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STRICTLY SUB ROW HADAMARD MAJORIZATION

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Abstract. Let $\mathbf{M}_{m,n}$ be the set of all m-by-n real matrices. A matrix R in $\mathbf{M}_{m,n}$ with nonnegative entries is called strictly sub row stochastic if the sum of entries on every row of R is less than 1. For $A, B \in \mathbf{M}_{m,n}$, we say that A is strictly sub row Hadamard majorized by B (denoted by $A \prec_{SH} B$) if there exists an m-by-n strictly sub row stochastic matrix R such that $A = R \circ B$ where $X \circ Y$ is the Hadamard product (entrywise product) of matrices $X, Y \in \mathbf{M}_{m,n}$. In this paper, we introduce the concept of strictly sub row Hadamard majorization as a relation on $\mathbf{M}_{m,n}$. Also, we find the structure of all linear operators $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ which are preservers (resp. strong preservers) of strictly sub row Hadamard majorization.

Keywords: Linear preserver, Strong linear preserver, Strictly sub row Hadamard majorization, Strictly sub row stochastic
2020 MSC: 15A04, 15A21

1. Introduction

The Hadamard product has been penetrated in many branches of mathematical sciences and other sciences such as linear algebra theory, programming languages, statistics, etc. See [1–4]. In this paper, with using the Hadamard product and a type of nonnegative matrices which are called strictly sub row stochastic matrices, we introduce a relation on $\mathbf{M}_{m,n}$ which is called strictly sub row Hadamard majorization or in brief SH-majorization. For $X, Y \in \mathbf{M}_{m,n}$, the Hadamard product (entrywise product) of $X = [x_{ij}]$ and $Y = [y_{ij}]$, is denoted by $X \circ Y$ and is defined by $X \circ Y = [x_{ij}y_{ij}]$. A matrix R in $\mathbf{M}_{m,n}$ with nonnegative entries is called strictly sub row stochastic if the sum of entries on every row of R is less than 1.

Definition 1.1. Let $X, Y \in \mathbf{M}_{m,n}$. We say that X is *SH-Hadamard majorized* by Y (denoted by $X \prec_{SH} Y$), if there exists a strictly sub row stochastic matrix $R \in \mathbf{M}_{m,n}$ such that $X = R \circ Y$.

For a linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$, it is said that T preserves (resp. strongly preserves) SH-Hadamard majorization if $T(X) \prec_{SH} T(Y)$ whenever

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$X \prec_{SH} Y$ (resp. $T(X) \prec_{SH} T(Y)$ if and only if $X \prec_{SH} Y$). In this paper, we characterize all linear operators on $\mathbf{M}_{m,n}$ that preserve (resp. strongly preserve) SH-majorization. The following convention will be fixed throughout the paper. $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ is the standard basis of $\mathbf{M}_{m,n}$. When we use E_{ij}, the positive integers i and j are either fixed or are understood from the context. The m-by-n matrix $\mathbf{J}$ is the matrix of all ones, $\mathbf{R}_{m,n}$ is the set of all m-by-n row stochastic matrices, and $\mathbf{sR}_{m,n}$ is the set of all m-by-n sub row stochastic matrices.

In the next proposition we investigate a useful result from [5]. For every $m \in \mathbb{N}$, let $\mathbb{N}_m = \{1, \ldots, m\}$.

Proposition 1.2. *[5, Theorem 2.6] Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. The following conditions are equivalent:*

- **(1):** *$T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(p, q) \neq (r, s)$.*
- **(2):** *There exist a function $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ and a matrix $A \in \boldsymbol{M}_{m,n}$ such that for every $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$,*

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{1}$$

where $x_{f(i,j)}$ means x_{pq} if $f(i, j) = (p, q)$.

2. Linear preservers of SH-Hadamard majorization

In this section, first we state and prove some properties of preservers of SH-Hadamard majorization on $\mathbf{M}_{m,n}$. Then we give some examples of linear preservers and strong linear preservers of SH-Hadamard majorization. Finally, we find the structure of all linear operators on $\mathbf{M}_{m,n}$ which preserve SH-Hadamard majorization. The next remark is helpful in the following.

Remark 2.1. The next results hold:

- **(i):** Let $A \in \mathbf{M}_{m,n}$. $A \prec_{SH} A$ if and only if $A = 0$.
- **(ii):** A linear operator $X \mapsto T(X)$ on $\mathbf{M}_{m,n}$, preserves $\prec_{SH}$ if and only if $X \mapsto PT(X)Q$ preserves $\prec_{SH}$, where $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ are arbitrary permutation matrices.
- **(iii):** For $A \in \mathbf{M}_{m,n}$ with no zero entries, the linear operator $X \mapsto T(X)$ is a linear preserver of $\prec_{SH}$ if and only if the linear operator $X \mapsto T(X) \circ A$ is a linear preserver of $\prec_{SH}$.

Now we give a useful proposition about linear preservers of $\prec_{SH}$ on $\mathbf{M}_{m,n}$.

Proposition 2.2. *If $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ is a linear preserver of $\prec_{SH}$, then $T(E_{pq}) \circ T(E_{rs}) = 0$, for every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(p, q) \neq (r, s)$.*

Proof. Assume if possible that $T(E_{pq}) \circ T(E_{rs}) \neq 0$ for some $(p,q) \neq (r,s)$. So $[T(E_{pq})]_{ij} = \lambda \neq 0$ and $[T(E_{rs})]_{ij} = \mu \neq 0$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $Y = \frac{1}{\lambda}E_{pq} - \frac{1}{\mu}E_{rs}$. Set $X = R \circ Y$, where $R = [r_{ij}]$ is a strictly sub row stochastic matrix such that r_{pq} and r_{rs} are $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Now, $X \prec_{SH} Y$ but $T(X) \not\prec_{SH} T(Y)$, which is a contradiction. So $T(E_{pq}) \circ T(E_{rs}) = 0$, for all $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(p,q) \neq (r,s)$. □

Definition 2.3. Let $A \in \mathbf{M}_{m,n}$. We say that A is dominated by a $(0,1)$-row stochastic matrix if there exists a $(0,1)$-row stochastic matrix $R \in \mathbf{M}_{m,n}$ such that $A = A \circ R$. The set of all matrices which are dominated by $(0,1)$-matrices is denoted by $\Pi_{m,n}$.

The next theorem gives important properties of linear preservers of SH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.4. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SH-Hadamard majorization, then the following conditions hold:*

- **(1):** *For every $1 \leq p \leq m$ and $1 \leq q \leq n$, $T(E_{pq}) \in \Pi_{m,n}$.*
- **(2):** *For every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $p \neq r$, $T(E_{pq})$ and $T(E_{rs})$ do not simultaneously have a nonzero entry in any row.*

Proof.

- **(1):** Assume if possible that $T(E_{pq}) \notin \Pi_{m,n}$ for some $1 \leq p \leq m$ and $1 \leq q \leq n$. So by using part (ii) and part (iii) of Remark 2.1, at least two entries of the first row of $T(E_{pq})$ are 1. Set $X = E_{pq}$ and $Y = 2E_{pq}$. Thus, $X \prec_{SH} Y$ but $T(X) \not\prec_{SH} T(Y)$.
- **(2):** Assume that $1 \leq p, r \leq m$, $1 \leq q, s \leq n$ with $p \neq r$ and let $T(E_{pq}) = [a_{ij}]$, $T(E_{rs}) = [b_{ij}]$. By part (ii) of Remark 2.1, without loss of generality we may assume that $a_{11} \neq 0$. Now by using Proposition 2.2, $b_{11} = 0$. We show $b_{1j} = 0$ for all $2 \leqslant j \leqslant n$. Let $b_{1j} \neq 0$ for some $2 \leq j \leq n$. Put $X = E_{pq} + E_{rs}$ and $Y = 2X$. So $X \prec_{SH} Y$. We show that $T(X) \not\prec_{SH} T(Y)$. If $T(X) \prec_{SH} T(Y)$ there exists a strictly sub row stochastic matrix R such that

$$\begin{pmatrix} a_{11} & \dots & b_{1j} & \dots & \star \\ & \vdots & \vdots & \vdots & \\ \star & \dots & \star & \dots & \star \end{pmatrix} = R \circ \begin{pmatrix} 2a_{11} & \dots & 2b_{1j} & \dots & \star \\ & \vdots & \vdots & \vdots & \\ \star & \dots & \star & \dots & \star \end{pmatrix},$$

which is imposible.

□

By using Proposition 1.2, we can prove the following theorem.

Theorem 2.5. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SH-Hadamard majorization, then the following conditions hold:*

(1): *There exist a function* $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ *and a matrix* $A \in \boldsymbol{M}_{m,n}$ *such that for every* $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$*,*

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{2}$$

where $x_{f(i,j)}$ *means* x_{pq} *if* $f(i,j) = (p,q)$*.*

(2): $T(X \circ Y) = T(X) \circ T(Y)$ *for all* $X, Y \in \boldsymbol{M}_{m,n}$ *if* $T(\boldsymbol{J})$ *is a* $(0,1)$*-matrix.*

Proof. **(1):** Since T is a linear preserver of $\prec_{SH}$, by using Proposition 2.2, we have $T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(p,q) \ne (r,s)$. Now the conclusion follows from the Proposition 1.2.

(2): Assume that T is a linear preserver of $\prec_{SH}$ and $T(\mathbf{J})$ is a $(0,1)$-matrix. By using Proposition 2.2, $T(E_{pq}) \circ T(E_{rs}) = 0$ for all $(p,q) \ne (r,s)$. So $T(E_{ij})$ is a $(0,1)$-matrix for each $1 \le i \le m$, $1 \le j \le n$ and $T(E_{ij}) \circ T(E_{ij}) = T(E_{ij})$. Let $X = \sum_{i,j} x_{ij} E_{ij}$ and $Y = \sum_{i,j} y_{ij} E_{ij}$ be arbitrary m-by-n real matrices. Now we have

$$\begin{aligned} T(X \circ Y) =& T(\sum_{i,j} x_{ij} E_{ij} \circ \sum_{i,j} y_{ij} E_{ij}) \\ =& T(\sum_{i,j} x_{ij} y_{ij} E_{ij}) \\ =& \sum_{i,j} x_{ij} y_{ij} T(E_{ij}) \\ =& \sum_{i,j} x_{ij} T(E_{ij}) \circ \sum_{i,j} y_{ij} T(E_{ij}) \\ =& T(X) \circ T(Y). \end{aligned}$$

□

To understanding the structure of the linear preservers of SH-Hadamard majorization, we present the following examples.

Example 2.6. *Assume that* P *is an* m*-by-*m *permutation matrix,* Q *is an* n*-by-*n *permutation matrix and* $A \in \mathbf{M}_{m,n}$*. The linear operator* $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ *defined by* $T(X) = (PXQ) \circ A$ *is a preserver of* $\prec_{SH}$*. Also,* T *strongly preserves* $\prec_{SH}$ *if* A *has no zero entry. But* $T(X) = (PX^tQ) \circ A$ *is not a preserver of* $\prec_{SH}$ *(*X^t *is the transpose of* X*).*

Example 2.7. *Let $X = [x_{ij}] \in \boldsymbol{M}_{m,n}$. Consider the linear operator $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ defined by*

$$T(X) = \begin{pmatrix} x_{11} & x_{11} & 0 & \cdots & 0 \\ 0 & & & & \\ & & \ddots & & \\ & & & & \\ 0 & & & & 0 \end{pmatrix}.$$

Now, $I \prec_{SH} 2I$ but $T(I) \not\prec_{SH} T(2I)$. So T is not a preserver of $\prec_{SH}$.

The following proposition is used to prove the main theorem of this section. For a subset X of $\mathbf{M}_{m,n}$, the set of extreme points of X is denoted by $\operatorname{ext}(X)$.

Proposition 2.8. *The set of all m-by-n real strictly sub row stochastic matrices is a strictly convex set that its extreme points are m-by-n, $(0,1)$-row stochastic matrices, i.e.*

$$\operatorname{ext}(\boldsymbol{sR}_{m,n}) = \{A \in \boldsymbol{R}_{m,n} : A \text{ is a } (0,1)\text{-row stochastic matrix}\}.$$

Proof. It is easy to see that every m-by-n, $(0,1)$-row stochastic matrix is an extreme point of $\mathbf{sR}_{m,n}$. Now we show that if $R \in \mathbf{sR}_{m,n}$, then R is not an extreme point of $\mathbf{sR}_{m,n}$. Without loss of generality we may assume that the first row of R has k nonzero components with $k \geqslant 2$. Let

$$R = \begin{pmatrix} r_{11} \dots r_{1n} \\ A \end{pmatrix},$$

and let $r_{1j_1}, \dots, r_{1j_k}$ be the nonzero components of the first row of R. Put

$$R_{j_1} = E_{j_1} + \begin{pmatrix} 0 \\ A \end{pmatrix}, \dots, R_{j_k} = E_{j_k} + \begin{pmatrix} 0 \\ A \end{pmatrix}.$$

So $R_{j_1}, \dots, R_{j_k} \in \mathbf{sR}_{m,n}$ and we have $R = r_{j_1} R_{j_1} + \cdots + r_{j_k} R_{j_k}$. Since $k \geqslant 2$, R is not an extreme point of $\mathbf{sR}_{m,n}$ and the proof is complete. □

The following theorem is the key to characterize the linear preservers of SH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.9. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T preserves $\prec_{SH}$ if and only if T satisfies the following conditions:*

- **(1):** *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(r,s) \ne (p,q)$.*
- **(2):** *For every $R \in \operatorname{ext}(\boldsymbol{sR}_{m,n})$ there exists a $(0,1)$-matrix $Z \in \boldsymbol{M}_{m,n}$ such that $Z \circ T(\boldsymbol{J}) = 0$ and $Z + T(R) \in \Pi_{m,n}$.*

Proof. By using part (iii) of Remark 2.1, without loss of generality we may assume that $T(\mathbf{J})$ is a $(0,1)$-matrix. Suppose that T preserves $\prec_{SH}$. By Proposition 2.2, (1) holds. Let $R \in \operatorname{ext}(\mathbf{sR}_{m,n})$. Since T satisfies (2), it is clear that $T(R)$ is a $(0,1)$-matrix. Also $R = R \circ \mathbf{J} \prec_{SH} 2\mathbf{J}$, and so there exists a strictly sub row stochastic matrix $D \in \mathbf{M}_{m,n}$ such that $T(R) = D \circ 2T(\mathbf{J})$. Thus,

$T(R) \in \Pi_{m,n}$ and there exist permutation matrices $P \in M_m$ and $Q \in M_n$ such that

$$T(R) = P \begin{pmatrix} U & 0 \\ & \\ & 0 \end{pmatrix} Q,$$

where U is a $k \times k$ $(0,1)$-row stochastic matrix for some $0 \leq k \leq \min\{\mathrm{m},\mathrm{n}\}$. By the use of part (ii) of Remark 2.1, we may assume that

$$T(R) = \begin{pmatrix} U & 0 \\ & \\ & 0 \end{pmatrix}.$$

Also by part (2) of Theorem 2.5, we have $T(R) \circ T(\mathbf{J}) = D \circ 2T(\mathbf{J})$. So $[T(R) - 2D] \circ T(\mathbf{J}) = 0$. Now we have

$$D = \begin{pmatrix} \frac{U}{2} & 0 \\ & \\ & V \end{pmatrix}, \quad \begin{pmatrix} & 0 \\ & \\ & V \end{pmatrix} \circ T(\mathbf{J}) = 0,$$

where, $V \in \mathbf{M}_{m-k,n}$ is strictly sub row stochastic. Now we can choose a $(0,1)$-matrix $W \in \Pi_{m-k,n}$ such that

$$\begin{pmatrix} 0 \\ \\ W \end{pmatrix} \circ T(\mathbf{J}) = 0.$$

Put

$$Z = \begin{pmatrix} 0 \\ \\ W \end{pmatrix}.$$

Therefore, $Z + T(R) \in \Pi_{m,n}$, and $Z \circ T(\mathbf{J}) = 0$.
Conversely, first similar to the necessary part and without loss of generality we can assume that $T(\mathbf{J})$ is a $(0,1)$-matrix. Let $X, Y \in \mathbf{M}_{m,n}$ and let $X \prec_{SH} Y$. Then there exists a strictly sub row stochastic matrix R in $\mathbf{M}_{m,n}$ such that $X = R \circ Y$ and hence by part (2) of Theorem 2.5, $T(X) = T(R) \circ T(Y)$. By Theorem 2.8, $R = \sum_{i=1}^{k} \lambda_i R_i$ for some matrices $R_1, \ldots, R_k \in \mathrm{ext}(\mathbf{sR}_{m,n})$ and some positive numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^{k} \lambda_i < 1$. By the use of part (2), for each $1 \leq i \leq k$, we can find $(0,1)$-matrices $Z_i \in \mathbf{M}_{m,n}$ such that $Z_i \circ T(\mathbf{J}) = 0$ and $T(R_i) + Z_i \in \Pi_{m,n}$. By part (2) of Proposition 1.2, $Z_i \circ T(R_i) = 0$ and hence $T(R_i) + Z_i$ is a $(0,1)$-matrix. Thus, $R' =$

$\sum_{i=1}^{k} \lambda_i(T(R_i) + Z_i) \in \mathbf{M}_{m,n}$ is strictly sub row stochastic . Now we have

$$\begin{aligned} T(X) =& T(R) \circ T(Y) \\ =& T\left(\sum_{i=1}^{k} \lambda_i R_i\right) \circ T(Y) \\ =& \left(\sum_{i=1}^{k} \lambda_i (T(R_i) + Z_i)\right) \circ T(Y) \\ =& R' \circ T(Y). \end{aligned}$$

Therefore, T preserves SH-Hadamard majorization. □

In the next Theorem we completely determine the structure of the linear operators $T : \mathbf{M}_{m,n} \to \mathbf{M}_{n,m}$, which preserves SH-Hadamard majorization.

Theorem 2.10. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T preserves $\prec_{SH}$ if and only if there exist $A \in \boldsymbol{M}_{m,n}$ and permutation matrices $Q_1, \ldots, Q_m \in \boldsymbol{M}_n$ such that*

$$T(X) = \begin{pmatrix} X_{i_1} Q_1 \\ X_{i_2} Q_2 \\ \vdots \\ X_{i_m} Q_m \end{pmatrix} \circ T(\boldsymbol{J}), \qquad \forall X \in \boldsymbol{M}_{m,n}, \tag{3}$$

where X_{i_j} are some rows of X for $1 \leqslant j \leqslant m$ (not necessarily distinct).

Proof. Assume that T is of the form (3) and $X \prec_{SH} Y$. Then there exists an m-by-n strictly sub row stochastic matrix R such that $X = R \circ Y$. Thus, $T(X) = S \circ T(Y)$ where

$$S = \begin{pmatrix} R_{i_1} Q_1 \\ R_{i_2} Q_2 \\ \vdots \\ R_{i_m} Q_m \end{pmatrix},$$

is an m-by-n strictly sub row stochastic matrix (R_{i_j} are some rows of R for $1 \leqslant j \leqslant m$). Therefore, $T(X) \prec_{SH} T(Y)$ and so T preserves $\prec_{SH}$.

Conversely, assume that T is a preserver of $\prec_{SH}$. By Proposition 1.2, there exist a function $\gamma : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ such that for every $X = [x_{i,j}] \in \mathbf{M}_{m,n}$,

$$T(X) = \begin{pmatrix} x_{\gamma(1,1)} \cdots & x_{\gamma(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{\gamma(m,1)} \cdots & x_{\gamma(m,n)} \end{pmatrix} \circ T(\mathbf{J}),$$

where $x_{\gamma(i,j)}$ means x_{uv} if $\gamma(i,j) = (u,v)$. Set $A = [a_{ij}] = T(\mathbf{J})$. So the rth row of $T(X)$ is $[a_{r1} x_{\gamma(i,1)} \ldots a_{rn} x_{\gamma(i,n)}]$. Now by part (ii) of Theorem 2.9, for every $(0,1)$-row stochastic matrix R, $T(R)$ has at most one nonzero entry in each row and hence for each $1 \leqslant j \leqslant n$, $\gamma(i,j) = (r,s)$. Thus, the nonzero entries of a row of $T(X)$ must be multiple of entries of a row of X. So the rth row of $T(X)$ is of the form $[a_{r1}x_{i_kj_1}\dots a_{rn}x_{i_kj_n}]$, where $1 \leqslant i_k \leqslant m$ and $\{j_1,\dots,j_n\} = \{1,\dots,n\}$. Therefore, T is of the form (3) and the proof is complete. □

3. Strong linear preservers of SH-Hadamard majorization

In this section, we characterize the linear operators on $\mathbf{M}_{m,n}$ which strongly preserve SH-Hadamard majorization. The next lemma shows that every strong linear preserver of $\prec_{SH}$ on $\mathbf{M}_{m,n}$ is invertible.

Lemma 3.1. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T strongly preserves $\prec_{SH}$, then T is invertible.*

Proof. Assume that $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a strong linear preserver of SH-majorization and $T(X) = 0$. Then, $T(X) \prec_{SH} 0$ and hence $X \prec_{SH} 0$. Therefore, $X = 0$ which implies that T is invertible. □

Lemma 3.2. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T strongly preserves $\prec_{SH}$, then $T(\boldsymbol{J})$ has no zero entry.*

Proof. Assume that the linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ strongly preserves $\prec_{SH}$. So by Theorem 2.5, T has the form 1.2 and by Lemma 3.1, T is invertible. Thus, $T(\mathbf{J})$ has no zero entry. □

The next proposision, gives necessary and sufficient conditions for a linear operator T on $\mathbf{M}_{m,n}$ that strongly preserves SH-Hadamard majorization.

Proposition 3.3. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T strongly preserves $\prec_{SH}$ if and only if T is invertible and T satisfies the following conditions:*

- **(1):** *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(r,s) \neq (p,q)$.*
- **(2):** *For every $R \in \mathrm{ext}(\boldsymbol{sR}_{m,n})$, $T(R)$ has exactly one nonzero entry in each row.*

Proof. Similar to the proof of Theorem 2.9, without loss of generality we can assume that $T(\mathbf{J})$ is a $(0,1)$-matrix. Assume that T strongly preserves $\prec_{SH}$. By Lemma 3.1, T is invertible and by part (1) of Theorem 2.9, (1) holds. Now, by part (2) of Theorem 2.9, for every $R \in \mathrm{ext}(\mathbf{sR}_{m,n})$ there exists a $(0,1)$-matrix $Y \in \mathbf{M}_{m,n}$ such that $Y \circ T(\mathbf{J}) = 0$ and $T(R) + Y$ has exactly one nonzero entry in each row. By Lemma 3.2, $T(\mathbf{J})$ has no zero entry. Hence $Y = 0$ and the conclusion is desired.
Conversely, since T is invertible and satisfies (2), $T^{-1}(\mathrm{ext}(\mathbf{sR}_{m,n})) \subseteq \mathrm{ext}(\mathbf{sR}_{m,n})$ and hence T^{-1} satisfies (2). For $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(r,s) \neq (p,q)$, assume that $A = T^{-1}(E_{rs})$ and $B = T^{-1}(E_{pq})$. Thus by part (2) of Theorem 2.5, $T(A \circ B) = T(A) \circ T(B) = E_{rs} \circ E_{pq} = 0$. This implies

that $A \circ B = 0$ and hence T^{-1} satisfies (1). Therefore, by Theorem 2.9, T^{-1} preserves $\prec_{SH}$ and so T strongly preserves $\prec_{SH}$. □

The following theorem characterizes the linear preservers of SH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 3.4. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T strongly preserves $\prec_{SH}$ if and only if there exist $A \in \boldsymbol{M}_{m,n}$ with no zero entry and permutation matrices $P \in \boldsymbol{M}_m$ and $Q_1, \ldots, Q_m \in \boldsymbol{M}_n$ such that*

$$T(X) = P \begin{pmatrix} X_1Q_1 \\ X_2Q_2 \\ \vdots \\ X_mQ_m \end{pmatrix} \circ A, \qquad \forall X \in \boldsymbol{M}_{m,n}, \tag{4}$$

where $X_1, \ldots, X_m$ are rows of X.

Proof. First assume that T strongly preserves $\prec_{SH}$. By Theorem 2.10, there are $A \in \mathbf{M}_{m,n}$ and permutation matrices $Q_1, \ldots, Q_m \in \mathbf{M}_n$ such that

$$T(X) = \begin{pmatrix} X_{i_1}Q_1 \\ X_{i_2}Q_2 \\ \vdots \\ X_{i_m}Q_m \end{pmatrix} \circ A, \qquad \forall X \in \mathbf{M}_{m,n},$$

where $X_{i_1}, \ldots, X_{i_m}$ are some rows of X. By Lemma 3.1, T is invertible and hence A has no zero entry and $X_{i_1}, \ldots, X_{i_m}$ are distinct rows of X. Therefore,

$$T(X) = P \begin{pmatrix} X_1Q_1 \\ X_2Q_2 \\ \vdots \\ X_mQ_m \end{pmatrix} \circ A, \qquad \forall X \in \mathbf{M}_{m,n},$$

where $P \in \mathbf{M}_m$ is a permutation matrix, as desired. For the proof of sufficiency, if T is of the form (4), we conclude that

$$T^{-1}(X) = P^{-1} \begin{pmatrix} X_1Q_1^{-1} \\ X_2Q_2^{-1} \\ \vdots \\ X_mQ_m^{-1} \end{pmatrix} \circ B, \qquad \forall X \in \mathbf{M}_{m,n},$$

where $B = [\frac{1}{a_{ij}}] \in \mathbf{M}_{m,n}$. Now, it is easy to check that T and T^{-1} preserve SH-Hadamard majorization. Therefore, T strongly preserves SH-Hadamard majorization and the proof is complete. □

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