Journal of Mahani Mathematical Research Center
Print ISSN: 2251-7952 Online ISSN: 2645-4505

INITIAL COEFFICIENT BOUNDS FOR INTERESTING SUBCLASSES OF MEROMORPHIC AND BI-UNIVALENT FUNCTIONS

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Article type: Research Article
(Received: 26 August 2021, Revised: 24 January 2022, Accepted: 24 January 2022)
(Available Online: 30 January 2022)
(Communicated by A. Askari hemmat)

Abstract. In this paper, we investigate an interesting subclass of univalent functions. Also, we introduce a new subclass of meromorphic bi-univalent functions. We obtain the estimates on the initial Taylor-Maclurin Coefficients for functions in the interesting subclass of meromorphically bi-univalent functions defined on $\Delta = \{z \in \mathbb{C} : \ 1 < |z| < \infty\}$.

Keywords: Univalent functions, Meromorphic functions, Meromorphic Bi-univalent functions, Coeffcient estimates, Vertical strip.
2020 MSC: 30C45.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit open disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, \ |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of functions of the form (1), that is, functions which are analytic and univalent in $\mathbb{U}$ and are normalized by the following conditions:

$$f(0) = 0, \quad f'(0) = 1.$$

The Koebe one-quarter theorem states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f) \quad , \qquad r_0(f) \geq \frac{1}{4}).$$

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DOI: 10.22103/jmmrc.2022.18124.1162

How to cite: H. Rahmatan, Sh. Najafzadeh and H. Haji, *Initial Coefficient Bounds for interesting Subclasses of Meromorphic and Bi-Univalent Functions*, J. Mahani Math. Res. Cent. 2022; 11(2): 9-17.

where

$$
\begin{aligned}
f^{-1}(w) &= e(w) \\
&= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots. \qquad (2)
\end{aligned}
$$

which implies that f^{-1} is analytic function.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$.

Let f and g be analytic in $\mathbb{U}$. Then f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$.

If g is univalent, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Suppose that $\mathcal{P}$ denote the class of analytic functions p of the type

$$
p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n \qquad (3)
$$

such that $Re\ p(z) > 0$.

Lemma 1.1 ([1]). *If $p \in \mathcal{P}$ and of the form 3, then for $n \in \mathbb{N} = \{1, 2, \dots\}$, the following sharp inequality holds*

$$
|p_n| \leq 2. \qquad (4)
$$

Let $\mathcal{S}_m$ denote the class of meromorphically univalent functions $g(z)$ of the form:

$$
g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \qquad (5)
$$

which are defined on the domain Δ given by

$$
\Delta = \{z \in \mathbb{C} : \ 1 < |z| < \infty\}.
$$

Since $g \in \mathcal{S}_m$ is univalent, it has an inverse $g^{-1} = h$ that satisfies the following condition:

$$
g^{-1}(g(z)) = z \qquad (z \in \Delta),
$$

and

$$
g(g^{-1}(w)) = w, \qquad (0 < M < |w| < \infty),
$$

where

$$
g^{-1}(w) = h(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n}, \quad (0 < M < |w| < \infty). \qquad (6)
$$

A simple computation shows that

$$
\begin{aligned}
w = g(h(w)) = (b_0 + B_0) &+ w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} \\
&+ \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3} + \dots \qquad (7)
\end{aligned}
$$

Comparing the initial coefficients in (7), we find that

$$\begin{aligned} b_0 + B_0 = 0 &\implies B_0 = -b_0 \\ b_1 + B_1 = 0 &\implies B_1 = -b_1 \\ B_2 - b_1B_0 + b_2 = 0 &\implies B_2 = -(b_2 + b_0b_1) \\ B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3 = 0 &\implies B_3 = -(b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2). \end{aligned}$$

By putting these values in the equation (6), we get

$$\begin{aligned} g^{-1}(w) &= h(w) \\ &= w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0b_1}{w^2} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} + \dots . \end{aligned} \tag{8}$$

Recently, some researchers for example, Janani and Murugusundaramoorthy [5] and Hamidi et al. [2, 3] introduced new subclasses of meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions in each of these subclasses.

Definition 1.2. Let $\mathcal{S}(\alpha, \beta)$ denote the class of all functions $f \in \mathcal{A}$ which satisfy the following two sided inequality

$$\alpha < Re \left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \quad (\alpha < 1,\ \beta > 1).$$

The class $\mathcal{S}(\alpha, \beta)$ was introduced in [4] and studied in [7]. By observation of subordination method and Definition 1.2, we conclude,

$$\frac{zf'(z)}{f(z)} \prec \mathcal{P}_{\alpha,\beta}(z) \quad (z \in \mathbb{U}), \tag{9}$$

where

$$\mathcal{P}_{\alpha,\beta}(z) := 1 + \frac{\beta - \alpha}{\pi} i\ log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right). \tag{10}$$

The function $\mathcal{P}_{\alpha,\beta}(z)$ is convex univalent in $\mathbb{U}$ and has the form

$$\mathcal{P}_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{11}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}} \right) \quad (n = 1, 2, ...) \tag{12}$$

and maps $\mathbb{U}$ onto a convex domain

$$\Omega_{\alpha,\beta} := \{ w \in \mathbb{C} :\ \alpha < Re\, w < \beta \}$$

conformally.

Recently, the function $\mathcal{P}_{\alpha,\beta}(z)$ has been studied by many works, see for example [4, 6, 7].

Theorem 1.3 ([4]). *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$, then*

$$|a_n| \leqq \prod_{k=2}^{n} \frac{k - 2 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{(n-1)!} \qquad (n = 2, 3, \cdots).$$

In our present investigation, the initial coefficients for certain subclasses of bi-univalent functions are given.

2. Coeffcient bounds for the function class $\mathcal{S}(\alpha, \beta)$

In this section, by using the subordination method we obtain estimates on the initial Taylor-Maclurin coefficients for functions in subclass of bi-univalent functions with bounded real part.

Definition 2.1. A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{S}(\alpha, \beta)$, if the following conditions are satisfied:

$$\alpha < Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ z \in \mathbb{U}), \tag{13}$$

and

$$\alpha < Re\left\{\frac{we'(w)}{e(w)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ w \in \mathbb{U}). \tag{14}$$

where the function e is the inverse of f.

Theorem 2.2. *Let f given by (1) be in the class $\mathcal{S}(\alpha, \beta)$. Then*

$$|a_2| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{|B_1^2 + B_1 - B_2|}} \tag{15}$$

and

$$|a_3| \leq 2|B_1| + |B_2|. \tag{16}$$

Proof. Let $f \in \mathcal{S}(\alpha, \beta)$ and $e = f^{-1}$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{zf'(z)}{f(z)} = \mathcal{P}_{\alpha,\beta}(u(z)) \tag{17}$$

and

$$\frac{we'(w)}{e(w)} = \mathcal{P}_{\alpha,\beta}(v(w)). \tag{18}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + ...$$

$$q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + ...$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + ...\right] \tag{19}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + ...\right]. \tag{20}$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$.
Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, hence from lemma1.1 we conclude, $|p_i| \le 2$ and $|q_i| \le 2$. Using (19) and (20) in (17) and (18) respectively, we have

$$\frac{zf'(z)}{f(z)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + ...\right]\right) \tag{21}$$

and

$$\frac{we'(w)}{e(w)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + ...\right]\right). \tag{22}$$

By using of (1), (2), (9)–(12), from (21) and (22), also from Definition 2.1, we have,

$$1 + a_2 z + (2a_3 - a_2^2)z^2 + ... = 1 + \frac{1}{2}B_1 p_1 z + [\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2]z^2 + ...$$

and

$$1 - a_2 w + (2a_2^2 - a_3)w^2 + ... = 1 + \frac{1}{2}B_1 q_1 w + [\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2]w^2 + ...$$

wherein $z, w \in \mathbb{U}$.
Which yields the following relations,

$$a_2 = \frac{1}{2}B_1 p_1, \tag{23}$$

$$2a_3 - a_2^2 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{24}$$

$$-a_2 = \frac{1}{2}B_1 q_1 \tag{25}$$

and

$$2a_2^2 - a_3 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{26}$$

From (23) and (25), it follows that

$$p_1 = -q_1 \tag{27}$$

and

$$8a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{28}$$

From (24), (26) and (28), we obtain

$$a_2^2 = \frac{B_1^3[p_2 + 2q_2]}{6[B_1^2 + B_1 - B_2]}.$$

Applying the properties of $p(z)$ and $q(z)$, for the coefficients p_2 and q_2, we immediately got the desired estimate on $|a_2|$ as asserted in (15).
By summing up the two sides of the (24) to (26) and using (27) and (28), we get

$$a_3 = \frac{1}{6}B_1[2p_2 + q_2] + \frac{1}{4}[B_2 - B_1]p_1^2.$$

Applying the properties of $p(z)$ and $q(z)$, once again for the coefficients p_1, p_2 and q_2, we get the desired estimate on $|a_3|$ as asserted in (16).

□

3. Coeffcient bounds for the function class $\mathcal{S}_m^{\Sigma}(\alpha, \beta)$

In this section, initial Taylor-Maclurin coefficients for functions in subclass of meromorphic bi-univalent functions with bounded real part are given.

Definition 3.1. A function $g(z) \in \Sigma_m$ is said to be in the class $\mathcal{S}_m^{\Sigma}(\alpha, \beta)$, if the following conditions are satisfied:

$$\alpha < Re\left\{\frac{zg'(z)}{g(z)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ z \in \Delta), \tag{29}$$

and

$$\alpha < Re\left\{\frac{wh'(w)}{h(w)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ w \in \Delta). \tag{30}$$

where the function h is the inverse of g.

Theorem 3.2. *Let g given by* (5) *be in the class $\mathcal{S}_m^{\Sigma}(\alpha, \beta)$. Then*

$$|b_0| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{|B_1^2 + B_1 - B_2|}} \tag{31}$$

and

$$|b_1| \leq \frac{1}{2}|B_1|. \tag{32}$$

Proof. Let $g \in \mathcal{S}(\alpha, \beta)$ and $h = g^{-1}$. Similar considerations apply to $\mathcal{S}_m^{\Sigma}(\alpha, \beta)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{zg'(z)}{g(z)} = \mathcal{P}_{\alpha,\beta}(u(z)) \tag{33}$$

and

$$\frac{wh'(w)}{h(w)} = \mathcal{P}_{\alpha,\beta}(v(w)). \tag{34}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + ...$$

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + ...$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + ...\right] \tag{35}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + ...\right], \tag{36}$$

where $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Using (35) and (36) in (33) and (34) respectively, we have

$$\frac{zg'(z)}{g(z)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + ...\right]\right) \tag{37}$$

and

$$\frac{wh'(w)}{h(w)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + ...\right]\right). \tag{38}$$

By using of (5), (6), (9)–(12) from (37) and (38), also from Definition 3.1, we have,

$$1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} + ... = 1 + \frac{1}{2}\frac{B_1 p_1}{z} + [\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2]\frac{1}{z^2} + ...$$

and

$$1 + \frac{b_0}{w} + \frac{b_0^2 + 2b_1}{w^2} + ... = 1 + \frac{1}{2}\frac{B_1 q_1}{w} + [\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2]\frac{1}{w^2} + ...$$

wherein $z \in \Delta$.
Which yields the following relations,

$$-b_0 = \frac{1}{2}B_1 p_1, \tag{39}$$

$$b_0^2 - 2b_1 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{40}$$

$$b_0 = \frac{1}{2}B_1 q_1 \tag{41}$$

and

$$b_0^2 + 2b_1 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{42}$$

From (39) and (41), it follows that

$$p_1 = -q_1 \tag{43}$$

and

$$8b_0^2 = B_1^2(p_1^2 + q_1^2). \tag{44}$$

From (40), (42) and (44), we obtain

$$b_0^2 = \frac{B_1^3[p_2 + q_2]}{4[B_1^2 + B_1 - B_2]}.$$

Applying the properties of $p(z)$ and $q(z)$, for the coefficients p_2 and q_2, we immediately got the desired estimate on $|b_0|$ as asserted in (31).
By subtracting (40) from (42) and using (43) and (44), we get

$$b_1 = -\frac{1}{8}B_1[p_2 - q_2].$$

Applying the properties of $p(z)$ and $q(z)$, once again for the coefficients p_2 and q_2, we get the desired estimate on $|b_1|$ as asserted in (32). □

4. Conclusion

The coefficient estimates for subclasses of analytic functions have always been the main interest of researchers in Univalent and bi-Univalent classes. Many studies related to this problem are around analytic normalized functions. Here the initial coefficients for certain subclasses of bi-univalent functions are given. Also, we may obtain bounds of Hankel and Toeplitz determinant for the classes in future.

References

[1] P. L. Duren, *Univalent Functions*, in: Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, (1983).

[2] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, *Faber polynomial coefficient estimates for meromorphic bi- starlike functions*, Internat. J. Math. Math. Sci. 1-4 (2013).

[3] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, *Coefficient estimates for a class of meromorphic biunivalent functions*, C. R. Acad. Sci. Paris Sér. I 351, 349-352(2013).

[4] K. Kuroki, S. Owa, *Notes on new class for certain analytic functions*, RIMS Kokyuroku Kyoto Univ. 1772, 21–25 (2011).

[5] T. Janani and G. Murugusundaramoorthy, *Coefficient estimates of meromorphic bi-starlike functions of complex order*, Internat. J. Anal. Appl. 4 (1), 68–77(2014).

[6] Y. J. Sim, O. S. Kwon, *Certain subclasses of meromorphically bi– univalent functions*, Bull. Malays. Math. Sci. Soc. 40, 841–855 (2017).

[7] Y. J. Sim, O. S. Kwon, *Notes on analytic functions with a bounded positive real part*, J.Inequal. Appl. 370, 1–6(2013).

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