



A NOTE ON PRODUCT TOPOLOGIES IN LOCALLY CONVEX CONES

M.R. MOTALLEBI*

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ABSTRACT. We consider the locally convex product cone topologies and prove that the product topology of weakly cone-complete locally convex cones is weakly cone-complete. In particular, we deduce that a product cone topology is barreled whenever its components are weakly cone-complete and carry the countable neighborhood bases.

Keywords: Products cone topologies, weak cone-completeness, barreledness.

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1. Introduction

In theory of locally convex cones, a linear functional may take the infinite values $+\infty$, which is one of the most important differences in comparison with the convex spaces. Further, we recall from [2] that any linear functional on product cone topology is written by a finite sum of functionals on its components. In this note, taking into account the above, we prove that under the product topology, the weak cone-completeness of locally convex cones is preserved. As a specific result; we infer that if the components of product topology are weakly-cone complete with countable bases, then it is barreled.

An *ordered cone* is a set \mathcal{P} endowed with an addition $(a, b) \mapsto a + b$ and a scalar multiplication $(\alpha, a) \mapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and the scalar multiplication is associative and distributive, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone \mathcal{P} carries a (partial) order, i.e., a reflexive transitive relation \leq that is compatible with the algebraic operations, i.e., $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. In any cone \mathcal{P} , equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone \mathcal{P} that contains an *abstract neighborhood system* \mathcal{V} , i.e., a subset of positive elements that is directed

*Corresponding author, ORCID: 0000-0002-1761-0601

E-mail: motallebi@uma.ac.ir

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downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of \mathcal{V} define *upper (lower) neighborhoods* for the elements of \mathcal{P} by $v(a) = \{b \in \mathcal{P} : b \leq a + v\}$ ($((a)v = \{b \in \mathcal{P} : a \leq b + v\}$), creating the *upper (lower) topologies* on \mathcal{P} . Their common refinement is called the *symmetric topology*. We assume all elements of \mathcal{P} to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system \mathcal{V} . For a locally convex cone $(\mathcal{P}, \mathcal{V})$ the collection of all sets $\tilde{v} \subseteq \mathcal{P}^2$, where $\tilde{v} = \{(a, b) : a \leq b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on \mathcal{P} . On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including \mathcal{P} as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2].

For cones \mathcal{P} and \mathcal{P}' , a map $T : \mathcal{P} \rightarrow \mathcal{P}'$ is called a *linear operator*, if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If \mathcal{V} and \mathcal{W} are abstract neighborhood systems on \mathcal{P} and \mathcal{P}' , a linear operator $T : \mathcal{P} \rightarrow \mathcal{P}'$ is called *uniformly continuous* if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on \mathcal{P} and \mathcal{P}' .

Remark 1.1. In the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. If \mathcal{P} is a locally convex cone, then the set of all uniformly continuous linear functionals $\mu : \mathcal{P} \rightarrow \overline{\mathbb{R}}$ is a cone called the *dual cone* of \mathcal{P} and denoted by \mathcal{P}^* . In particular, $\overline{\mathbb{R}}^* = \{\alpha \in \mathbb{R} : \alpha \geq 0\} \cup \{0\}$, where

$$\bar{0}(a) = \begin{cases} 0 & \text{for } a \in \mathbb{R}, \\ +\infty & \text{for } a = +\infty. \end{cases}$$

For details see [8, Example 2.3].

A convex set $U \subset \mathcal{P} \times \mathcal{P}$ is called *barrel*, if for every $b \in \mathcal{P}$ there is $v \in \mathcal{V}$ such that for every $a \in v(b) \cap (b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda U$; in addition if $(a, b) \notin U$, then there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in U$ and $\mu(a) > \mu(b) + 1$. The cone \mathcal{P} is *barreled* if for every barrel U and every element $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda U$ for all $a \in v(b) \cap (b)v$.

2. Products, weak cone-completeness and barreledness

Locally convex cones have been investigated from the point of view of products in [2-5]. In this section, we intend to study the product topologies in terms of the weak cone-completeness, which is one of the approaches in locally convex cones presented in [7]. First, we review the concept of locally convex

product cones: Let $\mathcal{P}_\gamma, \gamma \in \Gamma$, be cones and put $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$. For elements $a, b \in \mathcal{P}$, $a = \times_{\gamma \in \Gamma} a_\gamma, b = \times_{\gamma \in \Gamma} b_\gamma$ and $\alpha \geq 0$ we set $a + b = \times_{\gamma \in \Gamma} (a_\gamma + b_\gamma)$ and $\alpha a = \times_{\gamma \in \Gamma} (\alpha a_\gamma)$. With these operations \mathcal{P} is a cone which is called the *product cone* of \mathcal{P}_γ . The subcone of the product cone \mathcal{P} spanned by $\cup \mathcal{P}_\gamma$ (more precisely, by $\cup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma : \mathcal{P}_\gamma \rightarrow \mathcal{P}$ is the injection mapping) is said to be the *direct sum cone* of \mathcal{P}_γ and denoted by $\mathcal{Q} = \sum_{\gamma \in \Gamma} \mathcal{P}_\gamma$. It is worth remembering that here we use only positive scalars.

For $a, b \in \mathcal{P}$, we set $a \leq_\Gamma b$ if $a_\gamma \leq_\gamma b_\gamma$ for all $\gamma \in \Gamma$. As required, the order \leq_Γ is reflexive, transitive and compatible with the algebraic operations of \mathcal{P} . This is the weakest order on $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$ such that every projection mapping $\phi_\gamma : \mathcal{P} \rightarrow \mathcal{P}_\gamma$, given by $\phi_\gamma(a) = a_\gamma$, is monotone. For each $\gamma \in \Gamma$, let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ be a locally convex cone and let us denote by $\mathcal{V} = \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma$, the direct sum of the abstract neighborhoods \mathcal{V}_γ . In fact, \mathcal{V} is a subcone (with out zero) of the product cone of the all corresponding full cones containing \mathcal{V}_γ . The direct sum $\mathcal{V} = \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma$ leads us to define the *coarsest convex quasi-uniform structure*, as well as, the *coarsest locally convex cone topology* on the product cone \mathcal{P} such that the all projection mappings ϕ_γ are uniformly continuous:

Definition 2.1. For elements $a, b \in \mathcal{P}$, $a = \times_{\gamma \in \Gamma} a_\gamma, b = \times_{\gamma \in \Gamma} b_\gamma$ and $v \in \mathcal{V}, v = \sum_{\gamma \in \Delta} v_\gamma$, we define

$$a \leq_\Gamma b + v \quad \text{if} \quad a_\gamma \leq_\gamma b_\gamma + v_\gamma \quad (\text{for all } \gamma \in \Delta)$$

where Δ is a finite subset of Γ . The subsets $\{(a, b) \in \mathcal{P}^2 : a \leq_\Gamma b + v\}$ for all $v \in \mathcal{V}$ form the coarsest convex quasi-uniform structure on \mathcal{P} which makes all the projection mappings ϕ_γ uniformly continuous. Then, according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_0$, with abstract neighborhood system $V = \{0\} \oplus \mathcal{V}$, whose neighborhoods yield the same quasi-uniform structure on \mathcal{P} . The elements $v \in \mathcal{V}, v = \sum_{\gamma \in \Delta} v_\gamma$, form a basis for V in the following sense: For every $v \in V$ there is $v \in \mathcal{V}$ such that $a \leq_\Gamma b + v$ for $a, b \in \mathcal{P}$ implies that $a \leq_\Gamma b \oplus v$. The locally convex cone topology on \mathcal{P} induced by \mathcal{V} is called the *locally convex product cone of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$* and is denoted by $(\mathcal{P}, \mathcal{V})$ [2, Definition 2.1].

Proposition 2.2. *If $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma \in \Gamma} \mathcal{P}_\gamma, \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma)$ is the locally convex product cone, then for each neighborhood $v \in \mathcal{V}, v = \sum_{\gamma \in \Delta} v_\gamma$, we have*

$$\left(\sum_{\gamma \in \Delta} v_\gamma \right)^\circ \subseteq \sum_{\gamma \in \Delta} v_\gamma^\circ \subseteq n \left(\sum_{\gamma \in \Delta} v_\gamma \right)^\circ \quad \text{for some } n \in \mathbb{N},$$

in particular, $\mathcal{P}^* = \sum_{\gamma \in \Gamma} \mathcal{P}_\gamma^*$; where \mathcal{P}^* and \mathcal{P}_γ^* are dual cones of \mathcal{P} and \mathcal{P}_γ , respectively.

Proof. See [2, Proposition 2.5]. □

Proposition 2.3. *Suppose $(\mathcal{P}, \mathcal{V})$ is the locally convex product cone of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$. Then if $(a_i)_{i \in \mathcal{I}} \subset \mathcal{P}, a_i = \times_{\gamma \in \Gamma} a_{i\gamma}$ for all $i \in \mathcal{I}$ such that $(a_i)_{i \in \mathcal{I}}$ converges to*

$b \in \mathcal{P}$, $b = \times_{\gamma \in \Gamma} b_\gamma$ in the symmetric topology of \mathcal{P} , then $(a_{i\gamma})_{i \in \mathcal{I}}$ converges to b_γ in the symmetric topology of \mathcal{P}_γ for all $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$. For each $v_\gamma \in \mathcal{V}_\gamma$, $v := v_\gamma \in \mathcal{V}$, so there is $i_0 \in \mathcal{I}$ such that $a_i \in v(b) \cap (b)v$ for all $i \in \mathcal{I}$ with $i \geq i_0$, i.e., $a_{i\gamma} \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$. That is, $(a_{i\gamma})_{i \in \mathbb{N}}$ converges to b_γ in the symmetric topology of \mathcal{P}_γ . \square

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *weakly cone-complete* if for all $b \in \mathcal{P}$ and $v \in \mathcal{V}$, every sequence $(a_i)_{i \in \mathbb{N}}$ in $v(b) \cap (b)v$ that converges to b in the symmetric topology of \mathcal{P} and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$, there is $a \in v(b) \cap (b)v$ such that

$$(1) \quad \mu(a) = \sum_{i=1}^{\infty} \eta_i \mu(a_i)$$

for all $\mu \in \mathcal{P}^*$ with $\mu(b) < \infty$ [7].

Theorem 2.4. *A product cone topology $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma \in \Gamma} \mathcal{P}_\gamma, \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma)$ is weakly cone-complete, whenever $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is weakly cone-complete for all $\gamma \in \Gamma$.*

Proof. Suppose $b \in \mathcal{P}$, $b = \times_{\gamma \in \Gamma} b_\gamma$, $v \in \mathcal{V}$, $v = \sum_{\gamma \in \Delta} v_\gamma$ and let $(a_i)_{i \in \mathbb{N}} \subset v(b) \cap (b)v$, $a_i = \times_{\gamma \in \Gamma} a_{i\gamma}$ for all $i \in \mathbb{N}$ such that $(a_i)_{i \in \mathbb{N}}$ converges to b in the symmetric topology of \mathcal{P} and $\sum_{i=1}^{\infty} \eta_i = 1$, $\eta_i > 0$. By Proposition 2.3, for each $\gamma \in \Gamma$, the sequence $(a_{i\gamma})_{i \in \mathbb{N}} \subset v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ converges to b_γ in the symmetric topology of \mathcal{P}_γ , so from the weak cone-completeness of \mathcal{P}_γ there exists $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ such that

$$(2) \quad \mu_\gamma(a_\gamma) = \sum_{i=1}^{\infty} \eta_i a_{i\gamma} \quad \text{for all } \mu_\gamma \in \mathcal{P}_\gamma^* \text{ with } \mu_\gamma(b_\gamma) < \infty.$$

Suppose $a := \times_{\gamma \in \Gamma} a_\gamma \in v(b) \cap (b)v$. By Proposition 2.2, for every $\mu \in \mathcal{P}^*$, there is a finite set $\Theta \subset \Gamma$ such that $\mu = \sum_{\gamma \in \Theta} \mu_\gamma$, where $\mu_\gamma \in \mathcal{P}_\gamma^*$ for all $\gamma \in \Theta$. If $\mu(b) < \infty$, then $\mu_\gamma(b_\gamma) < \infty$ for all $\gamma \in \Theta$, hence (2) yields

$$\begin{aligned} \mu(a) &= \sum_{\gamma \in \Theta} \sum_{i=1}^{\infty} \eta_i \mu_\gamma(a_{i\gamma}) = \sum_{i=1}^{\infty} \eta_i \sum_{\gamma \in \Theta} \mu_\gamma(a_{i\gamma}) \\ &= \sum_{i=1}^{\infty} \eta_i \sum_{\gamma \in \Theta} \mu_\gamma(\times_{\gamma \in \Gamma} a_{i\gamma}) = \sum_{i=1}^{\infty} \eta_i \mu(a_i). \end{aligned}$$

That is, $(\mathcal{P}, \mathcal{V})$ is weakly cone-complete. \square

A *neighborhood base* for a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a subset \mathcal{U} of \mathcal{V} such that for every $v \in \mathcal{V}$ there exists some $u \in \mathcal{U}$ with $u \leq v$. We cite Theorem 2.3 from [7]:

Theorem 2.5. *Every weakly cone-complete locally convex cone $(\mathcal{P}, \mathcal{V})$ with a countable neighborhood base is barreled.*

As a consequence of Theorem 2.4 and Theorem 2.5, we have:

Corollary 2.6. *If, for each $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is weakly cone-complete and carries a countable base, then $(\mathcal{P}, \mathcal{V}) = (\times_{\gamma \in \Gamma} \mathcal{P}_\gamma, \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma)$ is barreled.*

Example 2.7. The cone $\overline{\mathbb{R}}$ with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ is weakly cone-complete. For, let $b \in \overline{\mathbb{R}}$, $\epsilon \in \mathcal{V}$, $(a_i)_{i \in \mathbb{N}} \subset \epsilon(b) \cap (b)\epsilon$ converges to b in the symmetric topology of $\overline{\mathbb{R}}$ and let $\sum_{i=1}^\infty \eta_i = 1$, $\eta_i > 0$. If $b = +\infty$, then for $a = +\infty$ the assertion holds. Let $b \in \mathbb{R}$, $a := \sum_{i=1}^\infty \eta_i a_i \in \epsilon(b) \cap (b)\epsilon$ and $\mu \in \overline{\mathbb{R}}^*$ with $\mu(b) < \infty$. If $\mu = \bar{0}$, then clearly (1) holds and if $\mu = \lambda$ for some $\lambda \geq 0$, then $\mu(a) = \lambda(\sum_{i=1}^\infty \eta_i a_i) = \sum_{i=1}^\infty \eta_i \mu(a_i)$.

Now, let $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$ and $\mathcal{V} = \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma$, where $\mathcal{P}_\gamma = \overline{\mathbb{R}}$ and $\mathcal{V}_\gamma = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ for all $\gamma \in \Gamma$. We show that $(\mathcal{P}, \mathcal{V})$ is weakly cone-complete. Suppose $b \in \mathcal{P}$, $b = \times_{\gamma \in \Gamma} b_\gamma$, $v \in \mathcal{V}$, $v = \sum_{\gamma \in \Delta} \epsilon_\gamma$ and let $(a_i)_{i \in \mathbb{N}} \subset v(b) \cap (b)v$, $a_i = \times_{\gamma \in \Gamma} a_{i,\gamma}$ for all $i \in \mathbb{N}$ such that $(a_i)_{i \in \mathbb{N}}$ converges to b in the symmetric topology of \mathcal{P} and $\sum_{i=1}^\infty \eta_i = 1$, $\eta_i > 0$. If we set $a = \times_{\gamma \in \Gamma} a_\gamma$; where $a_\gamma := \sum_{i=1}^\infty \eta_i a_{i,\gamma}$ for all $\gamma \in \Gamma$ then $a \in v(b) \cap (b)v$, since $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ for all $\gamma \in \Delta$.

For every $\mu \in \mathcal{P}^*$, $\mu = \sum_{\gamma \in \Theta} \mu_\gamma$ where $\mu_\gamma \in \overline{\mathbb{R}}^*$ for all $\gamma \in \Theta$. If $\mu(b) < +\infty$ then Remark 1.1 yields

$$\mu_\gamma = \begin{cases} \lambda_\gamma \quad (\text{some } \lambda_\gamma \geq 0) & \text{for } \gamma \in \Theta \setminus \Theta_{\bar{0}}, \\ \bar{0} & \text{for } \gamma \in \Theta_{\bar{0}}, \end{cases}$$

where $\Theta_{\bar{0}} = \{\gamma \in \Theta : \mu_\gamma = \bar{0}\}$, hence

$$\begin{aligned} \mu(a) &= \sum_{\gamma \in \Theta \setminus \Theta_{\bar{0}}} \lambda_\gamma \left(\sum_{i=1}^\infty \eta_i a_{i,\gamma} \right) + \sum_{\gamma \in \Theta_{\bar{0}}} \bar{0} \left(\sum_{i=1}^\infty \eta_i a_{i,\gamma} \right) \\ &= \sum_{i=1}^\infty \eta_i \left(\sum_{\gamma \in \Theta \setminus \Theta_{\bar{0}}} \lambda_\gamma (a_{i,\gamma}) \right) + \sum_{i=1}^\infty \eta_i \left(\sum_{\gamma \in \Theta_{\bar{0}}} \bar{0} (a_{i,\gamma}) \right) \\ &= \sum_{i=1}^\infty \eta_i \mu(a_i). \end{aligned}$$

Example 2.8. For each $\gamma \in \Gamma$, let $(E_\gamma, \|\cdot\|_\gamma)$ be a Banach space and $\mathcal{P}_\gamma = \overline{\text{Conv}}(E_\gamma)$ be the cone of all non-empty closed bounded convex subsets of E_γ with the usual multiplication operation of sets by non-negative scalars, a slightly modified addition $A_\gamma \oplus B_\gamma = \overline{A_\gamma + B_\gamma}$ and the set inclusion as order. If \mathbb{B}_γ denotes the unit ball of E_γ , then the neighborhood system on \mathcal{P}_γ is given by $\mathcal{V}_\gamma = \{\lambda_\gamma \mathbb{B}_\gamma : \lambda_\gamma > 0\}$ and $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is a locally convex cone, which is also weakly-cone complete [7, Example 2.2].

Now, let $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$ and $\mathcal{V} = \sum_{\gamma \in \Gamma} \mathcal{V}_\gamma$. For every $v \in \mathcal{V}$, $v = \sum_{\gamma \in \Delta} \lambda_\gamma \mathbb{B}_\gamma$ the corresponding product neighborhood on \mathcal{P} for elements $A, A' \in \mathcal{P}$, $A = \times_{\gamma \in \Gamma} A_\gamma, A' = \times_{\gamma \in \Gamma} A'_\gamma$ is given by

$$A \leq_\Gamma A' \oplus v \quad \text{if} \quad A_\gamma \subset A'_\gamma \oplus \lambda_\gamma \mathbb{B}_\gamma \quad (\text{for all } \gamma \in \Delta).$$

We show that \mathcal{P} is weakly-cone complete. Suppose $v \in \mathcal{V}$, $v = \sum_{\gamma \in \Delta} \lambda_\gamma \mathbb{B}_\gamma$, $B \in \mathcal{P}$, $B = \times_{\gamma \in \Gamma} B_\gamma$ and let $(A_i)_{i \in \mathbb{N}} \subset v(B) \cap (B)v$, $A_i = \times_{\gamma \in \Gamma} A_{i\gamma}$ for all $i \in \mathbb{N}$ such that $(A_i)_{i \in \mathbb{N}}$ converges to B in the symmetric topology of \mathcal{P} . For a convergent series $\sum_{i=1}^{\infty} \eta_i = 1$ in \mathbb{R} that $\eta_i \geq 0$ we set $A = \times_{\gamma \in \Gamma} A_\gamma$, where A_γ is the closure of the set

$$\left\{ \sum_{i=1}^{\infty} \eta_i a_{i\gamma} \mid (a_{i\gamma})_{i \in \mathbb{N}} \subset A_{i\gamma} \text{ is bounded in } E_\gamma \right\}.$$

For every $\gamma \in \Gamma$, there is $\rho_\gamma > 0$ such that $B_\gamma \cap \rho_\gamma \mathbb{B}_\gamma \neq \emptyset$, hence $A_{i\gamma} \cap (1 + \rho_\gamma) \mathbb{B}_\gamma \neq \emptyset$ for all $i \in \mathbb{N}$. Thus we may choose a sequence $(b_{i\gamma})_{i \in \mathbb{N}}$ in $A_{i\gamma} \cap (1 + \rho_\gamma) \mathbb{B}_\gamma$ which is bounded in E_γ . Then $a := \times_{\gamma \in \Gamma} a_\gamma \in A$, where $a_\gamma = \sum_{i=1}^{\infty} \eta_i b_{i\gamma} \in A_\gamma$; hence $A \neq \emptyset$. Since A_γ is closed bounded convex for all $\gamma \in \Gamma$, A is also closed bounded convex (cf. [6, Proposition 2.1]), so $A \in \mathcal{P}$. Fix $\gamma \in \Delta$. For each $a_\gamma \in A_\gamma$, there is a sequence $(a_\gamma^m)_{m \in \mathbb{N}} \subset E_\gamma$, where $a_\gamma^m = \sum_{i=1}^{\infty} \eta_i a_{i\gamma}^m$ and $(a_{i\gamma}^m)_{i \in \mathbb{N}} \subset A_{i\gamma}$ for all $m \in \mathbb{N}$ such that

$$a_\gamma = \lim_{m \rightarrow \infty} \sum_{i=1}^{\infty} \eta_i a_{i\gamma}^m \in B_\gamma \overline{\oplus} \lambda_\gamma \mathbb{B}_\gamma.$$

Consequently, $A \in v(B) \cap (B)v$. Now, let $\mu \in \mathcal{P}^*$ with $\mu(B) < \infty$. By Proposition 2.2, $\mu = \sum_{\gamma \in \Delta} \mu_\gamma$ where $\mu_\gamma \in \lambda'_\gamma \mathbb{B}_\gamma^\circ$ for some $\lambda'_\gamma > 0$ for all $\gamma \in \Delta$. Fix $i \in \mathbb{N}$. If we choose $a_k \in A_k$ for $k = 1, 2, \dots, i$ and $a_k = b_k$ for all $k > i$, then

$$\times_{\gamma \in \Gamma} \sum_{k=1}^i \eta_k a_{k\gamma} = \times_{\gamma \in \Gamma} \sum_{k=1}^{\infty} \eta_k a_{k\gamma} - \times_{\gamma \in \Gamma} \sum_{k=i+1}^{\infty} \eta_k b_{k\gamma},$$

so

$$\begin{aligned} \times_{\gamma \in \Gamma} \sum_{k=1}^i \eta_k A_{k\gamma} &\subset A \overline{\oplus} \left\{ - \times_{\gamma \in \Gamma} \sum_{k=i+1}^{\infty} \eta_k b_{k\gamma} \right\} \\ &\subset A \overline{\oplus} \left(\sum_{k=i+1}^{\infty} \eta_k \right) \left(1 + \sum_{\gamma \in \Delta} \rho_\gamma \right) \sum_{\gamma \in \Delta} \lambda_\gamma \mathbb{B}_\gamma, \end{aligned}$$

hence

$$\begin{aligned} \sum_{k=1}^i \eta_k \mu(A_k) &= \sum_{k=1}^i \eta_k \sum_{\gamma \in \Delta} \mu_\gamma(A_{k\gamma}) = \times_{\gamma \in \Gamma} \sum_{k=1}^i \eta_k \mu_\gamma(A_{k\gamma}) \\ &\leq \mu(A) + \left(\sum_{k=i+1}^{\infty} \eta_k \right) \left(1 + \sum_{\gamma \in \Delta} \rho_\gamma \right) \sum_{\gamma \in \Delta} \lambda_\gamma \lambda'_\gamma. \end{aligned}$$

On the other hand,

$$\times_{\gamma \in \Gamma} A_\gamma \subset \times_{\gamma \in \Gamma} \sum_{k=1}^i \eta_k A_{k\gamma} \overline{\oplus} \left(\sum_{k=i+1}^{\infty} \eta_k \right) (\times_{\gamma \in \Gamma} B_\gamma \overline{\oplus} \lambda_\gamma \mathbb{B}_\gamma)$$

which yields

$$\begin{aligned}\mu(A) &\leq \sum_{\gamma \in \Delta} \mu_{\gamma} \left(\sum_{k=1}^i \eta_k \mu_{\gamma}(A_{k_{\gamma}}) \right) + \left(\sum_{k=i+1}^{\infty} \eta_k \right) \left(\sum_{\gamma \in \Delta} \mu_{\gamma}(B_{\gamma}) + \lambda_{\gamma} \lambda'_{\gamma} \right) \\ &= \sum_{k=1}^i \eta_k \mu(A_k) + \left(\sum_{k=i+1}^{\infty} \eta_k \right) (\mu(B) + \sum_{\gamma \in \Delta} \lambda_{\gamma} \lambda'_{\gamma}).\end{aligned}$$

Thus $\mu(A) = \sum_{k=1}^{\infty} \eta_k \mu(A_k)$.

References

- [1] K. Keimel, W. Roth, *Ordered cones and approximation*, Lecture Notes in Mathematics, vol. 1517, Springer Verlag, Heidelberg-Berlin-New York, 1992.
- [2] M.R. Motallebi, *Locally convex product and direct sum cones*, Mediterr. J. Math., 11 **3** (2014), 913-927
- [3] M.R. Motallebi, *On weak completeness of products and direct sums in locally convex cones*, Period. Math. Hung., 75 **2**(2017), 322-329
- [4] M.R. Motallebi, *Weak compactness in locally convex cones*, Positivity, 23 **2**(2019), 303-313
- [5] M.R. Motallebi, *Weak compactness of direct sums in locally convex cones*, Stud. Sci. Math. Hung., 55 **4**(2018), 487-497
- [6] M.R. Motallebi, H. Saiflu, *Products and direct sums in locally convex cones*, Can. Math. Bull. vol. 55 **4**, (2012), 783-798.
- [7] W. Roth, *A uniform boundedness theorem for locally convex cones*, Proc. Amer. Math. Soc. 126 **7**(1998), 1973-1982
- [8] W. Roth, *Locally convex lattice cones*, J. Convex Anal. 16 **(1)**, (2009), 1-31.

MOHAMMAD REZA MOTALLEBI
 ORCID NUMBER: 0000-0002-1761-0601
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF MOHAGHEGH ARDABIL
 ARDABIL, IRAN
Email address: motallebi@uma.ac.ir