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ON THE EXISTENCE AND UNIQUNESS THEOREM OF THE GLOBAL SOLUTIONS FOR UFDES

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Abstract. The uncertain functional differential equation (UFDE) is a type of functional differential equations driven by a canonical uncertain process. Uncertain functional differential equation with infinite delay (IUFDE) have been widely applied in sciences and technology. In this paper, we prove an existence and uniqueness theorem for IUFDE intheinterval $[t_0, T]$, underuniform Lipschitz condition and weak condition. Also, the novel existence and uniqueness theorem under the linear growth condition and the local Lipschitz condition is proven. In the following, a more general type of UFDE considers, which the future state is determined by entire of the past states rather than some of them. Finally, the existence and uniqueness theorem is considered on theinterval $[t_0, \infty]$.

Keywords: Uncertain Functional differential equation, Canonical process, Uncertainty space.
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1. Introduction

The majority of life events such as attacks by terrorists, economic or political changes, tribal conflicts, governments' fall and wars takes place by coincident. For this reason, accurately anticipating or estimating of stocks or precious metal prices or exchange rates etc. are considered impossible. The only way to see in what way this factor influences the drop and growth of the value of corporations and companies, thus, can be concentrating on the stocks' prices. In order to carry out and figure out a more precise modeling of such phenomena, it is required to investigate the effects of the factors along with uncertainty theory. This idea, as mentioned, are according to self-duality, normality, subadditivity axioms and monotonicity. By proposing the uncertain process, Liu meant an uncertain process which possessed stationary and independent increment which is called canonical Liu's process which is applied in different other sciences like optimal control and economics. This process which is described by Brownian

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motion is similar to a stochastic process. [5], [7]. Further, the concepts of stochastic and its related process inspired Liu to introduce uncertain differential equations [5] driven by canonical Liu's process which contributes to better understanding the uncertain phenomena.
In connection with the significance of existence and uniqueness of a solution to uncertain differential equations driven by canonical Liu's process, the existence and uniqueness of solution to the uncertain differential equations were investigated Liu. He employed Lipschitz and Linear growth conditions [4], and also, Yao et al. presented the stability analysis of uncertain differential equations. [6]. A number of researchers attempted to manage finding analytic solutions for some particular types of uncertain differential equations like Chen and Liu [1]. Differential equations with memory identified as functional differential equations state the point that the velocity of the system pertains not only to the estate of the system at a specified moment but pertains to the history of the path to this instant. The range of differential equations with memory includes a vast category of differential equations. Such equations have an essential roleinan advancing number of models in biology ,engineering, physics, and other sciences. There is extensive research dealing with functional differential equations and their applications. We refer to the monographs [1], and references there. The existing paper proves a new theorem about the solution's existence and uniqueness of the uncertain functional differential equations under weaker conditions. Thispaperisorganizedasfollows: Some substantial concepts that are needed through out there maining sections givein section2. Section3 focuses on the main results including a new existence and uniqueness theorem for IUFDEs.This theorem provides us with the conditions to deal with some problems that are not previously solvable. In section 4, the global solution for IUFDEs is condider and, the final section includes a summary.

2. **Preliminaries**

The current section aims to briefly introduce several fundamental concepts in the theory of uncertainty.

Definition 2.1. [2]: Let Γ be a σ-algebra on a nonempty set Γ. Each element $A \in L$ is called an event and a set function $U : L \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

(1) Axiom1 (Normality) $U\{\Gamma\} = 1$.
(2) Axiom2 (Monotonicity) For every event A_1 and A_2 where $A_1 \subseteq A_2$, we have
$U\{A_1\} \leq U\{A_2\}$
(3) Axiom3 (Duality) $U\{A\} + U\{A^c\} = 1$ for any event A.
(4) Axiom4 (Subadditivity) For every countable sequence of events $A_1, A_2, \cdots$, we have

$$U\{\cup_{i=1}^{\infty} A_i\} \leq \Sigma_{i=1}^{\infty} U\{A_i\}$$

In addition, the triplet (Γ, L, U) is called an uncertainty space.

Definition 2.2. [2]: A uncertain process P_t is said to be a canonical process if:

- (i) $P_0 = 0$ and almost all sample paths are Lipschitz continuous,
- (ii) P_t has stationary and independent increments,
- (iii) every increment $P_{t+s} - P_s$ is a normally distributed uncertain variable $\aleph(0, t)$ with expected value 0 and variance t^2 whose uncertainty distribution is

$$\phi(x) = (1 + \ exp\ (\tfrac{\pi x}{\sqrt{3}t}))^{-1}.\ x \in \Re$$

3. The existence and uniqueness theorem for IUFDEs

In this section we consider UFDEs in $[t_0, T]$ with infinite delay as follow

$$dZ(t) = h(Z_t, t)dt + g(Z_t, t)dC(t),\ t_0 \leq t \leq T, \tag{1}$$

where $Z_t = \{Z(t + \delta) \mid -\infty < \delta \leq 0\}$ can be regarded as a $C((-\infty, 0], R^d)$-value uncertain process, where $h : C((-\infty, 0], R^d) \times [t_0, T] \to R^d$ and $g : C((-\infty, 0], R^d) \times [t_0, T] \to R^{d\times m}$ be uncertain measurable. We impose the initial data:

$$Z_{t_0} = \varsigma = \{\varsigma(\delta) \mid -\infty \leq \delta \leq t_0\} \tag{2}$$

where Z_{t_0} is an R_{t_0}-measurable $C((-\infty, 0], R^d)$-value uncertain variable such that $\varsigma \in M^2((-\infty, 0], R^d)$. In addition, let us state the following conditions:

- (i) (uniform Lipschitz condition) Function $h(Z, t)$ satisfies a uniform Lipschitz condition in the variable Z on a set $C((-\infty, 0], R^d)$ if for $t \in [t_0, T]$, a constant $L > 0$ exists with

$$\mid h(Z_t, t) - h(\overline{Z_t}, t) \mid^2 \bigvee \mid g(Z_t, t) - g(\overline{Z_t}, t) \mid^2 \leq \overline{L} \mid Z_t - \overline{Z_t} \mid^2, \tag{3}$$

- (II) Weak condition: There exists a positive constant L such that if $h(0, t), g(0, t) \in L^2[t_0, T]$ then $\mid h(0, t) \mid^2 \bigvee \mid g(0, t) \mid^2 \leq L$

Definition 3.1. [2] Z_t is a solution of equation (3.1) with initial data (3.2) if it is an R^d-value uncertain process for $t_0 \leq t \leq T$ and has the subsequent properties:

(1) it is continuous and $\{Z_t\}_{t_0 \leq t \leq T}$ is ρ_t-adapted
(2) $\int_{t_0}^{T} \mid h(Z, t)|dt < \infty$ and $\int_{t_0}^{T} |g(Z, t)|dt < \infty$
(3) $Z_{t_0} = \beta$ and, for every $t_0 \leq t \leq T$,

$$Z(t) = \varsigma_0 + \int_{t_0}^{t} h(Z_s, s)ds + \int_{t_0}^{t} g(Z_s, s)dC(s)\ a.s.$$

Lemma 3.2. *Let $Z(t)$ be the solution of (3.1) with initial data (3.2), and (I)-(II) hold then*

$$E(\sup_{-\infty<t\leq T} |Z(t)|)^2 \leq E\|\xi\|^2 + Pe^{6\overline{L}(T-t_0+1)(T-t_0)} \tag{4}$$

where $P = 3E\|\xi\|^2 + 6(T - t_0 + 1)(T - t_0)(L + \overline{L}E\|\xi\|^2)$.
In addition, $Z(t) \in M^2((-\infty, T], R^d)$.

Proof. : The stopping time is

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : \|Z_t\| \geq n\}.$$

For $n \geq 1$, Clearly, as $n \to \infty, \tau_n \uparrow T a.s.$ Let $Z_t^n = Z(t \wedge \tau_n), t \in [t_0, T]$. Then Z_t^n satisfy the next equation

$$Z_t^n = \xi(0) + \int_{t_0}^t h(Z_s^n, s)I_{[t_0,\tau_n]}(s)ds + \int_{t_0}^t g(Z_s^n, s)I_{[t_0,\tau_n]}(s)dC(s)$$

by utilizing $(e + f + g)^2 \leq 3(e^2 + f^2 + g^2)$, we have

$$|Z_t^n|^2 \leq 3|\xi(0)|^2 + 3|\int_{t_0}^t f(Z_s^n, s)I_{[t_0,\tau_n]}(s)ds|^2 + 3|\int_{t_0}^t g(Z_s^n, s)I_{[t_0,\tau_n]}(s)dC(s)|^2$$

By taking the expectation on both sides of the recent inequality, and using the Holder inequality and (II), one gets

$$E|Z_t^n|^2 \leq$$
$$3E|\xi(0)|^2 + 3E|\int_{t_0}^t h(Z_s^n, s)I_{t_0,\tau_n}(s)ds|^2 + 3E|\int_{t_0}^t g(Z_s^n, s)I_{[t_0,\tau_n]}(s)dC(s)|^2$$
$$\leq 3E\|\xi\|^2 + 3(t - t_0)E\int_{t_0}^t |h(Z_s^n, s)|^2 ds + 3E\int_{t_0}^t |g(Z_s^n, s)|^2 I_{[t_0,\tau_n]}(s)ds.$$

One further obtains that

$$E((\sup_{t_0<s\leq t}|Z_{(t)}^n|^2) \leq$$
$$3E\|\xi\|^2 + 3(t - t_0)E\int_{t_0}^t |h(Z_s^n, s)|^2 ds|^2 + 3E|\int_{t_0}^t |g(Z_s^n, s)|^2 d(s)$$
$$\leq 3E\|\xi\|^2 + 6(t - t_0 + 1)E\int_{t_0}^t (\overline{L}\|Z_s^n\|^2 + L)ds$$
$$\leq P_1 + 6\overline{L}(T - t_0 + 1)\int_{t_0}^t E(\|\xi\|^2 + \sup_{t_0<r\leq s}|Z_r^n|^2)ds$$
$$\leq P_2 + 6\overline{L}(T - t_0 + 1)\int_{t_0}^t E(\sup_{t_0<r\leq s}|Z_r^n|^2)ds,$$

where $P_1 = 3E\|\xi\|^2 + 6(t - t_0 + 1)(T - t_0)$, $P_2 = P_1 + 6\overline{L}(T - t_0 + 1)(T - t_0)E\|\xi\|^2$, By the Gronwall inequality,

$$E(\sup_{t_0<s\leq t}|Z_{(s)}^n|^2) \leq Pe^{6\overline{L}(T-t_0+1)(T-t_0)},\ t_0 \leq t \leq T.$$

Noting the fact that

$$(\sup_{-\infty<t\leq T}|Z(s)|)^2 \leq \|\xi\|^2 + (\sup_{t_0<s\leq t}|Z(s)|)^2,$$

therefore

$$E(\sup_{-\infty<s\leq t}|Z_s^n|^2) \leq E\|\xi\|^2 + E(\sup_{t_0<s\leq t}|Z_s^n|^2)$$
$$\leq E\|\xi\|^2 + Pe^{6\overline{L}(T-t_0+1)(T-t_0)}.$$

Letting $t = T$, it then follows that

$$E(\sup_{-\infty<s\leq T}|Z^n_s|^2)\leq E\|\xi\|^2+Pe^{6\overline{L}(T-t_0+1)(T-t_0)},$$

that is

$$E(\sup_{-\infty<s\leq T}|Z(s\wedge\tau_n)|^2)\leq E\|\xi\|^2+Pe^{6\overline{L}(T-t_0+1)(T-t_0)}.$$

Consequently

$$E(\sup_{-\infty<s\leq\tau_n}|Z(s)|^2)\leq E\|\xi\|^2+Pe^{6\overline{L}(T-t_0+1)(T-t_0)}.$$

If $n\to\infty$ then the following inequality is holds

$$E(\sup_{-\infty<s\leq T}|Z(s)|^2)\leq E\|\xi\|^2+Pe^{6\overline{L}(T-t_0+1)(T-t_0)}$$

□

Theorem 3.3. *Let the conditions* (I) *and* (II) *hold. Then problem (3.1) with initial data (3.2) has a unique solution* $Z_t\in M^2((\infty,T],R^d)$.

Proof. Let Z_t and Y_t be two solutions of equation (3.1)-(3.2). put $e(Z_s,Y_s)=h(s,Z_s)-h(s,Y_s)$ and $f(Z_s,Y_s)=g(s,Z_s)-g(s,Y_s)$. Then

$$Z_t-Y_t=\int_{t_0}^t eds+\int_{t_0}^t fdC_s.$$

Using inequality $(e+f)^2\leqslant 2(e^2+f^2)$, we obtain

$$|Z_t-Y_t|^2\leq 2|\int_{t_0}^t eds|^2+2|\int_{t_0}^t fdC_s|^2,$$

and by the Hölder inequality and Lipschitz condition I, we have

$$E|Z_s-Y_s|^2\leq 2\overline{L}(t-t_0)E\int_{t_0}^t|Z_s-Y_s|^2ds+2\overline{L}E\int_{t_0}^t|Z_s-Y_s|^2ds\leq$$

$$2\overline{L}(T-t_0+1)\int_{t_0}^t|Z_s-Y_s|^2ds,$$

from the fact $Z_{t_0}(s)=Y_{t_0}(s)=\varsigma(s), s\in(\infty,0]$, we get

$$E\sup_{t_0\leq s\leq t}|Z_s-Y_s|^2\leq 2\overline{L}(T-t_0+1)\int_{t_0}^t E\sup_{t_0\leq r\leq s}|Z_r-Y_r|^2ds.$$

According to Gronwall inequality, we have

$$E(\sup_{t_0\leq t\leq T}|Z_t-Y_t|^2)=0. \quad (5)$$

It means that $Z_{(t)}=Y_t$ for $t_0\leq t\leq T$.
Therefore, for all $-\infty<t\leq T, Z_t=Y_t$. So, the proof of uniqueness is complete
Now to consider the `existence`, let $Z^0_{t_0}=Z_0=\varsigma_0$, for $t_0\leq t\leq T$.
Let $Z^n_{t_0}=\xi$ for $n=1,2,\cdots$, also define Picard iterations sequence as follow

$$Z^n_t=\xi_0+\int_{t_0}^t h(Z^{n-1}_s,s)ds+\int_{t_0}^t g(Z^{n-1}_s,s)dC(s). \quad (6)$$

Clearly $Z^0(t) \in M^2((-\infty,T],R^d)$. By induction $Z^n(t) \in M^2((-\infty,T],R^d)$. From the Hölder inequality,and using equality $(e+f)^2 \leq 2e^2+2f^2$ and I we get

$$E|Z^n(t)|^2 \leq 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^t 2|h(Z_s^{n-1},s)-h(0,s)|^2 + 2|h(0,s)|^2 ds$$
$$+3E\int_{t_0}^t 2|g(Z_s^{n-1},s)-g(0,s)|^2 + 2|g(0,s)|^2)ds$$
$$\leq 3E\|\xi\|^2 + 3(t-t_0+1)E\int_{t_0}^t (2\overline{L}\|Z_s^{n-1}\|^2 + 2L)ds$$
$$\leq P_1 + 6\overline{L}(T-t_0+1)\int_{t_0}^t E(\sup_{t_0<r\leq s}|Z_r^{n-1}|^2)ds$$

$$\leq P_1 + 6\overline{L}(T-t_0+1)\int_{t_0}^t E(|Z_s^{n-1}|^2)ds, \tag{7}$$

where $P_1 = 3E\|\xi\|^2 + 6(t-t_0+1)(T-t_0)$. Consequently for any $k \geq 1$, one can obtain

$$\max_{1\leq n\leq k} E|Z^n(t)|^2 \leq P_1 + 6\overline{L}(T-t_0+1)\int_{t_0}^t \max_{1\leq n\leq k} E(|Z_s^{n-1}|^2)ds.$$
$$\leq P_2 + 6\overline{L}(T-t_0+1)E\int_{t_0}^t E(\max_{1\leq n\leq k}|Z_s^n|^2)ds,$$

where $P_2 = P_1 + 6\overline{L}(T-t_0+1)(T-t_0)E\|\xi\|^2$.
Of the Gronwall inequality, one gets that

$$\max_{1\leq n\leq k} E|Z^n(s)|^2 \leq P_2 e^{6\overline{L}(T-t_0+1)(T-t_0)}$$

Since k is arbitrary,

$$E|Z^n(s)|^2 \leq P_2 e^{6\overline{L}(T-t_0+1)(T-t_0)},\ t_0 \leq t \leq T\ n \geq 1 \tag{8}$$

Of the Hölder inequality and I, we have

$$E|Z^1(t)-Z^0(t)|^2 \leq 2E|\int_{t_0}^t h(Z_s^0,s)ds|^2 + 2E|\int_{t_0}^t g(Z_s^0,s)dC(s)|^2$$
$$\leq 2(t-t_0)E\int_{t_0}^t |h(Z_s^0,s)|^2 ds + E\int_{t_0}^t |g(Z_s^0,s)|^2 ds$$
$$\leq 2(t-t_0+1)E\int_{t_0}^t (2\overline{L}\|Z_s^0\|^2 + 2\overline{L})ds$$
$$\leq 4\overline{L}(t-t_0+1)(t-t_0) + 4\overline{L}(t-t_0+1)(t-t_0)E\|\xi\|^2,$$

that is

$$E(\sup_{t_0\leq s\leq t}|Z^1(t)-Z^0(t)|^2) \leq 4L(t-t_0+1)(t-t_0) + 4\overline{L}(t-t_0+1)(t-t_0)E\|\xi\|^2.$$

Setting $t = T$, then

$$E(\sup_{t_0\leq s\leq t}|Z^1(t)-Z^0(t)|^2) \leq$$
$$4L(T-t_0+1)(T-t_0) + 4\overline{L}(T-t_0+1)(T-t_0)E\|\xi\|^2 := P.$$

By the same manner, we compute

$$E|Z^1(t)-Z^0(t)|^2 \le 2E|\int_{t_0}^t [h(Z_s^1,s)-h(Z_s^0,s)]ds|^2$$

$$+2E|\int_{t_0}^t [g(Z_s^1,s)-g(Z_s^0,s]dC(s)|^2$$

$$\le 2(t-t_0)E\int_{t_0}^t |h(Z_s^1,s)-h(Z_s^0,s)|^2 ds + E\int_{t_0}^t |g(Z_s^1,s)-g(Z_s^0,s)|^2 ds$$

thus we derive that

$$E(\sup_{t_0\le r\le s}|Z^2(t)-Z^1(t)|^2) \le UE\int_{t_0}^t \|Z^1(s)-Z^0(s)\|^2 ds$$

$$\le U\int_{t_0}^t E(\sup_{t_0\le r\le s}|Z^1(r)-Z^0(r)|^2)ds \le U(t-t_0)P,$$

where $U=2\overline{L}(T-t_0+1)$. In the same way,

$$E(\sup_{t_0\le s\le t}|Z^3(t)-Z^2(t)|^2) \le U\int_{t_0}^t E(\sup_{t_0\le r\le s}|Z^2(r)-Z^1(r)|^2)ds$$

$$\le U\int_{t_0}^t U(s-t_0)Pds = \tfrac{P[U(t-t_0)]^2}{2},$$

continuing this process to find that,

$$E(\sup_{t_0\le s\le t}|Z^4(t)-Z^3(t)|^2) \le U\int_{t_0}^t E(\sup_{t_0\le s\le t}|Z^3(r)-X^2(r)|^2)ds$$

$$\le U\int_{t_0}^t \tfrac{[U(s-t_0)]^2P}{2}ds = \tfrac{P[U(t-t_0)]^3}{6}.$$

Now we claim that for all $n\ge 0$,

$$E(\sup_{t_0\le s\le t}|Z^{n+1}(s)-Z^n(s)|^2) \le \frac{P[U(t-t_0)]^n}{n!}\ t_0\le t\le T. \tag{9}$$

When $n=0,1,2,3$, inequality (3.9) is holds. We suppose that (3.9) holds holds check it for $n+1$. In fact,

$$E(\sup_{t_0\le s\le t}|Z^{n+2}(s)-Z^{n+1}(s)|^2) \le 2\overline{L}(t-t_0+1)\int_{t_0}^t E\|Z^{n+1}(s)-Z^n(s)\|^2 ds$$

$$\le UM\int_{t_0}^t E(\sup_{t_0\le r\le s}|Z^{n+1}(r)-Z^n(r)|^2)ds$$

By induction and (3.9),

$$E(\sup_{t_0\le s\le t}|Z^{n+2}(s)-Z^{n+1}(s)|^2) \le U\int_{t_0}^t \tfrac{[U(s-t_0)]^nP}{n!}ds = \tfrac{P[U(t-t_0)]^{n+1}}{(n+1)!}.$$

It is simple to observe that (3.9) holds for $n+1$. Accordingly, by induction, (3.9) holds for all $n\ge 0$. Now we verify Z_t is the solution of (3.1)-(3.2). Since Z_t^n converge to Z_t at the sense of L^2 and uncertainty on $M^2((-\infty,T],R^d)$. By setting $t=T$ in (3.9) , we have ,

$$E(\sup_{t_0\le t\le T}|Z^{n+1}(t)-Z^n(t)|^2) \le \frac{P[U(T-t_0)]^n}{(n)!}.$$

Using the Chebyshev inequality,

$$U\{\sup_{t_0\le t\le T}|Z^{n+1}(t)-Z^n(t)|^2 > \frac{1}{2^n}\} \le \frac{P[4U(T-t_0)]^n}{(n)!}.$$

From the fact $\sum_{n=0}^{\infty} P[4U(T-t_0)]^n/n! < \infty$, and by the Borel-Cantelli lemma, for almost all $w \in \Omega$, there exists a positive integer $n_0 = n_0(w)$ such that

$$\sup_{t_0 \le t \le T} |Z^{n+1}(t) - Z^n(t)|^2 \le \frac{1}{2^n} \ as\ n \ge n_0.$$

We know that the parial sums $Z_t^0 + \Sigma_{t=1}^n [Z_t^i - Z_t^{i-1}] = Z_t^n$ are uniformly in $[0,T]$. It is clear Z_t is-continuous and P_t- adapted. On the other hand from (3.9) the sequence $\{Z_t^n\}$ is Cauchy in L^2 , for every t. Therefore, in (3.8), $Z_t^2(-\infty, T]$. Let $n \to \infty$ then

$$E|Z_{(s)}^n|^2 \le P_2 e^{6\overline{L}(T-t_0+1)(T-t_0)} \ for\ all\ t_0 \le t \le T,$$

where $P_2 = P_1 + 6\overline{L}(T-t_0+1)(T-t_0)E\|\xi\|^2$.
Therefore, by use of the above result, we obtain that

$$E\int_{-\infty}^{T} |Z(s)|^2 ds = E\int_{-\infty}^{t_0} |Z(s)|^2 ds + E\int_{t_0}^{T} |Z(s)|^2 ds$$

$$\le E\int_{-\infty}^{0} |\xi(s)|^2 ds + \int_{t_0}^{T} P_2 e^{6\overline{L}(T-t_0+1)(T-t_0)} ds < \infty,$$

that is $Z_{(t)} \in M^2((-\infty, T], R^d)$. Now to show that $Z_{(t)}$ satisfy (3.1).

$$E|\int_{t_0}^{t} [h(Z_s^n, s) - h(Z_s, s)]ds|^2 + E|\int_{t_0}^{t} [g(Z_s^n, s) - g(Z_s, s)]dC(s)|^2$$

$$\le 2E|\int_{t_0}^{t} [h(Z_s^n, s) - h(Z_s, s)]ds|^2 + 2E|\int_{t_0}^{t} [g(Z_s^n, s) - g(Z_s, s)]dC(s)|^2$$

$$\le 2(t-t_0)E|\int_{t_0}^{t} [hf(Z_s^n, s) - h(Z_s, s)]ds|^2 + 2E|\int_{t_0}^{t} [g(Z_s^n, s) - g(Z_s, s)]dC(s)|^2$$

$$\le UE\int_{t_0}^{t} \|Z_s^n - Z_s\|^2 ds \le U\int_{t_0}^{t} E(\sup_{t_0 \le r \le s} |Z_r^n - Z_r|^2) ds$$

$$\le U\int_{t_0}^{T} E(|Z_s^n - Z_s|^2) ds.$$

Noting that sequence $Z_{(t)}^n$ is uniformly converge on $(-\infty, T]$, it means that for any given $\epsilon > 0$, there exists an n_0 such that as $n \le n_0$, for any $t \in (-\infty, T]$, one then deduces that $E(|Z_t^n - Z_t|^2 \le \epsilon$, further,

$$\int_{t_0}^{T} E(|Z_s^n - Z_s|^2) ds < (T-t_0)\epsilon.$$

In other words, for $t \in [t_0, T]$ one has

$$\int_{t_0}^{t} h(Z_s^n, s)ds \to^{L^2} \int_{t_0}^{t} h(Z_s, s)ds,\ \int_{t_0}^{t} g(Z_s^n, s)dCs \to^{L^2} \int_{t_0}^{t} g(Z_s, s)dCs$$

For $t_0 \le t \le T$, taking limits on both sides of (3.7),

$$\lim_{n\to\infty} Z_{(t)}^n = \xi(0) + \lim_{n\to\infty} \int_{t_0}^{t} h(Z_s^{n-1}, s)ds + \lim_{n\to\infty} \int_{t_0}^{t} g(Z_s^{n-1}, s)dCs$$

that is

$$Z_{(t)} = \xi(0) + \int_{t_0}^{t} h(Z_s, s)ds + \int_{t_0}^{t} g(Z_s, s)dCs\ t_0 \le t \le T$$

The expression mentioned above demonstrates that $Z_{(t)}$ is the solution of (3.1). So far, the existence of theorem is complete. □

Now, we consider the following conditions that are weaker than conditions (I) and(II).

(A) (Linear growth condition) For all $t \in [t_0, T]$ and $Z \in C((-\infty, 0], R^d)$, there exists a positive number L such that

$$|h(Z,t)|^2 \vee |g(Z,t)|^2 \leq L(1 + \|Z\|^2); \tag{10}$$

(B) (Local Lipschitz condition) For each integer $n \geq 1$, there exists a positive constant number L_n such that for all $t \in [t_0, T]$ and all $Z, Y \in C((-\infty, 0], R^d)$ with $\|Z\| \vee \|Y\| \leq n$, it follows that

$$|h(Z,t) - h(Y,t)|^2 \vee |g(Z,t) - g(Y,t)|^2 \leq L_n \|Z - Y\|^2; \tag{11}$$

Theorem 3.4. *Let conditions A and B hold.Then the initial value problem (3.1)-(3.2) has a unique $Z_{(t)}$. Moreover, $Z_{(t)} \in M^2((-\infty, T], R^d)$.*

Proof. For each $n \geq 1$, define truncation functions h_n and g_n as follows:

$$h_n(Z_t, t) = \{ \begin{array}{cc} h(Z_t, t) & \|Z_t\| \leq n \\ h(\frac{nZ_t}{\|Z_t\|}, t) & \|Z_t\| > n \end{array}$$

$$g_n(Z_t, t) = \{ \begin{array}{cc} g(Z_t, t) & \|Z_t\| \leq n \\ g(\frac{nZ_t}{\|Z_t\|}, t) & \|Z_t\| > n \end{array}$$

then h_n and g_n satisfy conditions (A) and (B). By Theorem 3.1, equation

$$Z_n(t) = \xi(0) + \int_{t_0}^{t} f_n((Z_n)_s, s)ds + \int_{t_0}^{t} g_n((Z_n)_s, s)dCs \; t_0 \leq t \leq T \tag{12}$$

has a unique solution $Z_n(t)$, moreover, $Z_n(t) \in M^2((-\infty, T], R^d)$. Of course,$Z_{n+1}(t)$ is the unique solution of equation

$$Z_{n+1}(t) = \xi(0) + \int_{t_0}^{t} h_{n+1}((Z_{n+1})_s, s)ds + \int_{t_0}^{t} g_{n+1}((Z_{n+1})_s, s)dCs \; t_0 \leq t \leq T,$$

and $Z_{n+1}(t) \in M^2((-\infty, T], R^d)$.

Define the stopping time $\tau_n = T \wedge \inf\{t \in [t_0, T] : \|(Z_n)_t\| \geq n\}$. Taking the expectation, and by the Hölder inequality, it deduces that

$$E|Z_{n+1}(t) - Z_n(t)|^2 \leq 2E|\int_{t_0}^{t} [f_{n+1}((Z_{n+1})_s, s)]ds - \int_{t_0}^{t} [h_n((Z_n)_s, s)]ds|^2$$

$$+2E|\int_{t_0}^{t} [g_{n+1}((Z_{n+1})_s, s)]dCs - \int_{t_0}^{t} [g_n((Z_n)_s, s)]dCs|^2 \leq$$

$$2(t - t_0)E\int_{t_0}^{t} |h_{n+1}((Z_{n+1})_s, s) - h_n((Z_n)_s, s)|^2 ds + E\int_{t_0}^{t} |g_{n+1}((Z_{n+1})_s, s) - g_n((Z_n)_s, s)|^2 ds$$

$$\leq 4(t - t_0)E\int_{t_0}^{t} |[h_{n+1}((Z_{n+1})_s, s) - h_{n+1}((Z_n)_s, s)|^2 + |h_{n+1}((Z_n)_s, s) - h_n((Z_n)_s, s)|^2]ds$$

$$+4E\int_{t_0}^{t} |[g_{n+1}((Z_{n+1})_s, s) - g_{n+1}((Z_n)_s, s)|^2 + |g_{n+1}((Z_n)_s, s) - g_n((Z_n)_s, s)|^2]ds.$$

For $t_0 \leq t \leq \tau_n$, we have known that

$$h_{n+1}((Z_n)_s, s) = h_n((Z_n)_s, s) = h((Z_n)_s, s),$$

$$g_{n+1}((Z_n)_s, s) = g_n((Z_n)_s, s) = g((Z_n)_s, s),$$

again by $Z_{n+1}(t_0 + s) = Z_n(t_0 + s) = \xi(s), s \in (-\infty, 0]$, one then gets that

$E(\sup_{t_0 \leq r \leq t} |Z_{n+1}(t) - Z_n(t)|^2) \leq$

$$\leq 4(t - t_0)E \int_{t_0}^{t} |h_{n+1}((Z_{n+1})_s, s) - h_{n+1}((Z_n)_s, s)|^2 ds$$

$$+4E \int_{t_0}^{t} |g_{n+1}((Z_{n+1})_s, s) - g_{n+1}((Z_n)_s, s)|^2 ds$$

$$4(t - t_0 + 1)E \int_{t_0}^{t} L_n \|(Z_{n+1})_s - (Z_n)_s\|^2 ds$$

$$\leq 4(t - t_0 + 1)L_n \int_{t_0}^{t} E(\sup_{t_0 \leq r \leq s} |(Z_{n+1})_r - (Z_n)_r|^2) ds.$$

From the Gronwall inequality, one sees that

$E(\sup_{t_0 \leq s \leq t} |Z_{n+1}(t) - Z_n(t)|^2) = 0 \; t_0 \leq t \leq \tau,$

this means that for $t_0 \leq t \leq \tau n$, we always have

$$Z_n(t) = Z_{n+1}(t). \tag{13}$$

It then deduces that τ_n is increasing, that is as $n \to \infty, \tau_n \uparrow T$ a.s. By linear growth condition, for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$ such that $\tau_n = T$ as $n \geq n_0$. Now define $Z(t)$ by $Z(t) = Z_{n0}(t), t \in [t_0, T]$. Next to verify that $Z_{(t)}$ is the solution of (3.1). By (3.13), $Z_{(t \wedge \tau_n)} = Z_n(t \wedge \tau_n)$, and by (3.12), it follows that

$$Z(t \wedge \tau_n) = \xi(0) + \int_{t_0}^{t \wedge \tau_n} h_n((Z)_s, s) ds + \int_{t_0}^{t \wedge \tau_n} g_n((Z)_s, s) dCs$$

$$= \xi(0) + \int_{t_0}^{t \wedge \tau_n} h((Z)_s, s) ds + \int_{t_0}^{t \wedge \tau_n} g((Z)_s, s) dCs.$$

Letting $n \to \infty$ then yields

$$X(t \wedge \tau_n) = \xi(0) + \int_{t_0}^{t \wedge \tau_n} h((Z)_s, s) ds + \int_{t_0}^{t \wedge \tau_n} g((Z)_s, s) dCs$$

that is

$$Z(t) = \xi(0) + \int_{t_0}^{t} h((Z)_s, s) ds + \int_{t_0}^{t} g((Z)_s, s) dCs.$$

It can be seen that $Z(t) \in M^2((-\infty, T], R^d)$ is the solution of (3.1) and $Z_n(t) \in M^2((-\infty, T], R^d)$. So far, the existence is complete. The uniqueness is obtained by stopping our process. The proof is complete. □

4. Global solution for IUFDEs

In the previous section, we proved the existence and uniqueness theorem of solutions for uncertain functional differential equation with initial date in $[t_0, T]$. Now, consider the assumptions of the existence and uniqueness theorem on every sub interval $[t_0, T]$ of $(-\infty, \infty)$. In follows, we consider the following UFDE:

$$dZ(t) = h(Z_t, t)dt + g(Z_t, t)dC(t), \qquad t \in [t_0, \infty), \tag{14}$$

with initial data (3.2), that equation (4.1) has a unique solution Z_t on $[t_0, \infty)$. Such a solution is called global solution. Now, Assume that for each real number $T > 0$ and each integer $n \geq 1$, there exists a positive constant $K_{T,n}$ such that for all $t \in [t_0, T]$ and all $Z, Y \in C((-\infty, 0], R^d)$ with $\|Z\| \bigvee \|Y\| n$, it follows that:

(a) $\mid f(Z,t) - f(Y,t) \mid^2 \bigvee \mid g(Z,t) - g(Y,t) \mid^2 \leq K_{T,n}\|Z - Y\|^2$

(b) There exists a positive number K_T such that for all $Z \in C((-\infty, 0], R^d)$ and $t \in [t_0, T]$, it then follows that

$$\mid f(Z,t) \mid^2 \bigvee \mid g(Z,t) \mid^2 \leq K_T(1 + \parallel Z \parallel^2)$$

Theorem 4.1. *Assume that* (*a*) *and* (*b*) *are hold. Then (4.1) has a unique global solution* Z_t*,more over,* $Z_t \in M^2((\infty, \infty), R^d)$.

Proof. The proof of Theorem 3 is similar to that of Theorem 2 . We omit it here. □

A kind of general uncertain functional differential equation will be considered next, its future state is dependent on all past states or part of them. For example, uncertain integral equation

$$dZ_{(t)} = h(Z_t, t)dt + \int_{t_0}^{t} \mid Z_s \mid G(Z_t, t)dC(t) \tag{15}$$

and uncertain functional equation

$$dZ_{(t)} = h(Z_t, t)dt + \sup_{t_0 \leq s \leq t} \mid Z_s \mid G(Z_t, t)dC(t) \tag{16}$$

Owing to the fact that there are some requirements for the formulation, some notations are primarily introduced. For each $t \geq 0$, let $C((-\infty, t], R^d)$ denote the family of bounded continuous functions $Z : (-\infty, t] \longrightarrow R^d$ with norm$\|Z\| = \sup_{-\infty<\delta\leq t} |Z(\delta)|$, and assume that $f(0,t)$ and $g(0,t)$ represent the mappings from $C((-\infty, t], R^d)$ toR^d and R^{dm}, respectively. Define $Z_t = Z(t + \delta) : -\infty < \delta t$. Consider a d-dimensional uncertain functional differential equation

$$dZ_{(t)} = h(Z_t, t)dt + g(Z_t, t)dC(t), \qquad t \in [t_0, \infty) \tag{17}$$

with initial value (3.2). Clearly, equations (15) and (16) are special cases of equation (17). Now we demonstrate the existence and uniqueness theorem for equation (4.17), its proof is similar to theorem 1, we omit it here and just state the theorem itself.

Theorem 4.2. *Assume that for each real number $T > 0$ and each integer $n \geq 1$, there exists a positive constant $K_{T,n}$ such that for all $t \in [t_0, T]$ and all $Z, Y \in C((-\infty, 0], R^d)$ with $\|Z\| \bigvee \|Y\| \leq n$, it follows that*

$$| h(Z,t) - h(Y,t) |^2 \bigvee | g(Z,t) - g(Y,t) |^2 \leq K_{T,n} \|Z - Y\|^2.$$

Assume further that for each $T > 0$, there exists a positive numberK_T such that for all $Z \in C((-infty, 0], R^d)$ and $t \in [t_0, T]$, it then follows that

$$| ch(Z,t) |^2 \bigvee | g(Z,t) |^2 \leq K_T(1 + \|Z\|^2)$$

Then equation (17) has a unique global solution Z_t, moreover, $Z_t \in M^2((-\infty, \infty), R^d)$.

5. Conclusion

In this study, the existence and uniqueness theorem for the solution of UFDEs with infinite delay are discussed by using uncertain space axioms. We provided a novel existence and uniqueness theorem under the local Lipschitz condition and the linear growth condition. Also, the global solution for IUFDEs is considered and the existence and uniqueness theorem under two conditions is proved.

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