) \quad & \int_{k_0}^{\infty} C_p^{-(1-n)} c^{-n} e^{\frac{(1-n){C_p}^2}{2c^2}} dc \\ \text{(6)} \qquad = \quad & \max_{C_p \geq c_0} C_{II}(C_p) \int_0^{k_0} C_p^{-(1-n)} c^{-n} e^{\frac{(1-n){C_p}^2}{2c^2}} dc. \end{aligned}$$

Proof. Regarding to the p.d.f. of $\hat{C}_p$ under normality assumption, which is presented in (3), it is obvious that $\hat{C}_p$ blongs to the one-dimentional exponential distributions family $f(x; \theta) = \omega(\theta)\psi(x)e^{\theta x}$, in which $x = \frac{1}{c^2}$ and $\theta = \frac{-(n-1){C_p}^2}{2}$.

Therefore, using Theorem 1 of [5] , one can assert that minimax critical region for testing hypotheses

$$H_0' : \frac{-(n-1)}{2}{C_p}^2 \geq \frac{-(n-1)}{2}{c_0}^2,$$
$$H_1' : \frac{-(n-1)}{2}{C_p}^2 < \frac{-(n-1)}{2}{c_0}^2,$$

under loss function (4) is $\frac{1}{c^2} < t$, or equivalently $\hat{c_p} > k_0$ (since $\hat{c_p}$ is the observed value of $\hat{C}_p$ which is denoted in (3) by c), where the constant $k_0 = t^{-\frac{1}{2}}$ is chosen so that

$$\max_{C_p \leq c_0} \mathcal{R}_{H_0}(C_p, d_{k_0}) = \max_{C_p \geq c_0} \mathcal{R}_{H_1}(C_p, d_{k_0}) \tag{7}$$

in which $\mathcal{R}_{H_j}(C_p, d_{k_0})$ is the risk of decision d_{k_0} under hypothesis H_j in C_p point for $j = 0, 1$ (for more details see Theorem 1 in [5]). In other words, by considering the fact that $C_p \in (0, \infty)$ one can assert that minimax critical region for capability testing H_0 vs H_1 under loss function (4) is $\hat{c_p} > k_0$ such that the constant k_0 is chosen from Eq. (7) in which

$$\begin{aligned}
\mathcal{R}_{H_0}(C_p, d_k) &= E_{C_p}\mathcal{L}_{H_0}(C_p, d_k(\underline{X})) \\
&= C_I(C_p)\ P_{C_p}(\hat{C}_p > k) \\
&= C_I(C_p) \int_k^\infty \frac{2(\sqrt{\frac{n-1}{2}}C_p)^{n-1}}{\Gamma(\frac{n-1}{2})} c^{-n} e^{\frac{-(n-1){C_p}^2}{2c^2}} dc, \quad C_p \leq c_0 \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{H_1}(C_p, d_k) &= E_{C_p}\mathcal{L}_{H_1}(C_p, d_k(\underline{X})) \\
&= C_{II}(C_p)\ P_{C_p}(\hat{C}_p \leq k) \\
&= C_{II}(C_p) \int_0^k \frac{2(\sqrt{\frac{n-1}{2}}C_p)^{n-1}}{\Gamma(\frac{n-1}{2})} c^{-n} e^{\frac{-(n-1){C_p}^2}{2c^2}} dc, \quad C_p \geq c_0. \qquad (9)
\end{aligned}$$

Hence, relation (7) can be rewritten as

$$\begin{aligned}
&\max_{C_p \leq c_0} C_I(C_p) \int_{k_0}^\infty \frac{2(\sqrt{\frac{n-1}{2}}C_p)^{n-1}}{\Gamma(\frac{n-1}{2})} c^{-n} e^{\frac{-(n-1){C_p}^2}{2c^2}} dc \\
&= \max_{C_p \geq c_0} C_{II}(C_p) \int_0^{k_0} \frac{2(\sqrt{\frac{n-1}{2}}C_p)^{n-1}}{\Gamma(\frac{n-1}{2})} c^{-n} e^{\frac{-(n-1){C_p}^2}{2c^2}} dc, \qquad (10)
\end{aligned}$$

which is equivalent to (6) and the proof is complete. □

Remark 3.2. The answer of Eq. (6) in Theorem 1 is positive and exists; since we have the following inequalities based on the left and the right sides of this equation

$$\max_{C_p \leq c_0} \mathcal{R}_{H_0}(C_p, d_{k_0}) > \max_{C_p \geq c_0} \mathcal{R}_{H_1}(C_p, d_{k_0}) = 0, \qquad \textit{when} \quad C_p \to 0,$$

$$0 = \max_{C_p \le c_0} \mathcal{R}_{H_0}(C_p, d_{k_0}) < \max_{C_p \ge c_0} \mathcal{R}_{H_1}(C_p, d_{k_0}), \qquad when \quad C_p \to +\infty.$$

Therefore, considering monotones of the left and the right sides/functions in Eq. (6) and also the continuity property of the C_p on domain $(0, +\infty)$, there exist one answer for Eq. (6). Moreover, regarding to the monotones of the left and the right sides, this answer is unique.

3.3. **Why and when minimax approach?** Some motivations and benefits of the proposed minimax capability test are presented in follows:

(1) In testing hypotheses, minimax criteria causes the equivalence of maximum risks under hypotheses H_0 and H_1, i.e.

$$\max_{C_p \le c_0} \mathcal{R}_{H_0}(C_p, d_{k_0}) = \max_{C_p \ge c_0} \mathcal{R}_{H_1}(C_p, d_{k_0}). \tag{11}$$

It means that the minimax criteria can be proposed an impartial and just procedure to producer and consumer for judging about the capability of a manufacturing process by C_p.

(2) From the practical point of view, "the loss of capable assessing for an incapable process" is not equivalent to "the loss of incapable assessing for a capable process". User can design an unequal and suitable loss function in minimax approach, but note that two above mentioned losses must be considered equivalent in UMP and p-value-based capability tests.

(3) Because of the importance of the capability tests, it is more appropriate that add more flexibility and sensitiveness to their inferences. In this regard, the proposed loss function in (4) is subjectively reasonable and it is completely flexible. In other words, the proposed loss function has the ability of mapping different values under H_0 (or equivalently, under H_1) to different losses. Therefore, it can easily model any objective/real situations for the consumer and producer's losses.

(4) Although the Bayesian procedures are introduced based on the loss function, there exist three weaknesses. First, one may has not a reliable and suitable prior density for unknown C_p index. Second, minimax does not require any knowledge about the chance that each of the states of the world will turn out to be true. Third, minimax statistical decisions are in many cases reasonable, and tend to err on the conservative side. For more details see [23].

(5) As presented in Subsection 3.2, the philosophy of minimax decision is minimizing the maximum risks. Therefore, this optimized decision rule can be helpful for the prudent consumers/producers which are wary to the point for which maximum risk occurred.

(6) From the perspective of the manufacturer, the minimax optimization method can be a useful approach for insurance of the factory. Because as you know, the minimax approach minimize the maximum risk, and

factory owners for fear of bankruptcy are interested in decision which is able to minimize its risk in the worst situation of the nature.

4. **An Industrial Application**

To illustrate the idea of this paper, we are going to quote a simplified example from [19] in this section. In manufacturing automobile engine piston rings, the inside diameter of the rings is a critical quality characteristic. Suppose we wish to evaluate whether the manufacturing process of automobile engine piston rings is capable or not. Twelve measurements were made on the inside diameter of forged piston rings (in millimeter) are 74.001, 73.994, 74.011, 74.012, 74.032, 74.001, 73.993, 74.008, 73.988, 74.025, 74.015, and 74.004. The sample mean of inside ring diameter is 74.007 millimeter, and its standard deviation is 0.01301049 millimeter. Shapiro-Wilk normality test strongly accepted the normality assumption of data with p-value $= 0.8853$.

Suppose that the specification limits on this piston ring are 74 ± 0.05 millimeter which is considered in [6]. Hence, the hypothesis $\mu = \frac{USL+LSL}{2}$ is acceptable on the basis of observations at significance level 0.05 with p-value $= 0.0892$. Therefore, we can use C_p index, with the estimated value $\hat{c}_p = 1.281018$, for evaluating the manufacture process of automobile engine piston rings. To determine a minimax critical value in testing capability by the method of Theorem 3.1, let us to choose the standard minimal lower boundary of C_p equal to $c_0 = 1.33$. In other words, we can design the hypotheses as $H_0 : C_p \leq 1.33$, v.s. $H_1 : C_p > 1.33$. Let the loss functions are defined as follows

$$\tag{12} \begin{cases} \mathcal{L}_{H_0}(C_p, a) = \begin{cases} \sqrt{1.33 - C_p}, & if \quad a = a_1 \\ 0, & if \quad a = a_0. \end{cases} \\ \mathcal{L}_{H_1}(C_p, a) = \begin{cases} \frac{C_p - 1.33}{3}, & if \quad a = a_0 \\ 0, & if \quad a = a_1. \end{cases} \end{cases}$$

As can be seen in Figure 1, the loss under hypothesis H_1 is defined by a linear function and the loss under hypothesis H_0 is considered as a nonlinear function. This unequal consideration of loss functions is one of the advantages of the proposed method in this paper which was also mentioned in Subsection 3.3. It means that the loss due to the increase in C_p - relative to the boundary of the hypotheses - increases linearly, but the loss due to the decrease in C_p relative to the boundary of the hypotheses increases nonlinearly. This nonlinear property of the considered loss under H_0, allows the rate of increase of losses not to be the same for different distances between C_p and c_0. Moreover, it must be mentioned that defining loss function has no certain mathematical rule and it must be defined by an expert who practically knows the losses caused by capability index C_p for each production process.

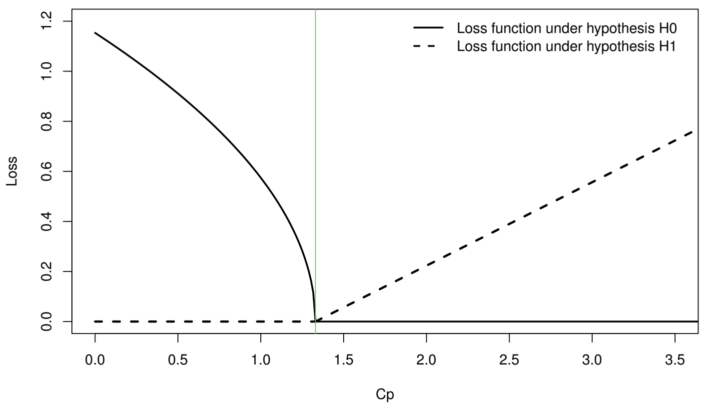

FIGURE 1. Loss functions under hypotheses H_0 and H_1 in industrial application

Considering Theorem 3.1, the minimax capability test reject H_0 if and only if $\hat{c_p} > k_0$, where the constant k_0 is chosen so that

$$\max_{C_p \leq 1.33} \quad \sqrt{1.33 - C_p} \int_{k_0}^{\infty} {C_p}^{-(1-n)} c^{-n} e^{\frac{(1-n){C_p}^2}{2c^2}} dc$$

$$(13) \qquad = \max_{C_p \geq 1.33} \frac{C_p - 1.33}{3} \int_0^{k_0} {C_p}^{-(1-n)} c^{-n} e^{\frac{(1-n){C_p}^2}{2c^2}} dc.$$

The left and right sides of Eq. (13) are drawn, as a function of k, in Figure 2 which are equivalent for critical value $k_0 = 1.6317$. Also, one can see the difference of left and right sides of Eq. (13) in the left shapes of Figure 2 for $c_0 = 1.00, 1.33, 1.50$ and 1.67 respectively. Therefore, the minimax test function for the inside diameter of the manufacturing automobile engine piston rings, under loss function (12) is

$$(14) \qquad \phi_{\hat{c}_p}(x_1, ..., x_{12}) = \begin{cases} 1 & if \quad \hat{c}_p > 1.6317, \\ 0 & if \quad \hat{c}_p \leq 1.6317. \end{cases}$$

It means that, by considering $\hat{c}_p = 1.281018$, we accept null hypothesis based on observed data and the process of the manufacturing piston rings is not capable from minimax point of view. It must be mentioned that all computations of this application done by a computer program in R software [29] which is available upon request on the basis of the presented approach in Section 3.

To show the behaviour of the proposed minimax test, a table is prepared for this example. Table 1 contains the critical values k_0 for the various c_0 and sample size, which are computed from Eq. (6). The calculated critical values imply to a reasonable reaction between c_0 and k_0 for the introduced minimax test in Theorem 3.1. By increasing the value of c_0 in each column of Table 1 (i.e. for each fixed sample size), an increase in the critical value k_0 is observed;

for instance see/check the column related to $n = 12$ in this table. Moreover, this reasonable fact that increasing c_0 causes increasing critical value k_0 shown in Figure 3 for the presented industrial application.

The relation between n and k_0 is depicted in Figure 4 for $n = 1, 2, ..., 100$ when $c_0 = 1.33$. This figure does not shown a monotone relationship between the sample size and the critical value in the minimax test, which of course is not expected. For example, in this study the sample size is 12 and the minimax critical value is 1.6317 which are shown by vertical and horizontal lines in Figure 4, respectively.

TABLE 1. Minimax critical value (k_0) for various c_0 and n in industrial application

	n				
c_0	6	12	18	24	30
1	1.3816	1.2432	1.1934	1.1663	1.1368
1.33	1.8003	1.6317	1.5710	1.5743	1.5334
1.50	2.0131	1.8302	1.7637	1.7849	1.7385
1.67	2.2241	2.0276	1.9565	1.9961	1.9438
2.00	2.6297	2.4083	2.3270	2.4064	2.3432

5. Conclusions and future works

To judge if the process satisfies the present capability requirement (i.e. being capable), one can consider the following capability test, procedure with the null hypothesis $H_0 : C_p \leq c_0$ (the process is not capable), versus the alternative $H_1 : C_p > c_0$ (the process is capable), where c_0 is a predetermined capability requirement. In this paper a capability test for index C_p is presented from minimax point of view. Then, some motivations and benefits of the presented minimax approach for capability test are listed and discussed. Finally, an industrial application is given to show the performance of the presented approach.

Regarding to the proposed approach in this paper, the following topics can be considered for future works:

(1) Investigation on the relationship between minimax capability test and Bayesian capability test is a potential subject for further research. It must be noted that both of these statistical tests are based on loss function and may be comparable for some special priors.

(2) As another related subject one can investigate on the minimax capability test based on not only one sample, but also for multiple samples with different sizes.

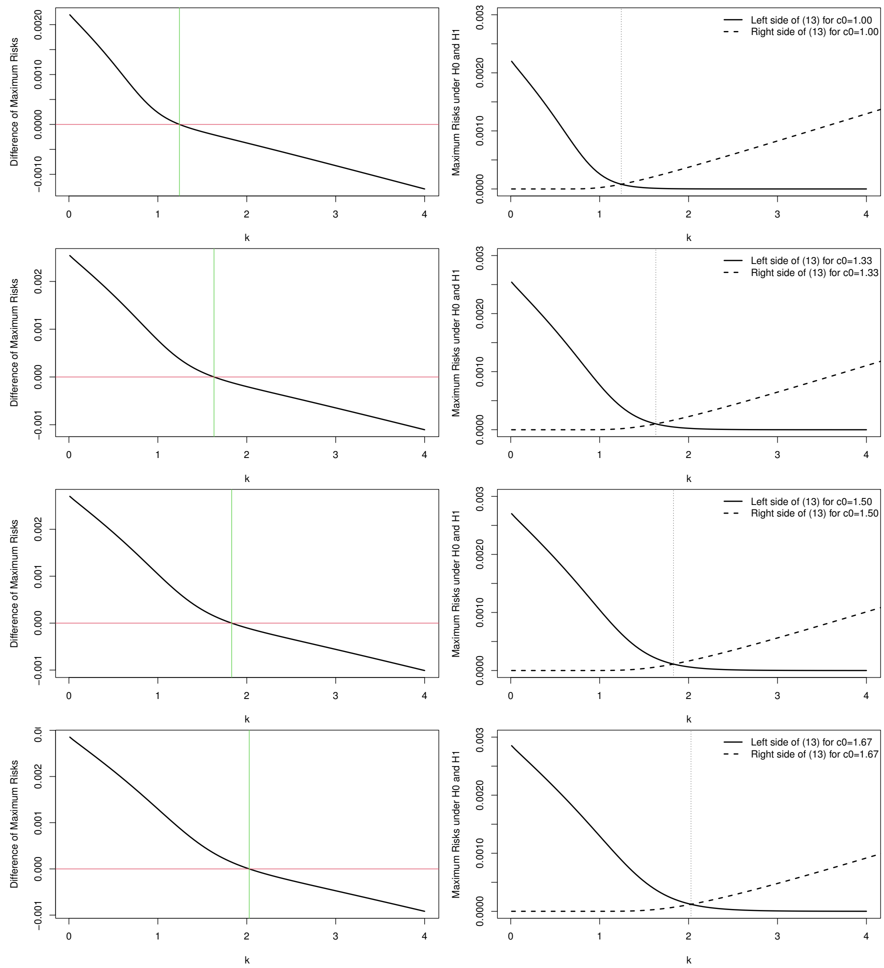

FIGURE 2. A proportion of maximum risks under hypotheses H_0 and H_1 is shown in right figures, and left figures show their difference for $n = 12$ and $c_0 = 1.00, 1.33, 1.50, 1.67$ in industrial application

(3) Moreover, the minimax capability test on the basis of other process capability indices, like C_{pk} and C_{pm}, are some other potential directions for future research.

6. Aknowledgement

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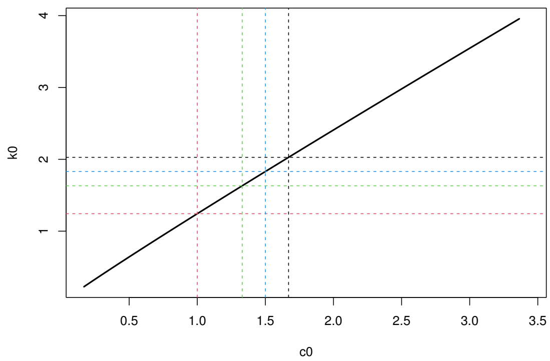

FIGURE 3. The relation between c_0 and k_0 in industrial application for $n = 12$

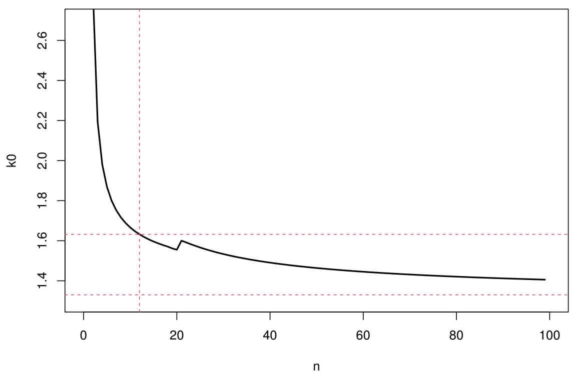

FIGURE 4. The relation between n and k_0 in industrial application for $c_0 = 1.33$

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