Journal of Mahani Mathematical Research Center
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

AN ALGORITHM FOR CONSTRUCTING INTEGRAL ROW STOCHASTIC MATRICES

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Article type: Research Article
(Received: 21 April 2019, Revised: 21 November 2021, Accepted: 03 December 2021)
(Available Online: 08 December 2021)
(Communicated by F. Abdollahi)

Abstract. Let $\mathbf{M}_n$ be the set of all n-by-n real matrices, and let $\mathbb{R}^n$ be the set of all n-by-1 real (column) vectors. An n-by-n matrix $R = [r_{ij}]$ with nonnegative entries is called row stochastic, if $\sum_{k=1}^{n} r_{ik}$ is equal to 1 for all i $(1 \leq i \leq n)$. In fact, $Re = e$, where $e = (1, \ldots, 1)^t \in \mathbb{R}^n$. A matrix $R \in \mathbf{M}_n$ is called integral row stochastic, if each row has exactly one nonzero entry, $+1$, and other entries are zero. In the present paper, we provide an algorithm for constructing integral row stochastic matrices, and also we show the relationship between this algorithm and majorization theory.

Keywords: Eigenvalue, Majorization, Integral row stochastic.
2020 MSC: Primary 34B24, 34B27.

1. Introduction

Let $\mathbf{M}_n$ be the set of all n-by-n real matrices, and $\mathbb{R}^n$ be the set of all n-by-1 real column vectors. A matrix $R \in \mathbf{M}_n$ with nonnegative entries is called row stochastic if $Re = e$, where $e = (1, \ldots, 1)^t \in \mathbb{R}^n$. If each row of a matrix R has exactly a nonzero entry, $+1$, and its other entries zero, R is called integral row stochastic. The collection of all n-by-n integral row stochastic matrices is denoted by $\mathcal{R}(n)$.

The X-ray is inspired by a region of discrete radiology that is used in medicine. The X-rays are vectors of dimension $2n - 1$ defined by summing entries a long diagonals or anti-diagonals of a matrix of order n. See [2]- [4]. The term L-ray is inspired by the notion of X-ray in area of discrete tomography [9]. They are interesting in mass distribution problems. The sets $L^{(k)}$ are shaped like an L (backward). The L-ray is defined in terms of sums of the entries in the blocks of a certain "L-shaped" partition of the positions of a matrix $A \in \mathbf{M}_n$.

For each $1 \leq k \leq n$ let

$$L^{(k)} = \{(k, 1), (k, 2), \ldots, (k, k), (k - 1, k), \ldots, (1, k)\}.$$

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DOI: 10.22103/jmmrc.2021.13883.1089

How to cite: A. Ilkhanizadeh Manesh, *An algorithm for constructing integral row stochastic matrices*, J. Mahani Math. Res. Cent. 2022; 11(1): 69-77.

We observe that $L^{(k)}$ consists of the first k positions in row k and column k.

Suppose as an example $n = 5$.

$$\begin{aligned}
L^{(1)} &= \{(1,1)\},\\
L^{(2)} &= \{(2,1),(2,2),(1,2)\},\\
L^{(3)} &= \{(3,1),(3,2),(3,3),(2,3),(1,3)\},\\
L^{(4)} &= \{(4,1),(4,2),(4,3),(4,4),(3,4),(2,4),(1,4)\},\\
L^{(5)} &= \{(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5)\}.
\end{aligned}$$

We define the linear function

$$\sigma : \mathbf{M}_n \to \mathbb{R}^n$$
$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))^t,$$

by

$$\sigma_k(A) = \sum_{(i,j)\in L^{(k)}} a_{ij}.$$

$\sigma(A)$ is called the L-ray of A.

In this paper, we give an algorithm for constructing integral row stochastic matrices. A main reference concerning majorization is [10]. For more information about majorization see [1], [6], [7], and [8].

2. Integral row stochastic matrices

In this section, we provide an algorithm for constructing integral row stochastic matrices.

Note that $x = (x_1, x_2, \ldots, x_n) \in \{0, 1, 2, \ldots, n\}^n$ means that $x_i \in \{0, 1, 2, \ldots, n\}$ for each $1 \le i \le n$.

Following [5] we use the following variation of majorization.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)^t$, $y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec^* y$ if $\sum_{i=1}^k x_i \le \sum_{i=1}^k y_i$ for $k < n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

Consider the following Algorithm. Theorem 2.2 ensures that Algorithm offers an integral row stochastic matrix A with $\sigma(A) = x$. e denotes an all ones vector.

Algorithm

Input: A vector $x = (x_1, x_2, \ldots, x_n) \in \{0, 1, 2, \ldots, n\}^n$ with $x \prec^* e$.

1. Initialize: Let $A = (a_{ij}) = 0_n$ (the zero matrix).

2. for $k = 1, 2, \ldots, n$ do

   (a) If $x_k = 1$, let $a_{kk} = 1$.

   Do not use the used rows.

(b) If $x_k = 2$, let l be maximal with $l < k$ and $\sigma_l(A) = 0$.
Let $a_{kl} = a_{lk} = 1$.
(c) If $x_k = j$ whenever $j = 3, 4, \ldots, n$, let l be maximal with $l < k$ and $\sigma_l(A) = 0$, and let $i_1, i_2, \ldots, i_{j-2}$ be the largest indices with $i_1, i_2, \ldots, i_{j-2} < k$, $i_1, i_2, \ldots, i_{j-2} \neq l$, and $x_{i_1}, x_{i_1}, \ldots, x_{i_{j-2}} = 0$.
Let $a_{kl} = a_{lk} = a_{i_1 k} = a_{i_2 k} = \cdots = a_{i_{j-2} k} = 1$.
Output: A.

Theorem 2.2. *Let $x \in \{0, 1, 2, \ldots, n\}^n$ and $x \prec^* e$. Then Algorithm offers an integral row stochastic matrix A with $\sigma(A) = x$.*

Proof. Suppose that $x \in \{0, 1, 2, \ldots, n\}^n$ and $x \prec^* e$. We claim that Algorithm constructs some $A \in \mathcal{R}(n)$ such that $x = \sigma(A)$.

Claim: After each iteration k (of step 2) the present matrix A has the property $\sigma_i(A) = x_i$ for each $i = 1, 2, \ldots, k$.

Proof of Claim: Use induction on k. For $k = 1$ there is nothing to prove. Suppose that $k \leq n$ and the statement holds for $k' < k$.
If $x_k = 0$, then A is not modified. So $\sigma_k(A) = 0$, and the induction statement holds in this case. We consider three cases.

Case 1. If $x_k = 1$, then $a_{kk} = 1$, and so $\sigma_k(A) = x_k$.

Case 2. If $x_k = 2$, as $x \prec^* e$, we see that $\sum_{i=1}^{k} x_i \leq k$, and hence $\sum_{i=1}^{k-1} x_i \leq k - 2$.

If for each $1 \leq i \leq k-1$ we have $\sigma_i(A) \neq 0$, the hypothesis induction ensures that $x_i \neq 0$. Since $x \in \{0, 1, 2, \ldots, n\}^n$, we observe that $1 \leq x_i$, for each $1 \leq i \leq k-1$.

Then

$$\begin{aligned} k - 1 &\leq \textstyle\sum_{i=1}^{k-1} x_i \\ &\leq k - 2, \end{aligned}$$

which is a contradiction. This means that there is some $1 \leq i \leq k-1$ such that $\sigma_i(A) = 0$.

Let l be maximal with $l < k$, and $\sigma_l(A) = 0$. Algorithm ensures that

$$a_{kl} = a_{lk} = 1.$$

Thus $\sigma_k(A) = 2$, as desired.

Case 3. If $x_k = j$ for $3 \leq j \leq n$, then

$$\begin{aligned} \textstyle\sum_{i=1}^{k-1} \sigma_i &= \textstyle\sum_{i=1}^{k-1} x_i \\ &\leq k - x_k \\ &= k - j, \end{aligned}$$

because $x \prec^* e$.

If for each $1 \leq i \leq k-1$ we have $\sigma_i(A) \neq 0$, then $1 \leq x_i$ and so

$$\begin{aligned} k-1 &\leq \textstyle\sum_{i=1}^{k-1} x_i \\ &\leq k-j, \end{aligned}$$

a contradiction. It follows that $\sigma_i(A) = 0$ for some $1 \leq i \leq k-1$.

Suppose that l is maximal with $l < k$, and $\sigma_l(A) = 0$. Algorithm states that

$$a_{kl} = a_{lk} = 1.$$

We claim that there exist

$$i_1, i_2, \ldots, i_{j-2} < k, \quad i_1, i_2, \ldots, i_{j-2} \neq l,$$

with

$$x_{i_1}, x_{i_1}, \ldots, x_{i_{j-2}} = 0.$$

Define

$$I_1 = \{1 \leq i \leq k-1 \mid i \neq l, x_i = 0\},$$

and

$$I_2 = \{1 \leq i \leq k-1 \mid i \neq l, x_i \neq 0\}.$$

We observe that

$$I_1 \cup I_2 = \{1, 2, \ldots, k-1\} \setminus \{l\},$$

and so

$$|I_1| + |I_2| = k-2.$$

We should prove that $|I_1| \geq j-2$. If $|I_1| < j-2$; it implies that $|I_2| > k-j$. On the other hand,

$$\begin{aligned} k-j &\geq \textstyle\sum_{i=1}^{k-1} x_i \\ &= \textstyle\sum_{i=1, i\neq l}^{k-1} x_i + x_l \\ &= \textstyle\sum_{i\in I_1} x_i + \sum_{i \in I_2} x_i \\ &= \textstyle\sum_{i \in I_2} x_i \\ &\geq |I_2| \\ &> k-j, \end{aligned}$$

a contradiction. So $|I_1| \geq j-2$. This shows the existence of $i_1, i_2, \ldots, i_{j-2}$ in step 2, and by putting

$$a_{i_1 k} = a_{i_2 k} = \cdots = a_{i_{j-2} k} = 1,$$

we have $\sigma_k(A) = x_k$. So the statement holds by the induction.

If $k = n$, then $\sigma_1(A) = x_1$, $\sigma_2(A) = x_2, \ldots, \sigma_n(A) = x_n$, and so $\sigma(A) = x$.

It remains to prove that $A \in \mathcal{R}(n)$. That is, every row of A has exactly a nonzero entry, $+1$, and the other entries are zero.

Algorithm ensures that A is $(0,1)$-matrix. It is enough to show that

$$r_1 = r_2 = \cdots = r_n = 1.$$

We claim that for each $1 \leq i \leq n$ the case $r_i = 0$ can not be happen. If $r_{i'} = 0$ for some $1 \leq i' \leq n$, then $x_{i'} = 0$, because if $x_{i'} \neq 0$;

- for $x_{i'} = 1$ we see $a_{i'i'} = 1$, and so $r_{i'} \neq 0$, a contradiction.
- for $x_{i'} \neq 1$, there exists some $1 \leq i_0 \leq i' - 1$ such that $x_{i_0} = 0$ (because if for each $1 \leq i \leq i' - 1$ we have $x_i \neq 0$, then

$$\begin{aligned} i' &\geq \textstyle\sum_{i=1}^{i'} x_i \\ &= x_{i'} + \textstyle\sum_{i=1}^{i'-1} x_i \\ &\geq i' + 1, \end{aligned}$$

a contradiction). In this case, let $1 \leq l \leq i' - 1$ be the maximal such that $x_l = 0$. So, $a_{i'l} = 1$, and then $r_{i'} \neq 0$, a contradiction.

We saw if there exists some $1 \leq k \leq n$ such that $r_k = 0$, then $x_k = 0$, and hence $\sigma_k(A) = 0$. It shows that

$$\begin{aligned} \textstyle\sum_{i=1}^{k-1} x_i &= \textstyle\sum_{i=1}^{k} x_i \\ &\leq k. \end{aligned}$$

If $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \leq 1$, then

$$\begin{aligned} n &= \textstyle\sum_{i=1}^{n} x_i \\ &\leq n - 1, \end{aligned}$$

which is a contradiction.

So there is some $t \in \{1, \ldots, k-1, k+1, \ldots, n\}$ such that $x_t \geq 2$. We claim that there exists some $t > k$ such that $x_t \geq 2$. Otherwise, for each $i > k$ we see $x_i = 0$ or $x_i = 1$. Hence

$$\begin{aligned} n &= \textstyle\sum_{i=1}^{k-1} x_i + \sum_{i=k+1}^{n} x_i \\ &\leq k - 1 + \textstyle\sum_{i=k+1}^{n} 1 \\ &= n - 1. \end{aligned}$$

It is a contradiction. It follows that there is $t > k$ such that $x_t \geq 2$. Let t be maximal with $t > k$, and $x_t \geq 2$. Algorithm states that $a_{kt} = 1$, and so $r_k \neq 0$, a contradiction. Thus,

$$r_1, r_2, \ldots, r_n \neq 0,$$

and hence

$$r_1, r_2, \ldots, r_n \geq 1.$$

If $r_k > 1$ for some $1 \leq k \leq n$; then

$$\begin{aligned} n &= \textstyle\sum_{i=1}^{n} 1 \\ &< \textstyle\sum_{i=1}^{n} r_i \\ &= n, \end{aligned}$$

a contradiction. So, $r_i \leq 1$ for each $1 \leq i \leq n$.
Now, we conclude that

$$r_1 = r_2 = \cdots = r_n = 1,$$

as desired. Therefore, $A \in \mathcal{R}(n)$.

□

Algorithm constructs an integral row stochastic matrix A with $\sigma(A) = x$. This A is shown by $A(x)$ and it is called the canonical integral row stochastic matrix with L-ray x.

Example 2.3. *(i) Consider $x = (1, 0, 1, 2)$. We see $x \prec^* e$. Algorithm offers the following matrix.*

$$A(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(ii) Let $x = (0, 0, 2, 1, 2)$. We observe that $x \prec^* e$ and the matrix constructed by Algorithm is

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(iii) Let $x = (0, 1, 2, 0, 0, 0, 0, 5, 1)$. So $x \prec^* e$ and the desired matrix is

$$A(x) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(iv) If $x = (0, 0, 4, 1, 1)$, then Algorithm stops in iteration $k = 3$.

We now want to talk about the eigenvalues of the canonical integral row stochastic matrices.

Define I_k as the k-by-k identity matrix, and

$$L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \cdots, L_k = \begin{pmatrix} & & 1 \\ & & 1 \\ & 0 & \vdots \\ & & 1 \\ & 1 & 0 \end{pmatrix} \in \mathbf{M}_k.$$

Suppose that $\lambda(A)$ is the set of eigenvalues of matrix A.

- Corresponding to any 1 in the vector $x \in \sigma(\mathcal{R}(n))$, we have $1 \in \lambda(A(x))$.
- Corresponding to any $(0, *, 0, *, \ldots, *, 0, *, k)$, $k-1$ zeros, in the vector $x \in \sigma(\mathcal{R}(n))$, we have $\{1, -1, 0, \ldots, 0\} \subseteq \lambda(A(x))$, $k-2$ zeros.

⧫ $A(x)$ is similar (after simultaneous row and column permutations) to a direct sum of a k-by-k identity matrix I_k and $\bigoplus_{i=1}^{t} L_{j_i}$, where

$$k = |\{i : x_i = 1\}|, t = |\{i : x_i > 1\}|, \{x_i : x_i > 1\} = \{j_1, j_2, \ldots, j_t\}.$$

Example 2.4. (i) *Let* $x = (0, 1, 1, 2)$. *Then*

$$\begin{aligned} A(x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad & \longrightarrow \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ & \longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ & = \quad I_2 \bigoplus L_2. \end{aligned}$$

(ii) Consider $x = (0,2,0,1,0,3)$. *We observe that*

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= L_2 \bigoplus I_1 \bigoplus L_3.$$

Finally, we will make an interesting point about L_i in the form of an example.

Example 2.5. *For* $n = 4$, $L_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ *and its eigenvalues are* $\{0,0,1,-1\}$, *whereas for* $L_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *its eigenvalues are* $\{0,1,-1\}$.
This shows that as the dimension of the matrix increases, only the number of zeros of the eigenvalues increases. The two values of 1 and -1 will have the same number of repetitions as 1.
The eigenvectors of this matrix are interesting. For $n = 4$, *the matrix of eigenvectors is as follows.*

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

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