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A NOTE ON PRODUCT TOPOLOGIES IN LOCALLY CONVEX CONES

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Abstract. We consider the locally convex product cone topologies and prove that the product topology of weakly cone-complete locally convex cones is weakly cone-complete. In particular, we deduce that a product cone topology is barreled whenever its components are weakly cone-complete and carry the countable neighborhood bases.

Keywords: Products cone topologies, weak cone-completeness, barreledness.
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1. Introduction

In theory of locally convex cones, a linear functional may take the infinite values $+\infty$, which is one of the most important differences in comparison with the convex spaces. Further, we recall from [2] that any linear functional on product cone topology is written by a finite sum of functionals on its components. In this note, taking into account the above, we prove that under the product topology, the weak cone-completeness of locally convex cones is preserved. As a specific result; we infer that if the components of product topology are weakly-cone complete with countable bases, then it is barreled.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a + b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and the scalar multiplication is associative and distributive, that is, $\alpha(\beta a) = (\alpha\beta)a,\ (\alpha + \beta)a = \alpha a + \beta a,\ \alpha(a + b) = \alpha a + \alpha b,\ 1a = a,\ 0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, i.e., $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars.

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The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} : b \le a + v\}$ ($(a)v = \{b \in \mathcal{P} : a \le b + v\}$), creating the *upper* (*lower*) *topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \le a + \rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. For a locally convex cone $(\mathcal{P}, \mathcal{V})$ the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \le b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2].

For cones $\mathcal{P}$ and $\mathcal{P}'$, a map $\mathrm{T} : \mathcal{P} \to \mathcal{P}'$ is called a *linear operator*, if $\mathrm{T}(a + b) = \mathrm{T}(a) + \mathrm{T}(b)$ and $\mathrm{T}(\alpha a) = \alpha \mathrm{T}(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \ge 0$. If $\mathcal{V}$ and $\mathcal{W}$ are abstract neighborhood systems on $\mathcal{P}$ and $\mathcal{P}'$, a linear operator $\mathrm{T} : \mathcal{P} \to \mathcal{P}'$ is called *uniformly continuous* if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $\mathrm{T}(a) \le \mathrm{T}(b) + w$ whenever $a \le b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{P}'$.

Remark 1.1. In the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. If $\mathcal{P}$ is a locally convex cone, then the set of all uniformly continuous linear functionals $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$. In particular, $\overline{\mathbb{R}}^* = \{\alpha \in \mathbb{R} : \alpha \ge 0\} \cup \{\bar{0}\}$, where

$$\bar{0}(a) = \begin{cases} 0 & \text{for} \quad a \in \mathbb{R}, \\ +\infty & \text{for} \quad a = +\infty. \end{cases}$$

For details see [8, Example 2.3].

A convex set $\mathrm{U} \subset \mathcal{P} \times \mathcal{P}$ is called *barrel*, if for every $b \in \mathcal{P}$ there is $v \in \mathcal{V}$ such that for every $a \in v(b) \cap (b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda \mathrm{U}$; in addition if $(a, b) \notin \mathrm{U}$, then there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in \mathrm{U}$ and $\mu(a) > \mu(b) + 1$. The cone $\mathcal{P}$ is *barreled* if for every barrel U and every element $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda \mathrm{U}$ for all $a \in v(b) \cap (b)v$.

2. **Products, weak cone-completeness and barreledness**

Locally convex cones have been investigated from the point of view of products in [2-5]. In this section, we intend to study the product topologies in terms of the weak cone-completeness, which is one of the approaches in locally convex cones presented in [7]. First, we review the concept of locally convex product cones: Let $\mathcal{P}_\gamma$, $\gamma \in \Gamma$, be cones and put $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$. For elements

$a, b \in \mathcal{P}$, $a = \times_{\gamma\in\Gamma} a_\gamma, b = \times_{\gamma\in\Gamma} b_\gamma$ and $\alpha \geq 0$ we set $a + b = \times_{\gamma\in\Gamma}(a_\gamma + b_\gamma)$ and $\alpha a = \times_{\gamma\in\Gamma}(\alpha a_\gamma)$. With these operations $\mathcal{P}$ is a cone which is called the *product cone* of $\mathcal{P}_\gamma$. The subcone of the product cone $\mathcal{P}$ spanned by $\cup\mathcal{P}_\gamma$ (more precisely, by $\cup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ is the injection mapping) is said to be the *direct sum cone* of $\mathcal{P}_\gamma$ and denoted by $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$. It is worth remembering that here we use only positive scalars.

For $a, b \in \mathcal{P}$, we set $a \leq_\Gamma b$ if $a_\gamma \leq_\gamma b_\gamma$ for all $\gamma \in \Gamma$. As required, the order $\leq_\Gamma$ is reflexive, transitive and compatible with the algebraic operations of $\mathcal{P}$. This is the weakest order on $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$ such that every projection mapping $\phi_\gamma : \mathcal{P} \to \mathcal{P}_\gamma$, given by $\phi_\gamma(a) = a_\gamma$, is monotone. For each $\gamma \in \Gamma$, let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ be a locally convex cone and let us denote by $\mathcal{V} = \sum_{\gamma\in\Gamma} \mathcal{V}_\gamma$, the direct sum of the abstract neighborhoods $\mathcal{V}_\gamma$. In fact, $\mathcal{V}$ is a subcone (with out zero) of the product cone of the all corresponding full cones containing $\mathcal{V}_\gamma$. The direct sum $\mathcal{V} = \sum_{\gamma\in\Gamma} \mathcal{V}_\gamma$ leads us to define the *coarsest convex quasi-uniform structure*, as well as, the *coarsest locally convex cone topology* on the product cone $\mathcal{P}$ such that the all projection mappings ϕ_γ are uniformly continuous:

Definition 2.1. For elements $a, b \in \mathcal{P}$, $a = \times_{\gamma\in\Gamma} a_\gamma, b = \times_{\gamma\in\Gamma} b_\gamma$ and $v \in \mathcal{V}$, $v = \sum_{\gamma\in\Delta} v_\gamma$, we define

$$a \leq_\Gamma b + v \quad \text{if} \quad a_\gamma \leq_\gamma b_\gamma + v_\gamma \quad (\text{for all } \gamma \in \Delta)$$

where Δ is a finite subset of Γ. The subsets $\{(a, b) \in \mathcal{P}^2 : a \leq_\Gamma b + v\}$ for all $v \in \mathcal{V}$ form the coarsest convex quasi-uniform structure on $\mathcal{P}$ which makes all the projection mappings ϕ_γ uniformly continuous. Then, according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_0$, with abstract neighborhood system $\mathrm{V} = \{0\} \oplus \mathcal{V}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{P}$. The elements $v \in \mathcal{V}, v = \sum_{\gamma\in\Delta} v_\gamma$, form a basis for V in the following sense: For every $\mathrm{v} \in \mathrm{V}$ there is $v \in \mathcal{V}$ such that $a \leq_\Gamma b + v$ for $a, b \in \mathcal{P}$ implies that $a \leq_\Gamma b \oplus \mathrm{v}$. The locally convex cone topology on $\mathcal{P}$ induced by $\mathcal{V}$ is called the *locally convex product cone of* $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ and is denoted by $(\mathcal{P}, \mathcal{V})$ [2, Definition 2.1].

Proposition 2.2. *If* $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma\in\Gamma}\mathcal{P}_\gamma, \sum_{\gamma\in\Gamma} \mathcal{V}_\gamma)$ *is the locally convex product cone, then for each neighborhood* $v \in \mathcal{V}, v = \sum_{\gamma\in\Delta} v_\gamma$, *we have*

$$\left(\sum_{\gamma\in\Delta} v_\gamma\right)^{\circ} \subseteq \sum_{\gamma\in\Delta} v_\gamma^{\circ} \subseteq n \left(\sum_{\gamma\in\Delta} v_\gamma\right)^{\circ} \quad \text{for some} \quad n \in \mathbb{N},$$

in particular, $\mathcal{P}^* = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma^*$; *where* $\mathcal{P}^*$ *and* $\mathcal{P}_\gamma^*$ *are dual cones of* $\mathcal{P}$ *and* $\mathcal{P}_\gamma$, *respectively.*

Proof. See [2, Proposition 2.5]. □

Proposition 2.3. *Suppose* $(\mathcal{P}, \mathcal{V})$ *is the locally convex product cone of* $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$. *Then if* $(a_i)_{i\in\mathcal{I}} \subset \mathcal{P}$, $a_i = \times_{\gamma\in\Gamma} a_{i\gamma}$ *for all* $i \in \mathcal{I}$ *such that* $(a_i)_{i\in\mathcal{I}}$ *converges to*

$b \in \mathcal{P}$, $b = \times_{\gamma\in\Gamma} b_\gamma$ in the symmetric topology of $\mathcal{P}$, then $(a_{i\gamma})_{i\in\mathcal{I}}$ converges to b_γ in the symmetric topology of $\mathcal{P}_\gamma$ for all $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$. For each $v_\gamma \in \mathcal{V}_\gamma$, $v := v_\gamma \in \mathcal{V}$, so there is $i_0 \in \mathcal{I}$ such that $a_i \in v(b) \cap (b)v$ for all $i \in \mathcal{I}$ with $i \geq i_0$, i.e., $a_{i\gamma} \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$. That is, $(a_{i\gamma})_{i\in\mathbb{N}}$ converges to b_γ in the symmetric topology of $\mathcal{P}_\gamma$. □

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *weakly cone-complete* if for all $b \in \mathcal{P}$ and $v \in \mathcal{V}$, every sequence $(a_i)_{i\in\mathbb{N}}$ in $v(b) \cap (b)v$ that converges to b in the symmetric topology of $\mathcal{P}$ and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$, there is $a \in v(b) \cap (b)v$ such that

$$\mu(a) = \sum_{i=1}^{\infty} \eta_i \mu(a_i) \tag{1}$$

for all $\mu \in \mathcal{P}^*$ with $\mu(b) < \infty$ [7].

Theorem 2.4. *A product cone topology $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma\in\Gamma} \mathcal{P}_\gamma, \sum_{\gamma\in\Gamma} \mathcal{V}_\gamma))$ is weakly cone-complete, whenever $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is weakly cone-complete for all $\gamma \in \Gamma$.*

Proof. Suppose $b \in \mathcal{P}$, $b = \times_{\gamma\in\Gamma} b_\gamma$, $v \in \mathcal{V}$, $v = \sum_{\gamma\in\Delta} v_\gamma$ and let $(a_i)_{i\in\mathbb{N}} \subset v(b) \cap (b)v$, $a_i = \times_{\gamma\in\Gamma} a_{i\gamma}$ for all $i \in \mathbb{N}$ such that $(a_i)_{i\in\mathbb{N}}$ converges to b in the symmetric topology of $\mathcal{P}$ and $\sum_{i=1}^{\infty} \eta_i = 1$, $\eta_i > 0$. By Proposition 2.3, for each $\gamma \in \Gamma$, the sequence $(a_{i\gamma})_{i\in\mathbb{N}} \subset v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ converges to b_γ in the symmetric topology of $\mathcal{P}_\gamma$, so from the weak cone-completeness of $\mathcal{P}_\gamma$ there exists $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ such that

$$\mu_\gamma(a_\gamma) = \sum_{i=1}^{\infty} \eta_i a_{i\gamma} \quad \text{for all}\, \mu_\gamma \in \mathcal{P}_\gamma^* \text{ with } \mu_\gamma(b_\gamma) < \infty. \tag{2}$$

Suppose $a := \times_{\gamma\in\Gamma} a_\gamma \in v(b) \cap (b)v$. By Proposition 2.2, for every $\mu \in \mathcal{P}^*$, there is a finite set $\Theta \subset \Gamma$ such that $\mu = \sum_{\gamma\in\Theta} \mu_\gamma$, where $\mu_\gamma \in \mathcal{P}_\gamma^*$ for all $\gamma \in \Theta$. If $\mu(b) < \infty$, then $\mu_\gamma(b_\gamma) < \infty$ for all $\gamma \in \Theta$, hence (2) yields

$$\begin{aligned} \mu(a) &= \sum_{\gamma\in\Theta} \sum_{i=1}^{\infty} \eta_i \mu_\gamma(a_{i\gamma}) = \sum_{i=1}^{\infty} \eta_i \sum_{\gamma\in\Theta} \mu_\gamma(a_{i\gamma}) \\ &= \sum_{i=1}^{\infty} \eta_i \sum_{\gamma\in\Theta} \mu_\gamma(\times_{\gamma\in\Gamma} a_{i\gamma}) = \sum_{i=1}^{\infty} \eta_i \mu(a_i). \end{aligned}$$

That is, $(\mathcal{P}, \mathcal{V})$ is weakly cone-complete. □

A *neighborhood base* for a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a subset $\mathcal{U}$ of $\mathcal{V}$ such that for every $v \in \mathcal{V}$ there exists some $u \in \mathcal{U}$ with $u \leq v$. We cite Theorem 2.3 from [7]:

Theorem 2.5. *Every weakly cone-complete locally convex cone $(\mathcal{P}, \mathcal{V})$ with a countable neighborhood base is barreled.*

As a consequence of Theorem 2.4 and Theorem 2.5, we have:

Corollary 2.6. *If, for each $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is weakly cone-complete and carries a countable base, then $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma\in\Gamma}\mathcal{P}_\gamma, \sum_{\gamma\in\Gamma}\mathcal{V}_\gamma)$ is barreled.*

Example 2.7. The cone $\overline{\mathbb{R}}$ with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ is weakly cone-complete. For, let $b \in \overline{\mathbb{R}}$, $\epsilon \in \mathcal{V}$, $(a_i)_{i\in\mathbb{N}} \subset \epsilon(b) \cap (b)\epsilon$ converges to b in the symmetric topology of $\overline{\mathbb{R}}$ and let $\sum_{i=1}^{\infty} \eta_i = 1$, $\eta_i > 0$. If $b = +\infty$, then for $a = +\infty$ the assertion holds. Let $b \in \mathbb{R}$, $a := \sum_{i=1}^{\infty} \eta_i a_i \in \epsilon(b) \cap (b)\epsilon$ and $\mu \in \overline{\mathbb{R}}^*$ with $\mu(b) < \infty$. If $\mu = \bar{o}$, then clearly (1) holds and if $\mu = \lambda$ for some $\lambda \geq 0$, then $\mu(a) = \lambda(\sum_{i=1}^{\infty} \eta_i a_i) = \sum_{i=1}^{\infty} \eta_i \mu(a_i)$.

Now, let $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$ and $\mathcal{V} = \sum_{\gamma\in\Gamma}\mathcal{V}_\gamma$, where $\mathcal{P}_\gamma = \overline{\mathbb{R}}$ and $\mathcal{V}_\gamma = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ for all $\gamma \in \Gamma$. We show that $(\mathcal{P}, \mathcal{V})$ is weakly cone-complete. Suppose $b \in \mathcal{P}$, $b = \times_{\gamma\in\Gamma} b_\gamma$, $v \in \mathcal{V}$, $v = \sum_{\gamma\in\Delta} \epsilon_\gamma$ and let $(a_i)_{i\in\mathbb{N}} \subset v(b) \cap (b)v$, $a_i = \times_{\gamma\in\Gamma} a_{i_\gamma}$ for all $i \in \mathbb{N}$ such that $(a_i)_{i\in\mathbb{N}}$ converges to b in the symmetric topology of $\mathcal{P}$ and $\sum_{i=1}^{\infty} \eta_i = 1$, $\eta_i > 0$. If we set $a = \times_{\gamma\in\Gamma} a_\gamma$; where $a_\gamma := \sum_{i=1}^{\infty} \eta_i a_{i_\gamma}$ for all $\gamma \in \Gamma$ then $a \in v(b) \cap (b)v$, since $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ for all $\gamma \in \Delta$.

For every $\mu \in \mathcal{P}^*$, $\mu = \sum_{\gamma\in\Theta} \mu_\gamma$ where $\mu_\gamma \in \overline{\mathbb{R}}^*$ for all $\gamma \in \Theta$. If $\mu(b) < +\infty$ then Remark 1.1 yields

$$\mu_\gamma = \begin{cases} \lambda_\gamma \quad (\text{some} \quad \lambda_\gamma \geq 0) & \text{for} \quad \gamma \in \Theta \setminus \Theta_{\bar{0}}, \\ \bar{0} & \text{for} \quad \gamma \in \Theta_{\bar{0}}, \end{cases}$$

where $\Theta_{\bar{0}} = \{\gamma \in \Theta : \mu_\gamma = \bar{0}\}$, hence

$$\begin{aligned} \mu(a) &= \sum_{\gamma\in\Theta\setminus\Theta_{\bar{0}}} \lambda_\gamma \left(\sum_{i=1}^{\infty} \eta_i a_{i_\gamma}\right) + \sum_{\gamma\in\Theta_{\bar{0}}} \bar{0} \left(\sum_{i=1}^{\infty} \eta_i a_{i_\gamma}\right) \\ &= \sum_{i=1}^{\infty} \eta_i \left(\sum_{\gamma\in\Theta\setminus\Theta_{\bar{0}}} \lambda_\gamma(a_{i_\gamma})\right) + \sum_{i=1}^{\infty} \eta_i \left(\sum_{\gamma\in\Theta_{\bar{0}}} \bar{0}(a_{i_\gamma})\right) \\ &= \sum_{i=1}^{\infty} \eta_i \mu(a_i). \end{aligned}$$

Example 2.8. For each $\gamma \in \Gamma$, let $(E_\gamma, ||.||_\gamma)$ be a Banach space and $\mathcal{P}_\gamma = \overline{Conv}(E_\gamma)$ be the cone of all non-empty closed bounded convex subsets of E_γ with the usual multiplication operation of sets by non-negative scalars, a slightly modefied addition $A_\gamma \overline{\oplus} B_\gamma = \overline{A_\gamma + B_\gamma}$ and the set inclusion as order. If $\mathbb{B}_\gamma$ denotes the unit ball of E_γ, then the neighborhood system on $\mathcal{P}_\gamma$ is given by $\mathcal{V}_\gamma = \{\lambda_\gamma \mathbb{B}_\gamma : \lambda_\gamma > 0\}$ and $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is a locally convex cone, which is also weakly-cone complete [7, Example 2.2].

Now, let $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$ and $\mathcal{V} = \sum_{\gamma\in\Gamma}\mathcal{V}_\gamma$. For every $v \in \mathcal{V}$, $v = \sum_{\gamma\in\Delta} \lambda_\gamma \mathbb{B}_\gamma$ the corresponding product neighborhood on $\mathcal{P}$ for elements $A, A' \in \mathcal{P}$, $A = \times_{\gamma\in\Gamma} A_\gamma$, $A' = \times_{\gamma\in\Gamma} A'_\gamma$ is given by

$$A \leq_\Gamma A' \overline{\oplus} v \qquad \text{if} \qquad A_\gamma \subset A'_\gamma \overline{\oplus} \lambda_\gamma \mathbb{B}_\gamma \quad (\text{for all } \gamma \in \Delta).$$

We show that $\mathcal{P}$ is weakly-cone complete. Suppose $v \in \mathcal{V}$, $v = \sum_{\gamma\in\Delta} \lambda_\gamma \mathbb{B}_\gamma$, $B \in \mathcal{P}$, $B = \times_{\gamma\in\Gamma} B_\gamma$ and let $(A_i)_{i\in\mathbb{N}} \subset v(B) \cap (B)v$, $A_i = \times_{\gamma\in\Gamma} A_{i_\gamma}$ for all $i \in \mathbb{N}$ such that $(A_i)_{i\in\mathbb{N}}$ converges to B in the symmetric topology of $\mathcal{P}$. For a convergent series $\sum_{i=1}^{\infty} \eta_i = 1$ in $\mathbb{R}$ that $\eta_i \geq 0$ we set $A = \times_{\gamma\in\Gamma} A_\gamma$, where A_γ is the closure of the set

$$\left\{ \sum_{i=1}^{\infty} \eta_i a_{i_\gamma} \;\middle|\; (a_{i_\gamma})_{i\in\mathbb{N}} \subset A_{i_\gamma} \text{ is bounded in } E_\gamma \right\}.$$

For every $\gamma \in \Gamma$, there is $\rho_\gamma > 0$ such that $B_\gamma \cap \rho_\gamma \mathbb{B}_\gamma \neq \emptyset$, hence $A_{i\gamma} \cap (1+\rho_\gamma)\mathbb{B}_\gamma \neq \emptyset$ for all $i \in \mathbb{N}$. Thus we may choose a sequence $(b_{i_\gamma})_{i\in\mathbb{N}}$ in $A_{i_\gamma} \cap (1+\rho_\gamma)\mathbb{B}_\gamma$ which is bounded in E_γ. Then $a := \times_{\gamma\in\Gamma} a_\gamma \in A$, where $a_\gamma = \sum_{i=1}^{\infty} \eta_i b_{i_\gamma} \in A_\gamma$; hence $A \neq \emptyset$. Since A_γ is closed bounded convex for all $\gamma \in \Gamma$, A is also closed bounded convex (cf. [6, Proposition 2.1]), so $A \in \mathcal{P}$. Fix $\gamma \in \Delta$. For each $a_\gamma \in A_\gamma$, there is a sequence $(a_\gamma^m)_{m\in\mathbb{N}} \subset E_\gamma$, where $a_\gamma^m = \sum_{i=1}^{\infty} \eta_i a_{i_\gamma}^m$ and $(a_{i_\gamma}^m)_{i\in\mathbb{N}} \subset A_{i_\gamma}$ for all $m \in \mathbb{N}$ such that

$$a_\gamma = \lim_{m\to\infty} \sum_{i=1}^{\infty} \eta_i a_{i_\gamma}^m \in B_\gamma \overline{\oplus} \lambda_\gamma \mathbb{B}_\gamma.$$

Consequently, $A \in v(B) \cap (B)v$. Now, let $\mu \in \mathcal{P}^*$ with $\mu(B) < \infty$. By Proposition 2.2, $\mu = \sum_{\gamma\in\Delta} \mu_\gamma$ where $\mu_\gamma \in \lambda'_\gamma \mathbb{B}_\gamma^\circ$ for some $\lambda'_\gamma > 0$ for all $\gamma \in \Delta$. Fix $i \in \mathbb{N}$. If we choose $a_k \in A_k$ for $k = 1, 2, ..., i$ and $a_k = b_k$ for all $k > i$, then

$$\times_{\gamma\in\Gamma} \sum_{k=1}^{i} \eta_k a_{k_\gamma} = \times_{\gamma\in\Gamma} \sum_{k=1}^{\infty} \eta_k a_{k_\gamma} - \times_{\gamma\in\Gamma} \sum_{k=i+1}^{\infty} \eta_k b_{k_\gamma},$$

so

$$\begin{aligned} \times_{\gamma\in\Gamma} \sum_{k=1}^{i} \eta_k A_{k_\gamma} \quad &\subset \quad A \overline{\oplus} \left\{ - \times_{\gamma\in\Gamma} \sum_{k=i+1}^{\infty} \eta_k b_{k_\gamma} \right\} \\ &\subset \quad A \overline{\oplus} \left(\sum_{k=i+1}^{\infty} \eta_k \right) \left(1 + \sum_{\gamma\in\Delta} \rho_\gamma \right) \sum_{\gamma\in\Delta} \lambda_\gamma \mathbb{B}_\gamma, \end{aligned}$$

hence

$$\begin{aligned} \sum_{k=1}^{i} \eta_k \mu(A_k) \quad &= \quad \sum_{k=1}^{i} \eta_k \sum_{\gamma\in\Delta} \mu_\gamma(A_{k_\gamma}) = \times_{\gamma\in\Gamma} \sum_{k=1}^{i} \eta_k \mu_\gamma(A_{k_\gamma}) \\ &\leq \quad \mu(A) + \left(\sum_{k=i+1}^{\infty} \eta_k \right) \left(1 + \sum_{\gamma\in\Delta} \rho_\gamma \right) \sum_{\gamma\in\Delta} \lambda_\gamma \lambda'_\gamma. \end{aligned}$$

On the other hand,

$$\times_{\gamma\in\Gamma} A_\gamma \subset \times_{\gamma\in\Gamma} \sum_{k=1}^{i} \eta_k A_{k_\gamma} \overline{\oplus} \left(\sum_{k=i+1}^{\infty} \eta_k \right) (\times_{\gamma\in\Gamma} B_\gamma \overline{\oplus} \lambda_\gamma \mathbb{B}_\gamma)$$

which yields

$$\begin{aligned}\mu(A) &\leq \sum_{\gamma\in\Delta}\mu_\gamma(\sum_{k=1}^{i}\eta_k\mu_\gamma(A_{k_\gamma})) + (\sum_{k=i+1}^{\infty}\eta_k)(\sum_{\gamma\in\Delta}\mu_\gamma(B_\gamma)+\lambda_\gamma\lambda'_\gamma)) \\ &= \sum_{k=1}^{i}\eta_k\mu(A_k) + (\sum_{k=i+1}^{\infty}\eta_k)(\mu(B)+\sum_{\gamma\in\Delta}\lambda_\gamma\lambda'_\gamma).\end{aligned}$$

Thus $\mu(A)=\sum_{k=1}^{\infty}\eta_k\mu(A_k)$.

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